International Journal of the Physical Sciences Vol. 6(12), pp. 2844-2851, 18 June, 2011
Available online at http://www.academicjournals.org/IJPS
DOI: 10.5897/IJPS11.361
ISSN 1992 - 1950 ©2011 Academic Journals

Full Length Research Paper

Stability analysis of a gradient elastic beam using finite element method

Mustafa Özgür YAYLI

Sakarya İl Özel İdaresi Erenler, Sakarya, 54100, Turkey. E-mail: ozguryayli@msn.com Tel: (90) 264 275 2312. Fax: (90) 264 275 2315

Accepted 24 March, 2011

Gradient elasticity models have been proposed in recent years in order to describe phenomena that cannot be described by classical elasticity models. In this contribution, the finite element model is conducted within the context of non-classical continuum mechanics, by introducing a material length scale parameter. The finite element method for the Euler-Bernoulli beam model is applied for the buckling analyzes of micro/nanobeams with the additional boundary conditions. Gradient elasticity stiffness and stability matrices are calculated to solve beam buckling problems. A qualitative discussion is given on this calculation process. The study contains two major parts, namely, finite element modeling of gradient elastic beams with the additional boundary conditions and gradient elasticity theory based approach for predicting buckling behavior of nanostructures. The gradient elasticity solutions are compared with their classical counterparts. The results illustrate the small length-scale effect in the stability expression or the surprisingly stiffening effect against buckling for some classes of beam buckling problems. The aim of this paper is to introduce some scale effects to derive relevant structural models which applies to nanostructures.

Key words: Carbon nanotubes, gradient elasticity, finite element method, additional boundary conditions, nanobeams.

INTRODUCTİON

Nanotechnology covers a broad range of topics in the field of applied sciences. The small-size effect and nano-scale surface effect associated with nanotechnology become significant and consequently the classical continuum theory can not predict the behavior of the nano-scale structures. Understanding the mechanism of how the size effects modulate the mechanical properties is fundamental to many potential applications of nanostructures and devices, especially for small-scale beam-like structures. Though researchers have commonly adopted the classic beam theories (Wong et al., 1997; Poncharal et al., 1999; Treacy et al., 1996; Krishnan et al., 1998; Ru, 2004) to treat the mechanical behavior of nanobeams, these theories essentially fail to capture the size-dependent properties of nanobeams.

Classical (local) continuum constitutive models possess no material/intrinsic length scale. The typical dimensions of length that appear in the corresponding boundary value problems are associated with the overall geometry of the domain under consideration. Although, the classical continuum models are efficient in carbon nanotubes mechanical analysis through relatively simple formula, its applicability to identify the small-scale effect on CNT's mechanical behaviors is questionable. The limitation of the applicability of the classical or local continuum modes at small length scales is partly due to the fact that the classical modeling does not admit intrinsic size dependence in elastic solutions of inclusions and inhomogeneities. Various size-dependent continuum theories, which capture small scale effects such as couple stress elasticity theory, strain gradient theory, modified couple stress theory are reported.

The authors attention was in fact focused on the search of an exact or closed form solution for a practical, even though very simple, mechanical problem and this is to provide a reference solution for numerical approaches. We should bear in mind that all methods of nano scale analysis are essentially concerned with solving the basic differential equations of equilibrium and compatibility, although, in some of the methods this fact may be obscured. Analytical solutions are limited to the cases where the load distribution, section properties and boundary conditions can be described by mathematical expressions. However, the numerical analysis methods are generally more practical for complex structures. Five general classes of numerical methods are available for solving ordinary and partial differential equations encountered in the various branches of science and engineering: finite-difference, finite-volume, finite element, boundary element, spectral and pseudo-spectral methods. Among these methods, Finite Difference Method (FDM), Rayleigh-Ritz Method, Galerkin Method, Least-Squares Method, Finite Element Methods (FEM) have dominated the applications to problems in engineering.

Beams of micro/nano dimensions are extensively used as sensors in various micro/nano technology applications, as well as for interpreting experimental measurements for material constants (e.g. elastic moduli) and assessing small scale phenomena (e.g. interfacial/ internal stresses). The gradient or micro/nano elasticity relations can be used to revisit various classical strength of materials or structural mechanics relationships and derive new modified ones, more suitable for design requirements of micro/nano components and devices. Two well-known approaches are the gradient approach, where higher order strain gradients are included in the strain energy density, and the Cosserat approach, where additional degrees of freedom of point rotation are considered along with the usual translational ones. Both theories result to additional, higher order terms in the governing equations and are, as a result, more complicated to treat numerically with the finite element method than classical elasticity. Gradient elasticity, in particular, contains strain gradient terms in the virtual work expression, leading to the requirement for C_1 interpolation if the displacement field only is discretised. Appropriate elements exist and perform very well, however the approach is often seen as complicated and/or computationally costly. Cosserat elasticity is simpler to implement, as the additional terms can be treated using standard shape functions. With respect to the application of Cosserat type elastic theories for interpreting size effects in torsion and bending of elastic materials with microstructure.

In this paper, a simple and practical method of the analysis of the finite element method by using gradient elasticity is presented. A qualitative discussion is given on the finite element formulation of gradient elasticity beams. Gradient elasticity stiffness matrix and stability matrix are calculated to solve beam buckling problems. A representative example is presented in order to assess the effect of the microstructure on the response of gradient elastic components to buckling loading. The modest goal of this article is to show that the finite element model conducted within the context of non-classical continuum mechanics can indeed be extended to describe buckling problems at the micro/nano regime.

Gradient elasticity nanobeam model

The literature on gradient elasticity theories is very rich, and many different versions of gradient elasticity have been formulated over the past decades. In order to describe and model the size effects, several strain gradient theories have been presented (Fleck and Hutchinson, 1993; Fleck and Hutchinson, 1997; Aifantis, 1984; Gao et al., 1999; Chen and Wang, 2001). In the strain gradient theories, several length parameters are included, and through them, the size effects are predicted and characterized. However, since the strain gradient terms are included in the constitutive equations and the displacement gradient terms appear in the boundary conditions, considerable complications and difficulties were encountered in solving the related problems (Engel et al., 2002; Shu et al., 1999; Matsushima et al., 2002).

Researchers have also extended Eringen’s nonlocal theory of elasticity to elucidate the size-dependent mechanical properties. The most general form of the constitutive equation for nonlocal elasticity involves an integral over the entire region of interest. This integral contains a kernel function which describes the relative influences of strains at various locations on the stress at a given location. An exact or approximate solution for the nonlocal integral function can be determined in some very special circumstances using the Green function and hence its use is rather limited. Although an equivalent but approximate expression of the nonlocal stress in a differential form was also derived by Eringen for different nonlocal moduli, this differential nonlocal stress relation seemed not to have attracted the attention for some time.

The Bernoulli/Euler beam model is based on the assumption that the beam consists of fibers parallel to the x axis, each in a state of uniaxial tension or compression. The same approach is used herein with the uniaxial Hooke's law replaced by

$$\sigma(x)-(e_0a)^2\sigma''(x)=E\varepsilon(x) \qquad (1)$$

where $\sigma(x)$ is the axial stress, $\varepsilon(x)$ is the axial strain, E is Young's modulus, a is the internal characteristic length, and e_0 is a constant. The small deflection Bernoulli/Euler relation between strain and curvature is

$$\varepsilon(x)=-yv''(x) \qquad (2)$$

where v denotes the beam's transverse displacement. Combining Eqs. (1) and (2) gives

$$\sigma(x)-(e_0a)^2\sigma''(x)=-Eyv''(x) \qquad (3)$$

The moment resultant of the corresponding stress distribution, and indicates that

$$M = -\int_A \sigma y dA \tag{4}$$

Multiplying Equation (1) by $-ydA$ and integrating the result over the area A yields

$$M(x) - (e_0 a)^2 M''(x) = EIv''(x) \tag{5}$$

Where;

$$I = \int_A y^2 dA \tag{6}$$

This model is an integral nonlocal model. There is an alternate interpretation of Equation (5).
Expansion of the general integral constitutive equation of nonlocal elasticity for $e_0 a / L << 1$, retention of only the first two terms, and simplification to the case of uniaxial stress produces (Peddieson et al., 2003);

$$\sigma(x) = E(\varepsilon(x) + (e_0 a)^2 \varepsilon''(x)) \tag{7}$$

Substituting Equations (2) into Equation (7), substituting the result into Equation (4), performing the indicated integration, and using Equation (6) leads to

$$M(x) = -EI(v''(x) + \gamma^2 v^{(4)}(x)) \tag{8}$$

where $e_0 a = \gamma^2$. It can also be noted that the local Bernoulli/Euler beam model can be obtained by setting the parameter γ^2 identically equal to zero and the governing equation for the elastic buckling behavior of a gradient elasticity beam is

$$-EI(v^{(4)}(x) + \gamma^2 v^{(6)}(x)) - Pv^{(2)}(x)) = 0 \tag{9}$$

The equation for an Euler-Bernoulli beam is expressed in terms of only one unknown, namely, the deflection of the beam, and neglects the effect of transverse shear deformation. Equation (9) can be thought of as constituting a "strain gradient" form of the Bernoulli/Euler beam model.

Equation (9) is of higher order than that of the local Bernoulli/Euler beam model and its solution would require additional boundary conditions which are not physically obvious. The gradient elasticity model based on Equation (9) of the present study has been investigated by Kumar et al. (2008).

VARIATIONAL PRINCIPLES FOR GRADIENT ELASTIC BEAMS

Weak formulation of the gradient elasticity beam

Compared to classical elasticity, there are two issues that need to be addressed in gradient elasticity. Firstly, the numerical discretisation with finite elements is less straightforward than classical elasticity, which is due to the additional spatial derivatives that are included in the differential equations. Secondly, through the introduction of the higher-order derivatives additional constitutive coefficients have appeared that must be identified and quantified, which will be discussed briefly in this section. All possible boundary conditions (classical and non-classical) can be obtained with the aid of variational formulations of the problems associated with these components. Thus, well posed boundary value problems can be solved.

The weak formulation of Equation (9) contains two types of expressions: those involving both the dependent variable v and the weight function w, and those involving only the latter. Multiplying Equation (9) by w and integrate over the domain

$$\int_0^L -w(EI\frac{d^4v}{dx^4} + EI\gamma^2\frac{d^6v}{dx^6} - P\frac{d^2v}{dx^2}) = 0 \tag{10}$$

and integrating by parts

$$\begin{aligned} 0 \; = \; & \int_0^L (EI\frac{d^2w}{dx^2}\frac{d^2v}{dx^2} - EI\gamma^2\frac{d^3w}{dx^3}\frac{d^3v}{dx^3})dx - \int_0^L (P\frac{dw}{dx}\frac{dv}{dx})dx \\ & + Pw\frac{dv}{dx}\Big|_0^L + EIw\frac{d^3v}{dx^3}\Big|_0^L - EI\frac{dw}{dx}\frac{d^2v}{dx^2}\Big|_0^L \\ & + EI\gamma^2 w\frac{d^5v}{dx^5}\Big|_0^L - EI\gamma^2\frac{dw}{dx}\frac{d^4v}{dx^4}\Big|_0^L - EI\gamma^2\frac{d^2w}{dx^2}\frac{d^3v}{dx^3}\Big|_0^L \end{aligned} \tag{11}$$

We shall denote these types of expressions by $B(w,v)$.

$$B(w,v) = \int_0^L (EI\frac{d^2w}{dx^2}\frac{d^2v}{dx^2} - EI\gamma^2\frac{d^3w}{dx^3}\frac{d^3v}{dx^3})dx - \int_0^L (P\frac{dw}{dx}\frac{dv}{dx})dx \tag{12}$$

Presented Equation (11) is a weak derivation of the governing equations and boundary conditions for the beams based on gradient elasticity theory and The Bernoulli/Euler beam theory. From Equation (11), small scale effect essential boundary conditions for a gradient elastic beam are

$$v|_{x=0} = 0 \quad v|_{x=L} = 0 \quad \frac{dv}{dx}|_{x=0} = 0 \quad \frac{dv}{dx}|_{x=L} = 0 \quad \frac{d^3v}{dx^3}|_{x=0} = 0 \quad \frac{d^3v}{dx^3}|_{x=L} = 0 \tag{13}$$

Interpolating functions for gradient elastic beams

The variational form of Equation (11) requires that the interpolation functions of an element be continuous with nonzero derivatives up to order three. The approximation of the primary variables over a finite element should be such that it satisfies the interpolation properties that it satisfies the essential boundary conditions

$$w(x_e)=v_1 \quad w(x_{e+1})=v_2 \quad \theta(x_e)=\theta_1 \quad \theta(x_{e+1})=\theta_2 \quad \lambda(x_e)=\lambda_1 \quad \lambda(x_{e+1})=\lambda_2 \tag{14}$$

In satisfying the essential boundary conditions (14), the approximation automatically satisfies the continuity conditions. Hence, we pay attention to the satisfaction of (14), which form the basis for the interpolation procedure.

Since there is a total of six conditions in an element (three per node), a six-parameter polynomial must be selected for v:

$$w(x)=c_6x^5+c_5x^4+c_4x^3+c_3x^2+c_2x+c_1 \tag{15}$$

Note that the continuty conditions are automatically met since the existence of a nonzero third derivative of v in the element. The next step involves expressing c_i in terms of the primary nodal variables

$$\begin{aligned} &v_1^e \equiv w(x_e) \quad v_2^e \equiv (-\frac{dw}{dx})|_{x=x_e} \quad v_3^e \equiv (-\frac{d^3w}{dx^3})|_{x=x_e} \\ &v_4^e \equiv w(x_{e+1}) \quad v_5^e \equiv (-\frac{dw}{dx})|_{x=x_{e+1}} \quad v_6^e \equiv (-\frac{d^3w}{dx^3})|_{x=x_{e+1}} \end{aligned} \tag{16}$$

Such that the conditions (14) are satisfied

$$v_1^e = w(x_e) = c_6x_e^5+c_5x_e^4+c_4x_e^3+c_3x_e^2+c_2x_e+c_1 \tag{17}$$

$$v_2^e = (-\frac{dw}{dx})|_{x=x_e} = -5c_6x_e^4-4c_5x_e^3-3c_4x_e^2-2c_3x_e-c_2 \tag{18}$$

$$v_3^e = (-\frac{d^3w}{dx^3})|_{x=x_e} = -60c_6x_e^2-24c_5x_e-6c_4 \tag{19}$$

$$v_4^e = w(x_{e+1}) = c_6x_{e+1}^5+c_5x_{e+1}^4+c_4x_{e+1}^3+c_3x_{e+1}^2+c_2x_{e+1}+c_1 \tag{20}$$

$$v_5^e = (-\frac{dw}{dx})|_{x=x_{e+1}} = -5c_6x_{e+1}^4-4c_5x_{e+1}^3-3c_4x_{e+1}^2-2c_3x_{e+1}-c_2 \tag{21}$$

$$v_6^e = (-\frac{d^3w}{dx^3})|_{x=x_{e+1}} = -60c_6x_{e+1}^2-24c_5x_{e+1}-6c_4 \tag{22}$$

Matrix form of above equations

$$\begin{pmatrix} v_1^e \\ v_2^e \\ v_3^e \\ v_4^e \\ v_5^e \\ v_6^e \end{pmatrix} = \begin{pmatrix} 1 & x_e & x_e^2 & x_e^3 & x_e^4 & x_e^5 \\ 0 & -1 & -2x_e & -3x_e^2 & -4x_e^3 & -5x_e^4 \\ 0 & 0 & 0 & -6 & -24x_e & -60x_e^2 \\ 1 & x_{e+1} & x_{e+1}^2 & x_{e+1}^3 & x_{e+1}^4 & x_{e+1}^5 \\ 0 & -1 & -2x_{e+1} & -3x_{e+1}^2 & -4x_{e+1}^3 & -5x_{e+1}^4 \\ 0 & 0 & 0 & -6 & -24x_{e+1} & -60x_{e+1}^2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{pmatrix} \tag{23}$$

Inverting this matrix equation to express c_i in terms of v_1^e, v_2^e, v_3^e, v_4^e, v_5^e and v_6^e, and substituting the result into (15), we obtain

$$w^e(x) = v_1^e\phi_1^e+v_2^e\phi_2^e+v_3^e\phi_3^e+v_4^e\phi_4^e+v_5^e\phi_5^e+v_6^e\phi_6^e = \sum_{j=1}^{6} v_j^e\phi_j^e \tag{24}$$

and the interpolation functions ϕ_i^e in Eq.(24) can be expressed in terms of the local coordinate x

$$\phi_1^e = -\frac{x^5}{h_e^5}+\frac{5x^4}{2h_e^4}-\frac{5x^2}{2h_e^2}+1 \qquad \phi_2^e = \frac{x^5}{2h_e^4}-\frac{5x^4}{4h_e^3}+\frac{7x^2}{4h_e}-x \tag{25}$$

$$\phi_3^e = -\frac{x^5}{24h_e^2}+\frac{7x^4}{48h_e}+\frac{x^2h_e}{16}-\frac{x^3}{6} \qquad \phi_4^e = \frac{x^5}{h_e^5}-\frac{5x^4}{2h_e^4}+\frac{5x^2}{2h_e^2} \tag{26}$$

$$\phi_5^e = \frac{x^5}{2h_e^4}-\frac{5x^4}{4h_e^3}+\frac{3x^2}{4h_e} \qquad \phi_6^e = -\frac{x^5}{24h_e^2}+\frac{x^4}{16h_e}-\frac{x^2h_e}{48} \tag{27}$$

Where $x_{e+1}=x_e+h_e$.

EIGENVALUE ALGORITHM

The determination of the eigen values is of engineering as well as mathematical importance. In structural problems the eigen values denote either natural frequencies or buckling loads. In this section, we develop finite element models of eigenvalue problem for the buckling loads. The study of buckling of beam-column leads to an eigenvalue problem.

Finite element model

Finite element model of the nonlocal beam is obtained by subsitituting the finite element interpolation (24) for w and the ϕ_j.

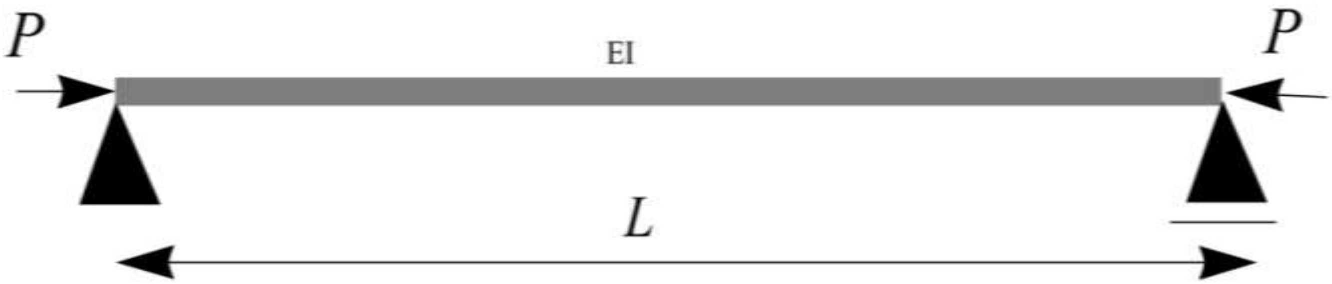

Figure 1. A simply supported beam subject to an axial loading.

$$0=\sum_{i=1}^{6}\sum_{j=1}^{6}\left(\int_{x_e}^{x_{e+1}}\left(EI\,\frac{d^2\phi_i^e}{dx^2}\frac{d^2\phi_j^e}{dx^2}-\gamma^2 EI\,\frac{d^3\phi_i^e}{dx^3}\frac{d^3\phi_j^e}{dx^3}\right)dx\right)v_j^e+\sum_{i=1}^{6}\sum_{j=1}^{6}\left(\int_{x_e}^{x_{e+1}}\left(P\,\frac{d\phi_i^e}{dx}\frac{d\phi_j^e}{dx}\right)dx\right)v_j^e \tag{28}$$

where

$$K_{ij}^e=\int_{x_e}^{x_{e+1}}\left(EI\,\frac{d^2\phi_i^e}{dx^2}\frac{d^2\phi_j^e}{dx^2}-\gamma^2 EI\,\frac{d^3\phi_i^e}{dx^3}\frac{d^3\phi_j^e}{dx^3}\right)dx \qquad G_{ij}^e=\int_{x_e}^{x_{e+1}}\left(P\,\frac{d\phi_i^e}{dx}\frac{d\phi_j^e}{dx}\right)dx \tag{29}$$

Note that the coefficients K_{ij}^e and G_{ij}^e are symmetric: By using interpolating functions K_{ij}^e can be written as

$$[K^e]=\begin{pmatrix}
\frac{5b\left(17h_e^2-168\gamma^2\right)}{7h_e^5} & b\left(\frac{60\gamma^2}{h_e^4}-\frac{85}{14h_e^2}\right) & \frac{b}{168} & \frac{5b\left(168\gamma^2-17h_e^2\right)}{7h_e^5} & b\left(\frac{60\gamma^2}{h_e^4}-\frac{85}{14h_e^2}\right) & \frac{b}{168} \\
b\left(\frac{60\gamma^2}{h_e^4}-\frac{85}{14h_e^2}\right) & b\left(\frac{113}{28h_e}-\frac{30\gamma^2}{h_e^3}\right) & -\frac{bh_e}{336} & b\left(\frac{85}{14h_e^2}-\frac{60\gamma^2}{h_e^4}\right) & b\left(\frac{57}{28h_e}-\frac{30\gamma^2}{h_e^3}\right) & -\frac{bh_e}{336} \\
\frac{b}{168} & -\frac{bh_e}{336} & b\left(\frac{11h_e^3}{6720}-\frac{\gamma^2h_e}{8}\right) & -\frac{b}{168} & -\frac{bh_e}{336} & \frac{b\left(840\gamma^2h_e-23h_e^3\right)}{20160} \\
\frac{5b\left(168\gamma^2-17h_e^2\right)}{7h_e^5} & b\left(\frac{85}{14h_e^2}-\frac{60\gamma^2}{h_e^4}\right) & -\frac{b}{168} & \frac{5b\left(17h_e^2-168\gamma^2\right)}{7h_e^5} & b\left(\frac{85}{14h_e^2}-\frac{60\gamma^2}{h_e^4}\right) & -\frac{b}{168} \\
b\left(\frac{60\gamma^2}{h_e^4}-\frac{85}{14h_e^2}\right) & b\left(\frac{57}{28h_e}-\frac{30\gamma^2}{h_e^3}\right) & -\frac{bh_e}{336} & b\left(\frac{85}{14h_e^2}-\frac{60\gamma^2}{h_e^4}\right) & b\left(\frac{113}{28h_e}-\frac{30\gamma^2}{h_e^3}\right) & -\frac{bh_e}{336} \\
\frac{b}{168} & -\frac{bh_e}{336} & \frac{b\left(840\gamma^2h_e-23h_e^3\right)}{20160} & -\frac{b}{168} & -\frac{bh_e}{336} & b\left(\frac{11h_e^3}{6720}-\frac{\gamma^2h_e}{8}\right)
\end{pmatrix} \tag{30}$$

where $b = EI$ is constant, h_e is the elements length. The coefficient matrix G_{ij}^e is known as the stability matrix. By using interpolating functions G_{ij}^e can be written as

$$[G^e] = \begin{pmatrix} \frac{155P}{126h_e} & -\frac{29P}{252} & \frac{Ph_e^2}{1512} & -\frac{155P}{126h_e} & -\frac{29P}{252} & \frac{Ph_e^2}{1512} \\ -\frac{29P}{252} & \frac{71Ph_e}{504} & -\frac{13Ph_e^3}{7560} & \frac{29P}{252} & -\frac{13Ph_e}{504} & \frac{Ph_e^3}{945} \\ \frac{Ph_e^2}{1512} & -\frac{13Ph_e^3}{7560} & \frac{13Ph_e^5}{362880} & -\frac{Ph_e^2}{1512} & \frac{Ph_e^3}{945} & -\frac{11Ph_e^5}{362880} \\ -\frac{155P}{126h_e} & \frac{29P}{252} & -\frac{Ph_e^2}{1512} & \frac{155P}{126h_e} & \frac{29P}{252} & -\frac{Ph_e^2}{1512} \\ -\frac{29P}{252} & -\frac{13Ph_e}{504} & \frac{Ph_e^3}{945} & \frac{29P}{252} & \frac{71Ph_e}{504} & -\frac{13Ph_e^3}{7560} \\ \frac{Ph_e^2}{1512} & \frac{Ph_e^3}{945} & -\frac{11Ph_e^5}{362880} & -\frac{Ph_e^2}{1512} & -\frac{13Ph_e^3}{7560} & \frac{13Ph_e^5}{362880} \end{pmatrix} \quad (31)$$

A beam subject to an axial loading

In this section a representative example is presented in order to assess the effect of the microstructure on the response of gradient elastic components to citical buckling load.

Consider the beam shown in Figure 1. The differantial equation (9) is valid. We will use one Euler-Bernoulli element in the half beam. Since EI is element wise constant, the element stiffness matrix is given by Equation (30) and the stability matrix is given by Equation (31). Because of the symmetry about $x = 0.5L$, we consider only half of the beam for finite element modeling. In this case, the boundary condition at $x = 0.5L$ is $\partial v / \partial x(0.5L) = 0$. Determination of the values of the parameter P such that the equation

$$K(v) = PG(v) \quad (32)$$

Where K and G denote linear differantial operators, has nontrivial solutions v is called an eigenvalue problem. The values of P are called eigen values and the associated functions V are called eigenfunctions. The characteristic polynomial of the above eigen value problem is obtained by setting the determinant of the coefficient matrix equal to zero

$$\begin{vmatrix} \frac{5b(17h^2-168\gamma^2)}{7h^5}-\frac{155P}{126h} & b\left(\frac{85}{14h^2}-\frac{60\gamma^2}{h^4}\right)-\frac{29P}{252} & \frac{h^2P}{1512}-\frac{b}{168} \\ b\left(\frac{85}{14h^2}-\frac{60\gamma^2}{h^4}\right)-\frac{29P}{252} & b\left(\frac{113}{28h}-\frac{30\gamma^2}{h^3}\right)-\frac{71hP}{504} & \frac{13h^3P}{7560}-\frac{bh}{336} \\ \frac{h^2P}{1512}-\frac{b}{168} & \frac{13h^3P}{7560}-\frac{bh}{336} & b\left(\frac{11h^3}{6720}-\frac{h\gamma^2}{8}\right)-\frac{13h^5P}{362880} \end{vmatrix} = 0 \quad (33)$$

or

$$0 = \frac{15b^3\gamma^4}{h^5}-\frac{12b^3\gamma^2}{7h^3}+\frac{5b^3}{252h}-\frac{5b^2\gamma^4P}{h^3}+\frac{1103b^2\gamma^2P}{1512h} -\frac{575b^2hP}{63504}+\frac{47bh^3P^2}{112896}-\frac{481bh\gamma^2P^2}{22680}-\frac{h^5P^3}{435456} \quad (34)$$

The smallest value of P called the critical buckling load P_{cr}

$$P_{cr} = \frac{C_3}{3C_4}+\frac{\sqrt[3]{2}\left(-C_3^2-3C_2C_4\right)}{3C_4\sqrt[3]{-2C_3^3-9C_2C_4C_3-27C_1C_4^2+\sqrt{\left(2C_3^3+9C_2C_4C_3+27C_1C_4^2\right)^2-4\left(C_3^2+3C_2C_4\right)^3}}} -\frac{\sqrt[3]{-2C_3^3-9C_2C_4C_3-27C_1C_4^2+\sqrt{\left(2C_3^3+9C_2C_4C_3+27C_1C_4^2\right)^2-4\left(C_3^2+3C_2C_4\right)^3}}}{3\sqrt[3]{2}C_4} \quad (35)$$

Where C_1, C_2, C_3 and C_4 constants can be listed as follows

$$C_1 = \frac{480b^3\gamma^4}{L^5}-\frac{96b^3\gamma^2}{7L^3}+\frac{5b^3}{126L} \qquad C_2 = -\frac{40b^2\gamma^4}{L^3}+\frac{1103b^2\gamma^2}{756L}-\frac{575b^2L}{127008} \quad (36)$$

$$C_3 = \frac{47bL^3}{903168}-\frac{481bL\gamma^2}{45360} \qquad C_4 = \frac{L^5}{13934592} \quad (37)$$

Figure 2 shows a comparison about critical buckling loads of the finite element method solution in gradient

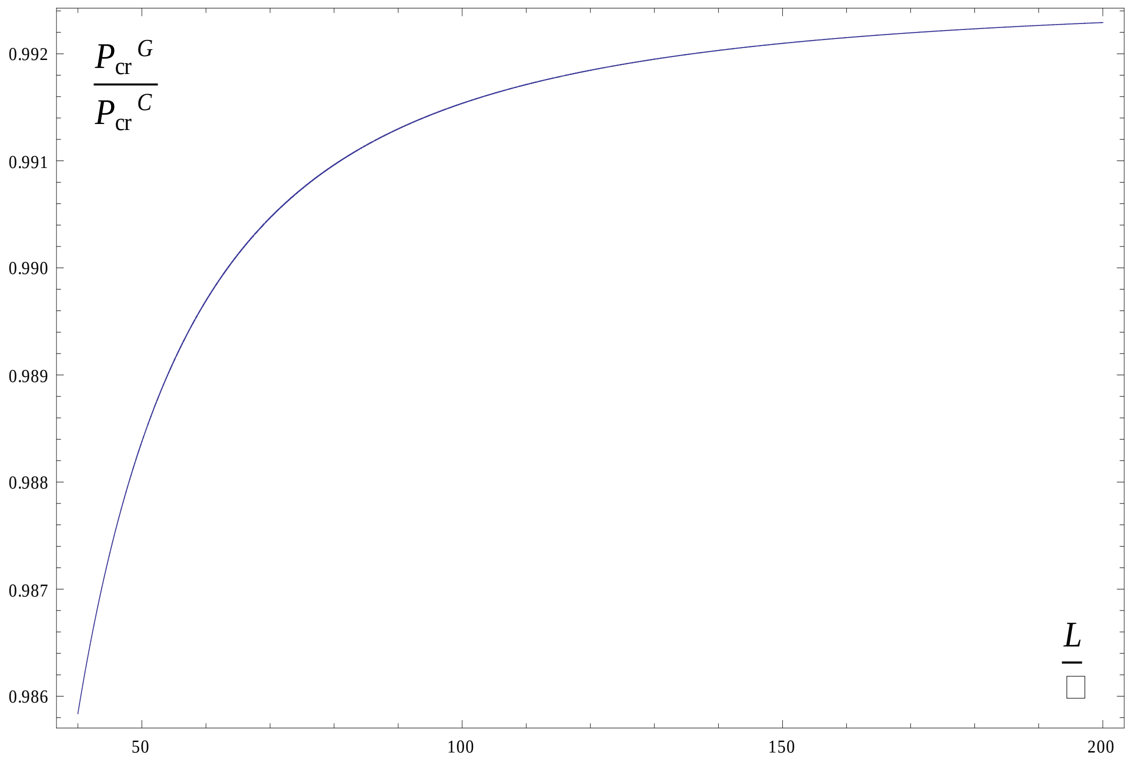

Figure 2. Comparison about critical buckling loads of the finite element method solution in gradient elasticity with the finite element method solution in classical elasticity of a simply supported beam.

elasticity with the finite element method solution in classical elasticity of a simply supported beam.

CONCLUSIONS

Present work has developed that the non classical continuum mechanics is combined with the finite element method to simulate the mechanical properties of the carbon nanotubes and nanostructures. The study contains two major parts, namely, finite element modeling of gradient elastic beams with the additional boundary conditions and gradient elasticity theory based approach for predicting buckling behavior of nanostructures. The governing equations of equilibrium for gradient elastic beams are derived and found to be ordinary for beams differential equations of an order which is higher by two than in the corresponding classical cases. As a result of that, one expects additional non-classical boundary conditions to the classical ones for a well posed boundary value problem. All possible boundary conditions (classical and non-classical)can be rationally obtained with the aid of variational principles. Gradient elasticity interpolation functions, stiffness and stability matrices are calculated by using geometrical boundary conditions. A representative example is presented in order to assess the effect of the microstructure on the response of gradient elastic components to critical buckling load. The results illustrate that the finite element method is successfully applied to analyze the nanomechanical characteristics of nanostructures and the small length-scale effect in the stability expression surprisingly stiffening effect against buckling for some classes of beam buckling problems. It is hoped that this paper will pave the way toward a better understanding of the application of non classical continuum models in the finite element method analysis of carbon nanotubes and nanostructures.

REFERENCES

Aifantis EC (1984). On the microstructural origin of certain inelastic models. Trans. ASME J. Engrg. Mater. Tech., 106: 326-330.
Chen SH, Wang TC (2001). Strain gradient theory with couple stress for crystalline solids. Eur. J. Mech. A Solids, 20: 739-756.
Engel G, Garikipati K, Hughes, TJR, Larson MG, Mazzei L, Taylor RL (2002). Continuous/discontinuous finite element approximations of fourth-order elliptic problems in structural and continuum mechanics with applications to thin beams and plates, and strain gradient elasticity. Comput. Methods Appl. Mech. Engrg., 191: 3669-3750.
Fleck NA, Hutchinson JW (1993). A phenomenological theory for strain gradient effects in plasticity. J. Mech. Phys. Solids. 41: 1825-1857.
Fleck NA, Hutchinson JW (1997). Strain gradient plasticity. Adv. Appl. Mech., 33: 295-361.
Gao H, Huang Y, Nix WD, Hutchinson JW (1999). Mechanism-based strain gradient plasticity - I. Theory. J. Mech. Phys. Solids, 47: 1239-1263.

Krishnan A, Dujardin E, Ebbesen TW, Yianilos PN, and Treacy MMJ (1998). Young's modulus of singlewalled nanotubes. Phys. Rev. B., 58: 14013-14019.

Kumar D, Heinrich C, Waas A M(2008). Buckling analysis of carbon nanotubes modeled using nonlocal continuum theories. J. Appl. Phys., 103: 073521.

Matsushima T, Chambon R, Caillerie D (2002). Large strain finite element analysis of a local second gradient model: Application to localization. Int. J. Numer. Methods Engrg., 54: 499-521.

Peddieson J, Buchanan GR, and McNitt RP (2003). Application of nonlocal continuum models to nanotechnology. Int. J. Eng. Sci., 41: 305-312.

Poncharal P, Wang ZL, Ugarte D, Heer WA (1999). Electrostatic deflctions and electromechanical resonances of carbon nanotubes. Sci., 283: 1513-1516.

Ru CQ (2004). Elastic models for carbon manutubes. Encyclopedia of Nanosci. Nanotechnol., pp. 731-744.

Shu JY, King WE, Fleck NA (1999). Finite elements for materials with strain gradient effects. Int. J. Numer. Methods Engrg., 44: 373-391.

Treacy MM, Ebbesen TW, Gibson JM (1996). Exceptionally high Young's modulus observed for individual carbon nanotubes. Nature, 381: 678-681.

Wong EW, Sheehan PE, Lieber CM (1997). Nanobeam mechanics: elasticity, strength, and toughness of nanorods and nanotubes and carbon nanotubes. Science, 277: 1971-1973.