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Full Length Research Paper

Rarely generalized ideal (gI) continuous functions in ideal topological spaces

M. Khan and Murad Hussain*

Department of Mathematics, COMSATS Institute of Information Technology, Park Road, Islamabad, Pakistan.

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In this paper, we introduce and study a new notion of functions in ideal topological spaces known as rarely generalized ideal (gI)- continuous functions and investigate some of their properties. This type of continuity is a generalization of rarely g −continuity.

Key words: Rarely continuous functions, rarely g −continuous functions, rarely generalized ideal (gI)-continuous functions.

INTRODUCTION

Weak continuity was generalized by Popa (1979) as a rare continuity which was further investigated by Long and Herrington (1982), Jafari (1995, 1997). Caldas and Jafari (2005) further generalized rare continuity as rare g −continuity in topological spaces and investigated some of its properties.

In this paper, we introduce and study the concept of rarely gI −continuous functions in ideal topological spaces as a generalization of rare continuity by Popa (1979) and rarely g −continuity by Caldas and Jafari (2005).

PRELIMINARIES

Recall that a rare set is a set R such that $Int(R) = \emptyset$. Let (X, τ) be a topological space with no separation properties assumed. An ideal I on a topological space (X, τ) is a non-empty collection of subsets of X which satisfies the following properties:

1. $A \in I$ and $B \subset A$ implies $B \in I$.
2. $A \in I$ and $B \in I$ implies $A \cup B \in I$.

An ideal topological space is a topological space (X, τ) with an ideal I on X and is denoted by (X, τ, I). For a subset $A \subseteq X$, $A^*(I,\tau) = \{x \in X | A \cap U \notin I, \forall U \in \tau(X,x)\}$ is called the local function of A with respect to I and τ (Jankovic et al., 1990; Kuratowski, 1933). For a subset $A \subseteq X$, $A_*(I,\tau) = \{x \in X | A \cap U \notin I, \forall U \in SO(X,x)\}$ is called the semi-local function of A with respect to I and τ (Khan and Noiri, 2010), where $SO\,(X,x) = \{U \in SO(X) | x \in U\}$. We simply write A_* instead of $A_*(I)$ in case there is no ambiguity. For every ideal topological space (X, τ, I), there exists a topology $\tau^*(I)$, finer than τ. For a subset $A \subseteq X$, $Cl^*(A)$ and $Int^*(A)$ will, respectively, denote the closure and interior of A in (X, τ^*).

Definition 1

Let (X, τ, I) be an ideal topological space and A be a subset of X.

1. A is called g −closed (Levine, 1963) if $Cl(A) \subset G$ whenever $A \subset G$ and G is open in X.
2. A is called s^*g − closed (Khan et al., 2008) if

*Corresponding author. E-mail: murad@comsats.edu.pk.

$Cl(A) \subset U$ whenever $A \subset U$ and U is semi-open in X.

3. A is called gI −closed (Khan and Noiri, 2010) if $A_* \subset U$ whenever $A \subset U$ and U is open in X. The complement of gI −closed set is gI −open in X.

Definition 2

A function $f: (X, \tau_X) \longrightarrow (Y, \tau_Y, I)$ is called:

1. Weakly continuous (Levine, 1961) (respectively weakly-g-continuous (Caldas, Jafari and Noiri's preprint)) if for each $x \in X$ and each open set G containing $f(x)$, there exists $U \in O(X, x)$ (resprctively $U \in GO(X, x)$) such that $f(U) \subset Cl(G)$.
2. g −continuous (Balachandran et al., 1991) if the inverse image of every closed set in Y is g −closed in X.
3. Rarely continuous (Popa, 1979) if for each $x \in X$ and each $G \in O(Y, f(x))$, there exists a rare set R_G with $G \cap Cl(R_G) = \emptyset$ and $U \in O(X, x)$ with $f(U) \subset G \cup R_G$.
4. Rarely g −continuous (Caldas and Jafari, 2005) if for each $x \in X$ and each $G \in O(Y, f(x))$, there exists a rare set R_G with $G \cap Cl(R_G) = \emptyset$ and $U \in GO(X, x)$ with $f(U) \subset G \cup R_G$.
5. Weakly I −continuous (Jeyanthi et al., 2006) if for each $x \in X$ and each open set V in Y containing $f(x)$, there exists an open set U containing x such that $f(U) \subset Cl^*(V)$.
6. I_g −continuous (Caldas and Jafari, 2005) at $x \in X$ if for each set $G \in O(Y, f(x))$, there exists $U \in GO(X, x)$ such that $Int(f(U)) \subset G$.

Rarely gI −continuous functions

Here, we will introduce Rarely gI −continuous functions in ideal topological spaces and give some characterizations.

Definition 3

A function $f: (X, \tau_X, I) \longrightarrow (Y, \tau_Y)$ is known as rarely gI −continuous if for each $x \in X$ and each $G \in O(Y, f(x))$, there exists a rare set R_G with $G \cap Cl(R_G) = \emptyset$ and gI −open set U in X containing x with $f(U) \subset G \cup R_G$.

Note that, every weakly I −continuous function is weakly continuous, every weakly continuous function is rarely continuous, every rarely continuous function is rarely g −continuous and every rarely g −continuous function is rarely gI −continuous.

The following diagram depicts the inter relation between various continuities in topological and ideal topological spaces.

$$\text{weakly } I-\text{cont.} \rightarrow \text{weakly cont.} \rightarrow \text{rarely cont.} \rightarrow \text{rarely} - g - \text{cont.} \rightarrow I_g - \text{cont.}$$

$$\text{rarely} - g - \text{cont.} \longrightarrow \text{rarely} - gI - \text{cont.}$$

Remark

1. A rarely gI −continuous function need not be continuous (I −continuous).
2. RarelygI −continuous function need not be I_g −continuous.
3. Rarely gI −continuous function need not be rarely continuous.

Example

Let $X = \{a, b, c, d\}$ with $\tau_X = \{\emptyset, \{a, b, c\}, \{b, c\}, \{a\}, X\}$ and $I = \{\emptyset, \{a\}, \{a, b\}, \{b\}\}$. Let $Y = \{1,2,3,4\}$ with $\tau_Y = \{\emptyset, \{1\}, \{2\}, \{1,2\}, Y\}$. Let $f: (X, \tau_X, I) \longrightarrow (Y, \tau_Y)$ be defined by $f(a) = 2$, $f(b) = 1$, $f(c) = 3$, $f(d) = 4$. Then f is rarely gI −continuous but not continuous because for an open set $U = \{1\}$in Y, $f^{-1}(U) = \{b\}$ is not open in X.

Theorem 1

The following statements are equivalent for a function $f: (X, \tau_X, I) \longrightarrow (Y, \tau_Y)$:

(1) The function f is rarely gI −continuous at $x \in X$.

(2) For each set $G \in O(Y, f(x))$, there exists a gI −open set U in X containing x such that

$Int[f(U) \cap (Y - G)] = \emptyset$.

(3) For each set $G \in O(Y, f(x))$, there exists a gI −open set U in X containing x such that $Int[f(U)] \subset Cl(G)$.

Proof

$(i) \Longrightarrow (ii)$. Let $G \in O(Y, f(x))$. By $f(x) \in G \subset Int(Cl(G))$ and the fact that $Int(Cl(G)) \in O(Y, f(x))$, there exists a rare set R_G with $Int(Cl(G)) \cap Cl(R_G) = \emptyset$ and a gI −open set $U \subset X$ containing x such that $f(U) \subset Int(Cl(G)) \cup R_G$. We have $Int[f(U) \cap (Y - G)] \subset [Int(Cl(G)) \cup R_G] \cap (Y - Cl(G)) = [Int(Cl(G)) \cap (Y - Cl(G))] \cup [R_G \cap (Y - Cl(G))] = \emptyset \cup [R_G \cap (Y - Cl(G))] = R_G \cap (Y - Cl(G))$
$Int[f(U) \cap (Y - Cl(G))] = \emptyset$.

$(ii) \Longrightarrow (iii)$. It is straightforward.

$(iii) \Longrightarrow (i)$. Let $x \in X$ and $G \in O(Y, f(x))$, then by (iii) there exists a gI −open set U in X containing x such that $Int[f(U)] \subset Cl(G)$. We have $f(U) = [f(U) - Int[f(U)]] \cup Int[f(U)] \subset [f(U) - Int[f(U)]] \cup Cl(G) = [f(U) - Int[f(U)]] \cup [G \cup (Cl(G) - G)] = [f(U) - Int[f(U)]] \cap (Y - G) \cup G \cup (Cl(G) - G)$ Let $R_1 = [f(U) - Int[f(U)]] \cap (Y - G)$ and $R_2 = Cl(G) - G$. Then R_1 and R_2 are rare sets. Moreover $R_G = R_1 \cup R_2$ is a rare set such that $Cl(R_G) \cap G = \emptyset$. This proves that $f(U) \subset G \cup R_G$. Hence f is rarely gI −continuous. This completes the proof.

Definition 5

A function $f: X \longrightarrow Y$ is gI −continuous at $x \in X$ if for each set $G \in O(Y, f(x))$, there exists a gI −open set U in X containing x such that $Int[f(U)] \subset G$. If f has this property at each point $x \in X$, then we say that f is gI −continuous on X.

Note that, every I_g −continuous function is gI −continuous and every gI −continuous function is rarely gI −continuous.

Theorem 2

Let (X, τ_X) be a space and (Y, τ_Y) be a regular space. A function $f: X \longrightarrow Y$ is rarely gI −continuous, if and only if f is gI −continuous.

Proof

We prove the necessity only since sufficiency is evident Let $x \in X$ and $f(x) \in G$ where G is an open set in Y. By regularity of Y, there exists a set $G_1 \in O(Y, f(x))$ such that $Cl(G_1) \subset G$. Since f is rarely gI −continuous, by theorem 6, there exists a gI −open set U in X containing x such that $Int[f(U)] \subset Cl(G_1) \subset G$. This implies $Int[f(U)] \subset G$. This proves that f is gI −continuous.

Lemma 1

If $g: Y \longrightarrow Z$ is continuous and one-to-one, then g preserves rare sets (Long and Herrington, 1982).

Theorem 3

If $f: X \longrightarrow Y$ is rarely gI −continuous and $g: Y \longrightarrow Z$ is a continuous surjection, then $g \circ f: X \longrightarrow Z$ is rarely gI −continuous.

Proof

Suppose $x \in X$ and $(g \circ f)(x) \in V$, where V is an open set in Z. By hypothesis, g is continuous, therefore $G = g^{-1}(V)$ is an open set in Y containing $f(x)$ such that $g(G) \subset V$. Since f is rarely gI −continuous, there exists a rare set R_G with $G \cap Cl(R_G) = \emptyset$ and a gI −open set U containing x such that $f(U) \subset G \cup R_G$. By Lemma 1, $g(R_G)$ is a rare set in Z. Since R_G is a subset of $Y - G$ and g is injective, we have $Cl(g(R_G)) \cap V = \emptyset$. This implies that $(g \circ f)(U) \subset V \cup g(R_G)$. This completes the proof.

Definition 6

A function $f: X \longrightarrow Y$ is gI −open if $f(U)$ is gI − open in Y for every gI −open set Uin X.

Theorem 4

Let $f: X \longrightarrow Y$ be a gI −open surjection and $g: Y \longrightarrow Z$ be a function such that $g \circ f: X \longrightarrow Z$ is rarely gI −continuous. Then g is rarely gI −continuous.

Proof

Let $y \in Y$ and $x \in X$ such that $f(x) = y$. Let $G \in O(Z, (g \circ f)(x))$. Since $g \circ f$ is rarely gI −continuous, there exists a rare set R_G with $G \cap Cl(R_G) = \emptyset$ and U a gI −open in X containing x such that $(g \circ f)(U) \subset G \cup R_G$. But $f(U)(\text{say } V)$ is a gI −open set containing $f(x)$. Therefore, there exists a rare set R_G with $G \cap Cl(R_G) = \emptyset$ and a gI −open set V in Y containing y such that $g(V) \subset G \cup R_G$. This proves that g is rarely gI −continuous.

Theorem 5

If $f: (X, \tau_X) \longrightarrow (Y, \tau_Y, I)$ is rarely gI −continuous function, then the graph function $g: X \longrightarrow X \times Y$, defined by $g(x) = (x, f(x))$ for every x in X, is rarely gI −continuous.

Proof

Suppose that $x \in X$ and W is any open set containing $g(x)$. It follows that there exist open sets U and V in X and Y, respectively, such that $(x, f(x)) \in U \times V \subset W$. Since f is rarely gI −continuous, there exists gI −open set G containing$x \in X$ such that $Int[f(G)] \subset Cl(V)$. Let $E = U \cap G$. By Theorem 3 (Khan and Noiri, 2010), E is gI − open set in X containing x and we have $Int[g(E)] \subset Int(U \times f(G)) \subset U \times Cl(V) \subset Cl(W)$.

Therefore, g is rarely gI −continuous.

Definition 6

Let $A = \{G_i\}$ be a class of subsets of X. By rarely union sets (Jafari, 1997) of A we mean $\{G_i \cup R_{G_i}\}$, where each R_{G_i} is rare set such that each of $G_i \cap Cl(R_{G_i})$ is empty.

Definition 7

A subset B of X is said to be rarely almost compact relative to X (Jafari, 1997) if every cover of B by open sets of$X$, there exists a finite subfamily whose rarely union sets cover B. A topological space X is said to be rarely almost compact if the set X is rarely almost compact relative to X.

Definition 8

A subset K of a space X is said to be gI −compact relative to X if every cover of K by gI −open sets in X has a finite subcover. A space X is said to be gI −compact if X is gI −compact relative to X.

Theorem 6

Let $f: X \longrightarrow Y$ be rarely gI −continuous and K be gI −compact relative to X. Then $f(K)$ is rarely almost compact relative to Y.

Proof

Suppose that Ω is an open cover of $f(K)$. Let B be the set of all V in Ω such that $V \cap f(K) \neq \emptyset$. Then B is an open cover of $f(K)$. Hence for each $k \in K$, there is some $V_k \in B$ such that $f(k) \in V_k$. Since f is rarely gI −continuous, there exists a rare set R_{V_k} with $V_k \cap Cl(R_{V_k}) = \emptyset$ and a gI −open set U_k containing k such that $f(U_k) \subseteq V_k \cap R_{V_k}$. Hence there is a finite subfamily $\{U_k\}_{k \in \Delta}$ which covers K, where Δ is a finite subset of K. The family

$\{V_k \cup R_{V_k}\}_{k\in\Delta}$ also covers $f(K)$. This proves that $f(K)$ is rarely almost compact relative to Y.

Conclusion

Rarely gI −continuity is a generalization of rarely g −continuity.

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