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Decomposition method for fractional partial differential equation (PDEs) using Laplace transformation

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In this Paper, we propose an efficient combination for the solution of partial differential equations (PDEs). In a similar context, decomposition coupled with Laplace transform is applied to solve partial differential equations of fractional order. It is observed that the proposed technique is highly suitable for such problems. The results of the proposed scheme are highly encouraging and efficient. It is also observed that same may be extended to other partial differential equations also.

Key words: Laplace transformation method, fractional differential equations, wave equation, burgers equation, fluid mechanics.

INTRODUCTION

The idea of derivatives of non integer order initially appeared in a letter from Leibniz to L'Hospital in 1695. For three centuries, studies on the theory of fractional order were mainly constraint to the field of pure theoretical mathematics, which were only useful for mathematicians. In the last several decades, many researchers found that derivatives of non-integer order are very suitable for the description of various physical phenomena such as damping laws, diffusion process, etc. These findings evoked the growing interest of studies of fractional calculus in various fields such as physics, chemistry and engineering. For these reasons, we need reliable and efficient techniques for the solution of fractional differential equations (Hosseini et al., 2011; Gejji and Jafari, 2007; Miller and Ross, 1993; Oldham and Spanier, 1974; Podlubny, 1999; Mohyud-Din et al., 2009). The existence and uniqueness of solutions for fractional differential equations have been investigated by many authors such as (Podlubny, 1999). Most fractional differential equations do not have exact analytic solution, therefore approximation and numerical techniques must be used. In the last decades, several methods have been used to solve fractional differential equations, fractional partial differential equations, fractional integro-differential equations and dynamic systems containing fractional derivatives, such as Adomian's decomposition method (Gejji and Jafari, 2007; Momani and Noor, 2006; Momani and Shawagfeh, 2006; Ray et al., 2006; Khan and Faraz, 2011), He's variational iteration method (Momani and Odibat, 2006), Homotopy perturbation method (Mohyud-Din and Noor, 2009; Momani and Odibat, 2007; Sweilam et al., 2007), Laplace transform method (Jumarie, 2009; Duan and Xu, 2004; Jumarie, 2006). In the present article, we use traditional Decomposition method coupled with Laplace transform to construct appropriate solutions to multi dimensional wave, Burger's and Klein-Gordon equations of fractional order. Numerical results are highly encouraging.

ANALYSIS OF PROPOSED SCHEME

To illustrate the basic idea of this method we consider a general fractional nonlinear non homogeneous fractional partial differential equation with initial conditions of the form:

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$$D_t^{\alpha}u(x,t) + Ru(x,t) + Nu(x,t) = g(x,t), \quad (1)$$

$$u(x,0)=h(x), u_t(x,o) = f(x), \quad (2)$$

where g(x) is the source term, N represents the nonlinear term and R is the linear differential operator, where h(x),f(x) are algebraic conditions and $D_t^{\alpha}u(x,t)$ is the Caputo fractional derivative of the function $u(x,t)$ which is defined as:

$$D_t^{\alpha}u(x,t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{u^{n}(x,\zeta)}{t-\zeta^{\alpha+1-n}}d\zeta, (n-1 < Re(\alpha) \le n, n \in N)), \quad (3)$$

where $\Gamma(.)$ denotes the Gamma function. The properties of fractional derivative can be found in Equations (1) and (2). Laplace transform (denoted throughout this paper by L) of the Caputo operator is an important property will be used in this paper:

$$\mathcal{L}[D_t^{\alpha}u(x,t)] = s^{\alpha}(x,s) - \sum_{k=0}^{n-1} u^k(x,0)s^{\alpha-1-k}, (n-1 < \alpha \le n). \quad (4)$$

Taking the Laplace transform on both sides of Equation (1), we get:

$$\mathcal{L}D_t^{\alpha}u(x,t) + \mathcal{L}Ru(x,t) + \mathcal{L}Nu(x,t) = \mathcal{L}g(x,t). \quad (5)$$

Using the property of the Laplace transform, we have:

$$\mathcal{L}u(x,t) = \frac{h(x)}{s} + \frac{f(x)}{s^2} - \frac{1}{s^{\alpha}}\mathcal{L}Ru(x,t) - \frac{1}{s^{\alpha}}\mathcal{L}Nu(x,t) + \frac{1}{s^{\alpha}}\mathcal{L}g(x,t)) \quad (6)$$

Operating with the Laplace inverse on both sides of Equation (6) gives:

$$u(x,t) = G(x,t) - \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha}}\mathcal{L}Ru(x,t) + \frac{1}{s^{\alpha}}\mathcal{L}Nu(x,t)\right], \quad (7)$$

where $G(x,t)$ represent the term arising from the source term and the prescribed initial conditions. Then, we apply the Adomians Decomposition Method (ADM) the basic assumption is that the solutions can be written as a series:

$$u(x,t) = \sum_{n=o}^{\infty} u_n(x,t). \quad (8)$$

NUMERICAL APPLICATIONS

We applied the proposed algorithm to construct the solution of fractional partial differential equation (PDEs). Numerical results obtained by the suggested scheme are encouraging. To check the efficiency, few examples are presented thus:

Example 1

Consider the following one–dimensional liner inhomogeneous fractional wave equation:

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}} + \frac{\partial u}{\partial x} = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}sinx + tcosx, t > 0, x \in R, 0 < \alpha \le 1, \quad (9)$$

subject to the initial condition:

$$u(x,0) = 0. \quad (10)$$

Applying Laplace operator on both side of (3.1), we get:

$$\mathcal{L}\left(\frac{\partial^{\alpha}u}{\partial t^{\alpha}} + \frac{\partial u}{\partial x}\right) = \mathcal{L}\left(\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}sinx + tcosx\right),$$

$$s^{\alpha}u(x,s) - s^{\alpha-1}u(x,0) = \mathcal{L}\left(\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}sinx + tcosx\right) - \mathcal{L}\left(\frac{\partial u}{\partial x}\right),$$

$$u(x,s) = \frac{1}{s^{\alpha}}\mathcal{L}\left(\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}sinx + tcosx\right) - \frac{1}{s^{\alpha}}\mathcal{L}\left(\frac{\partial u}{\partial x}\right).$$

By using ADM, we have the following recurrence relation:

$$u_o(x,s) = \frac{1}{s^{\alpha}}\mathcal{L}\left(\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}sinx + tcosx\right),$$

$$u_{k+1}(x,s) = -\frac{1}{s^{\alpha}}\mathcal{L}\left(\frac{\partial u_k}{\partial x}\right), k \ge 0.$$

Consequently,

$$u_0(x,s) = \frac{1}{s^2}sinx + cosx\frac{1}{s^{\alpha+2}},$$

$$u_1(x,s) = -cosx\frac{1}{s^{\alpha+2}} + sinx\frac{1}{s^{2\alpha+2}}.$$

Applying the inverse Laplace operator on $u_0(x,s)$, $u_1(x,s)$, we have:

$$u_o(x,t) = sinxt + cosx\frac{t^{\alpha+1}}{\Gamma(2+\alpha)},$$

$$u_1(x,t) = -cosx\frac{t^{\alpha+1}}{\Gamma(2+\alpha)} + sinx\frac{t^{2\alpha+1}}{\Gamma(2+2\alpha)},$$

$$u_2(x,t) = -sinx\frac{t^{2\alpha+1}}{\Gamma(2+2\alpha)} - cosx\frac{t^{3\alpha+1}}{\Gamma(2+3\alpha)} \ldots.$$

In this manner the rest of components of the decomposition series can be obtained. The solution in series form is given as:

$$u(x,t) = tsinx + cosx\frac{t^{\alpha+1}}{\Gamma(2+\alpha)} - cosx\frac{t^{\alpha+1}}{\Gamma(2+\alpha)} + sinx\frac{t^{2\alpha+1}}{\Gamma(2+2\alpha)} - sinx\frac{t^{2\alpha+1}}{\Gamma(2+2\alpha)} - cosx\frac{t^{3\alpha+1}}{\Gamma(2+3\alpha)} \ldots.$$

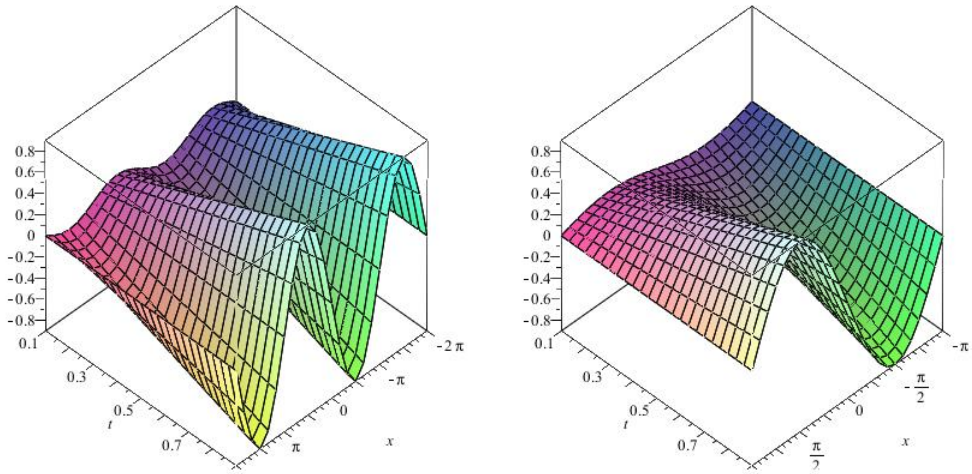

Figure 1. 3D Plot of $u(x,t) = tsinx$ for $x = -2\pi\ to 2\pi, x = -\pi\ to\ \pi$ for example 1.

Canceling the noise terms and keeping the non-noise terms yield the exact solution of Equation (9) is (Figure 1):

$$u(x,t) = tsinx.$$

Example 2

Consider the one–dimensional linear inhomogeneous fractional Burgers equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2, t > 0, x \in R, 0 < \alpha \le 1, \quad (11)$$

subject to the initial condition

$$u(x,0) = x^2. \quad (12)$$

Applying Laplace operator on both side of Equation (11):

$$\mathcal{L}\left(\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2}\right) = \mathcal{L}\left(\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2\right),$$

$$s^\alpha u(x,s) - s^{\alpha-1}u(x,0) = \mathcal{L}\left(\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2\right) - \mathcal{L}\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\right),$$

$$u(x,s) = \frac{1}{s}x^2 + \frac{1}{s^\alpha}\mathcal{L}\left(\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2\right) - \frac{1}{s^\alpha}\mathcal{L}\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\right).$$

By using ADM, we have the following recurrence relation:

$$u_0(x,s) = \frac{1}{s}x^2 + \frac{1}{s^\alpha}\mathcal{L}\left(\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2\right),$$

$$u_{k+1}(x,s) = \frac{1}{s^\alpha}\mathcal{L}\left(\frac{\partial^2 u_k}{\partial x^2} - \frac{\partial u_k}{\partial x}\right), k \ge 0.$$

Consequently,

$$u_0(x,s) = \frac{1}{s}x^2 + \frac{2}{s^3} + 2(x-1)\frac{2}{s^{\alpha+1}},$$

$$u_1(x,s) = -2(x-1)\frac{2}{s^{\alpha+1}} + 2\frac{t^{2\alpha}}{s^{2\alpha+1}}.$$

Applying the inverse Laplace operator on $u_0(x,s)$, $u_1(x,s)$, we have:

$$u_o(x,t) = x^2 + t^2 + 2(x-1)\frac{t^\alpha}{\Gamma(1+\alpha)},$$

$$u_1(x,t) = -2(x-1)\frac{t^\alpha}{\Gamma(1+\alpha)} - 2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)},$$

The solution in series form is given as

$$u(x,t) = x^2 + t^2 + 2(x-1)\frac{t^\alpha}{\Gamma(1+\alpha)} - 2(x-1)\frac{t^\alpha}{\Gamma(1+\alpha)} - 2\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \ldots$$

.

Canceling the noise terms and keeping the non-noise terms yield the exact solution of Equation (11) is (Figure 2):

$$u(x,t) = x^2 + t^2.$$

Example 3

Consider the one–dimensional linear inhomogeneous fractional klein-Goldon equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^2 u}{\partial x^2} + u = 6x^3 t + (x^3 - 6x)t^3, t > 0, x \in R, 1 < \alpha \le 2, \quad (13)$$

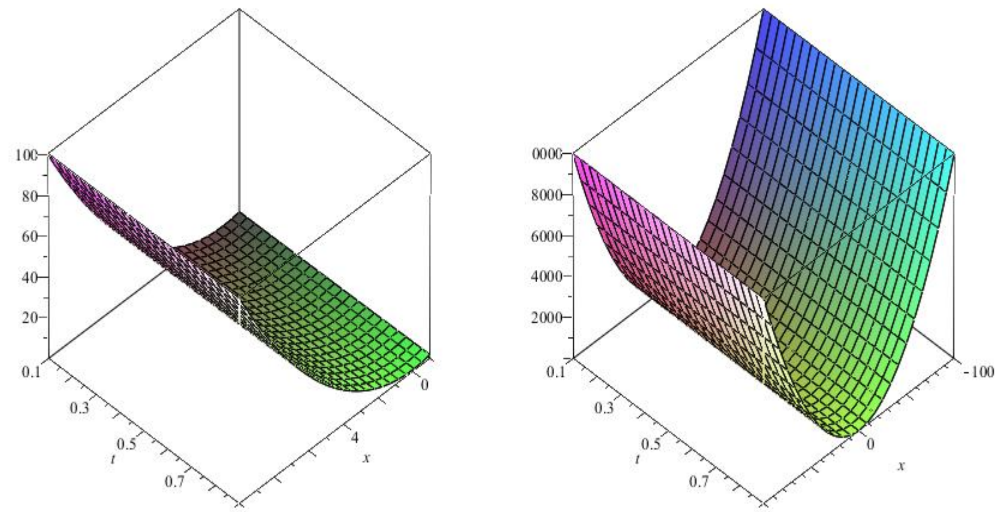

Figure 2. 3D Plot of $u(x,t) = x^2 + t^2$, for $x = -100\ to 100, x = 0\ \ to\ 10$, t= 0.1 to 0.10, for example 2.

subject to the initial condition:

$$u(x,0) = 0\,, \quad u_t(x,0) = 0\,. \tag{14}$$

Applying Laplace operator on both side of Equation(13), we get:

$$\mathcal{L}\left(\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^2 u}{\partial x^2} + u\right) = \mathcal{L}(\ 6x^3 t + (x^3 - 6x)t^3,$$

$$s^\alpha u(x,s) - s^{\alpha-1}u(x,0) - s^{\alpha-2}u(x,0) = \mathcal{L}(6x^3 t + (x^3 - 6x)t^3) + \mathcal{L}\left(\frac{\partial^2 u}{\partial x^2} - u\right),$$

$$u(x,s) = \frac{1}{s^\alpha}\ \mathcal{L}(6x^3 t + (x^3 - 6x)t^3) + \frac{1}{s^\alpha}\ \mathcal{L}\left(\frac{\partial^2 u}{\partial x^2} - u\right).$$

By using ADM, we have the following recurrence relation:

$$u_o(x,s) = \frac{1}{s^\alpha}\ \mathcal{L}(6x^3 t + (x^3 - 6x)t^3),$$

$$u_{k+1}(x,s) = \frac{1}{s^\alpha}\ \mathcal{L}\left(\frac{\partial^2 u_k}{\partial x^2} - u_k\right),\ k \geq 0.$$

Consequently,

$$u_o(x,s) = \left(6x^3 \frac{1}{s^{\alpha+2}} + (x^3 - 6x)\frac{3!}{s^{\alpha+4}}\right),$$

$$u_1(x,s) = 36x\frac{1}{s^{2\alpha+2}} - 36\frac{1}{s^{2\alpha+4}} - 18\frac{1}{s^{2\alpha+2}} -$$

$$6(3x^2 - 6)\frac{1}{s^{2\alpha+4}}.$$

Applying the inverse Laplace operator on $u_0(x,s)$, $u_1(x,s)$, we have:

$$u_o(x,t) = \ 6x^3\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + (x^3 - 6x)\frac{3!\,t^{\alpha+3}}{\Gamma(\alpha+4)},$$

$$u_1(x,t) = 36x\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - 18x^2\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - (3x^2 - 6)\frac{t^{2\alpha+3}}{\Gamma(2\alpha+4)}$$

The solution in the series is given as:

$$u(x,t) = \ 6x^3\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + (x^3 - 6x)\frac{3!\,t^{\alpha+3}}{\Gamma(\alpha+4)} + 36x\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - 18x^2\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - (3x^2 - 6)\frac{t^{2\alpha+3}}{\Gamma(2\alpha+4)} + \cdots$$

The closed form solution for $\alpha = 2$, is given as (Figure 3):

$$u(x,t) = x^3 t^3.$$

Conclusions

Laplace transform method is applied to find appropriate solutions of one dimensional wave equation, Burger's and Klein-Gordon equations. It may be concluded that the proposed technique is very powerful and efficient in finding the analytical solutions for a large class of partial differential equations of fractional order. Numerical results explicitly reveal the complete reliability and efficiency of the proposed algorithm. In our work we use MAPLE 13 package to show the solution graphically.

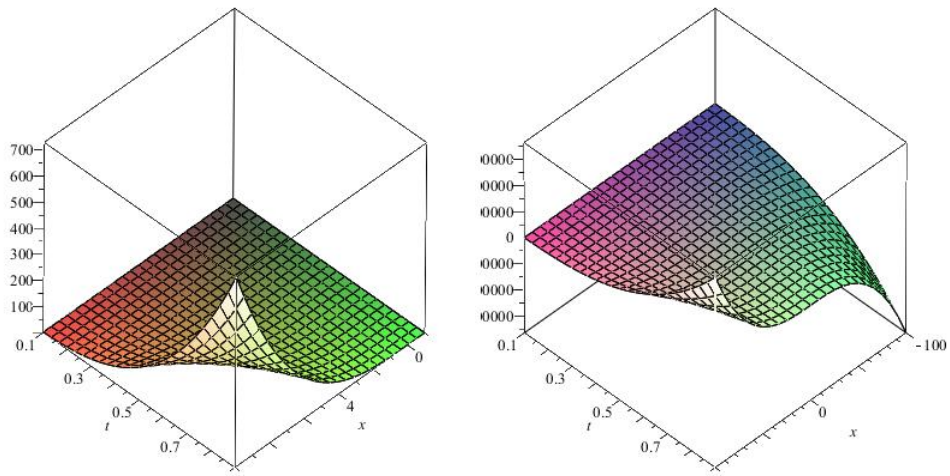

Figure 3. 3D Plot of $u(x,t)x^3t^3 =$ for $x = 0\ to 10, x = \ -100 to\ 100$ for example 3.

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