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Full Length Research Paper

Determination of periodic solution for the Helmholtz-Duffing oscillators by Hamiltonian approach and coupled homotopy-variational formulation

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This paper aims to directly extend the Hamiltonian approach and coupled homotopy-variational formulation to study the periodic solutions of the Helmholtz-Duffing oscillator. The results of numerical example are presented and only a few terms are required to obtain accurate solutions. Results derived from this method are shown graphically. The behaviors of the solutions in the positive and negative directions are quite different, the asymmetric equation is separated into two auxiliary equations and the auxiliary equations are solved by two different method. Finally, the phase plane to show the stability of systems is plotted and discussed. An excellent agreement between the approximate periods with the exact period is achieved. The numerical results presented show that these methods are very accurate.

Key words: Helmholtz-Duffing oscillator; nonlinear oscillation, analytical methods, periodic solution.

INTRODUCTION

Most of engineering problems, especially some oscillation equations are nonlinear, and in most cases it is difficult to solve such equations, especially analytically. Large number of oscillation problems applications in mathematical interpretation of engineering problems, such as ship dynamics, oscillation of the human eardrum, dynamics of a particle moving in cubic potential and oscillations of one dimensional structural system with an initial curvature. In general, such problems are not amenable to exact treatment. Amongst these, the perturbation methods (Nayfeh, 1993; Hagedorn, 1988) are in common use. Perturbation methods are based on the existence of small parameters, the so-called perturbation quantity.

Recently, considerable attention has been paid towards approximate solutions for analytically solving nonlinear differential equation. Many nonlinear problems do not contain such perturbation quantity, so to overcome the shortcomings. Many new techniques have appeared in open literature such as: variational iteration method (He et et al., 2010; Hosseini et al., 2010; Yilmaz and Inc, 2010; Herişanu and Marinca, 2010), energy balance method (Ganji et al., 2009; Yazdi et al., 2010; He, 2002; Akbarzade et al., 2008; Yiming et al., 2011), Hamiltonian approach (He, 2010; Xu and He, 2010; Khan et al., 2010), coupled homotopy-variational formulation (Akbarzade and Langari, 2011; Akbarzade and Ganji, 2010), variational approach (He, 2007; Khan et al., 2011), amplitude-frequency formulation (Ganji and Akbarzade, 2010; Khan et al., 2011) and other classical methods (He, 2009, 2006, 2005, 2000, 2008; Marinca, 2006; Marinca and Herisanu, 2010; Khan and Wu, 2011; Khan et al., 2011; Turkyilmazoglu, 2011a, b; Ganji and Kachapi, 2011; Marinca and Herişanu, 2011; Herişanu and Marinca, 2010a, b; Nahe et al., 2011; Khan and Austin, 2010; Šmarda and Archalousova, 2010; Khan et al., 2011; Usman et al., 2011; Biazar and Eslami, 2011a, b).

In this paper, the basic idea of Hamiltonian approach and coupled homotopy-variational formulation are introduced and then their applications are studied for the following model of nonlinear oscillations (Leung and Guo, 2009):

$$\frac{d^2u}{dt^2}+u+sgn(u)\left(1-\sigma\right)u^2+\sigma u^3=0 \qquad (1)$$

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$$u(0) = A\,,\ \frac{du}{dt}(0) = 0 \quad (2)$$

where $sgn(u)$ is the sign function, the result equals to +1, if $u > 0$, 0 if $u = 0$ and −1 if $u < 0$.

THE APPLICATION OF THE HAMILTONIAN APPROACH (HA)

Previously, He (2011) had introduced energy balance method based on collocation and Hamiltonian. Recently, in 2010 it was developed into the Hamiltonian approach (He, 2010). This approach is a kind of energy method with a vast application in conservative oscillatory systems.

If $u > 0$:

$$\frac{d^2u}{dt^2} + u + (1-\sigma)u^2 + \sigma u^3 = 0 \quad (3)$$

In order to clarify this approach, the Hamiltonian of the Equation 3 can be written in the form:

$$H(u) = \frac{1}{2}u'^2 + \frac{1}{2}u^2 + (1-\sigma)\frac{1}{3}u^3 + \sigma\frac{1}{4}u^4 \quad (4)$$

Equation 4 implies that the total energy keeps unchanged during the oscillation. According to Equation 4:

$$\frac{\partial H}{\partial A} = 0 \quad (5)$$

Introducing a new function, $\overline{H}(u)$, is defined as (He, 2010):

$$\overline{H}(u) = \int_0^{\frac{T}{4}} \left(\frac{1}{2}u'^2 + \frac{1}{2}u^2 + (1-\sigma)\frac{1}{3}u^3 + \sigma\frac{1}{4}u^4\right)dt = \frac{1}{4}TH \quad (6)$$

It is obvious that:

$$\frac{\partial \overline{H}}{\partial T} = \frac{1}{4}H \quad (7)$$

Equation 2 is equivalent to the following equation:

$$\frac{\partial}{\partial A}\left(\frac{\partial \overline{H}}{\partial T}\right) = 0 \quad (8)$$

or

$$\frac{\partial}{\partial A}\left(\frac{\partial \overline{H}}{\partial (1/\omega)}\right) = 0 \quad (9)$$

Assume that the solution can be expressed as:

$$u(t) = A\cos(\omega t) \quad (10)$$

Substituting it to Equation 9:

$$\frac{\partial}{\partial A}\left(\frac{\partial \overline{H}}{\partial (1/\omega)}\right) = \frac{3}{16}A^3\sigma\pi - \frac{2}{3}A^2(\sigma-1) + \frac{1}{4}\pi A\left(1-\omega^2\right) = 0 \quad (11)$$

Consequently, approximate frequency can be found from Equation 11:

$$\omega_{HA1} = \frac{\sqrt{3\pi\left(9\sigma A^2\pi - 32A\sigma + 32A + 12\pi\right)}}{6\pi} \quad (12)$$

The approximate period is:

$$\mathrm{T}_1 = \frac{2\pi}{\omega_{HA1}} \quad (13)$$

If $u < 0$:

$$\frac{d^2u}{dt^2} + u - (1-\sigma)u^2 + \sigma u^3 = 0 \quad (14)$$

Similarly, the approximate frequency can be obtained in the approximate period ω_{HA1}:

$$\omega_{HA2} = \frac{\sqrt{3\pi\left(9\sigma A^2\pi + 32A\sigma - 32A + 12\pi\right)}}{6\pi} \quad (15)$$

The approximate period is:

$$T_2 = \frac{2\pi}{\omega_{HA2}} \quad (16)$$

The approximate period T is:

$$T_{HA} = \frac{T_1 + T_2}{2} \quad (17)$$

THE APPLICATION OF THE COUPLED HOMOTOPY-VARIATIONAL (CHV) FORMULATION

The coupled method of homotopy perturbation method (He, 2000) and variational formulation (He, 2007), couples the homotopy perturbation method with the variational method. The method first constructs a homotopy equation, and then the solution is expanded into a series of p. As the zeroth order approximate solution is easy to be obtained, the second term is solved using the variational approach, where the frequency of the non-linear oscillator can be obtained. The first-order solution is the best among all possible solutions when the trial solution is chosen in cosine or sine function. This technology is very much similar to Marinca's work where the unknown parameters are identified using least squares technology (Marinca, 2006; Marinca and Herisanu, 2010).

In Equation 1, if $u > 0$, the following homotopy can be constructed:

$$u'' + \omega^2 u + p[\sigma u^3 + (1-\sigma)u^2 + (1-\omega^2)u] = 0\,,\; p \in [0,1] \tag{18}$$

When $p = 0$, Equation 18 becomes the linearized equation, $u'' + \omega^2 u = 0$, when $p = 1$, it turns out to be the original one. Assume that the periodic solution to Equation 3 may be written as a power series in p :

$$u = u_0 + pu_1 + p^2 u_2 + ... \tag{19}$$

Substituting Equation 19 into Equation 18, collecting terms of the same power of p , gives:

$$u_0'' + \omega^2 u_0 + = 0\,,\; u_0(0) = A\,,\; u_0'(0) = 0 \tag{20}$$

and

$$u_1'' + \omega^2 u_1 + \sigma u_0{}^3 + (1-\sigma)u_0{}^2 + (1-\omega^2)u_0 = 0\,,\; u_1(0) = 0\,,\; u_1'(0) = 0. \tag{21}$$

The solution of Equation 20 is $u_0 = A\cos\omega t$, where ω will be identified from the variational formulation for u_1, which reads:

$$J(u_1) = \int_0^T \left\{ -\frac{1}{2}u_1'^2 + \frac{1}{2}\omega^2 u_1{}^2 + (1-\omega^2)\,u_0 u_1 + \sigma u_0{}^3 u_1 + (1-\sigma)u_0{}^2 u_1 \right\} dt\,,\; T = \frac{2\pi}{\omega} \tag{22}$$

To better illustrate the procedure, a simple trail function can be chosen:

$$u_1 = B_1(\cos\omega t - \cos 2\omega t) + B_3(\cos 2\omega t - \cos 6\omega t) \tag{23}$$

Substituting u_1 into the functional Equation 22 results in:

$$J(A, B_1, B_3, \omega) = \frac{3}{4}\frac{\sigma B_1 \pi A^3}{\omega} + \frac{\pi A^2}{2\omega}(B_3 - B_1 - \sigma B_3 + \sigma B_1) + \frac{\pi A}{\omega}B_1(1-\omega^2) + \pi\omega\left(3B_1B_3 - \frac{3}{2}\omega^2 B_1{}^2 - 19\omega^2 B_3{}^2\right) \tag{24}$$

Setting:

$$\frac{\partial J}{\partial B_1} = 0\,,\; \frac{\partial J}{\partial \omega} = 0\,,\; \frac{\partial J}{\partial B_3} = 0 \tag{25}$$

Solving the foregoing equations, approximate frequency as a function of amplitude equals to:

$$\omega_{CHV1} = \frac{1}{38}\Big(-1444 - 1083A^2\sigma + 655A - 655\sigma A + 19\big(5215A^2 - 21280A + 24226A^2\sigma - 15960A^3\sigma + 23104 + 21280A\sigma + 5215A^2\sigma^2 + 15960A^3\sigma^2 + 12996A^4\sigma\big)^{1/2}\Big)^{1/2} \tag{26}$$

The approximate period is:

$$T_1 = \frac{2\pi}{\omega_{CHV1}} \tag{27}$$

If $u < 0$:

$$\frac{d^2u}{dt^2} + u - (1-\sigma)u^2 + \sigma u^3 = 0 \tag{28}$$

Similarly, the approximate frequency can be obtained as the approximate period ω_{CHV2}:

$$\omega_{CHV2} = \frac{1}{38}\Big(-1444 - 1083A^2\sigma - 655A + 655\sigma A + 19\big(5215A^2 + 21280A + 24226A^2\sigma - 15960A^3\sigma + 23104 - 21280A\sigma + 5215A^2\sigma^2 + 15960A^3\sigma^2 + 12996A^4\sigma\big)^{1/2}\Big)^{1/2} \tag{29}$$

The approximate period is:

$$T_2 = \frac{2\pi}{\omega_{CHV2}} \tag{30}$$

The approximate period T is:

$$T_{CHV} = \frac{T_1 + T_2}{2} \tag{31}$$

RESULTS AND DISCUSSION

Here, the applicability, accuracy and effectiveness of the proposed approaches are illustrated by comparing the analytical approximate frequency with the exact solutions (Leung and Guo, 2009) in Tables 1 and 2.

Finally, for selected constant parameters, the stability of the system is as shown in Figures 1 and 2. Generally, the system possesses no damping, and therefore, any excitation will cause instability of the system.

Conclusions

In this paper, the two powerful and simple methods are

Table 1. Comparison of approximate periods with the exact period for $\sigma = 0.5$.

A	T_{HA}	T_{CHV}	T_{Exact}
0.01	6.2831	6.2831	6.2831
0.02	6.2829	6.2828	6.2829
0.05	6.2813	6.2805	6.2818
0.1	6.2756	6.2726	6.2777
0.2	6.2530	6.2410	6.2599
0.4	6.1626	6.1186	6.1746
0.8	5.8104	5.6855	5.7281
1	5.5642	5.4104	5.3950
2	4.1616	4.0207	3.8411
5	1.9822	1.9590	1.8943
10	1.0173	1.0139	1.0041
20	0.5119	0.5115	0.51434
50	0.2051	0.2051	0.20827
100	0.1026	0.1026	0.10452

Table 2. Comparison of approximate periods with the exact period for $\sigma = 0.9$.

A	T_{HA}	T_{CHV}	T_{Exact}
0.01	6.2830	6.2830	6.2830
0.1	6.2623	6.2621	6.2622
0.5	5.8148	5.8127	5.8065
1	4.8595	4.8561	4.8413
5	1.4864	1.4862	1.5054
10	0.7592	0.7592	0.77273
50	0.1529	0.1529	0.15618
100	0.0765	0.0765	0.07813

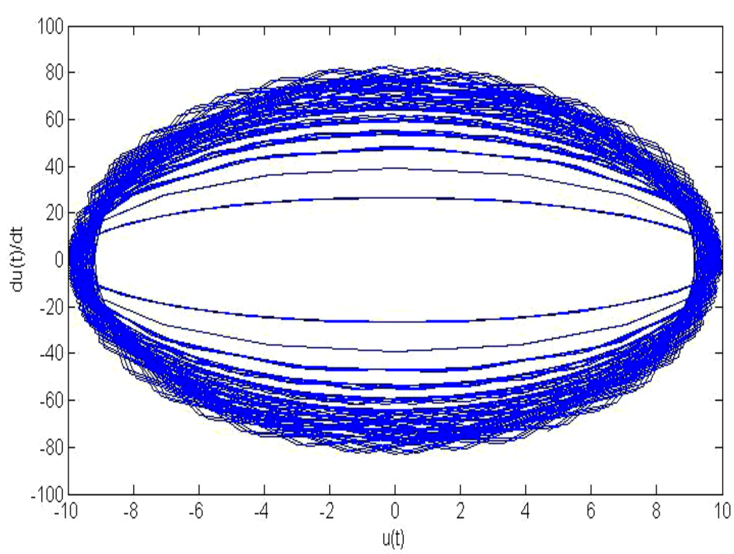

Figure 1. Phase plane maps showing the system stability, influence of σ in the stability $0.1 < \sigma < 0.9$.

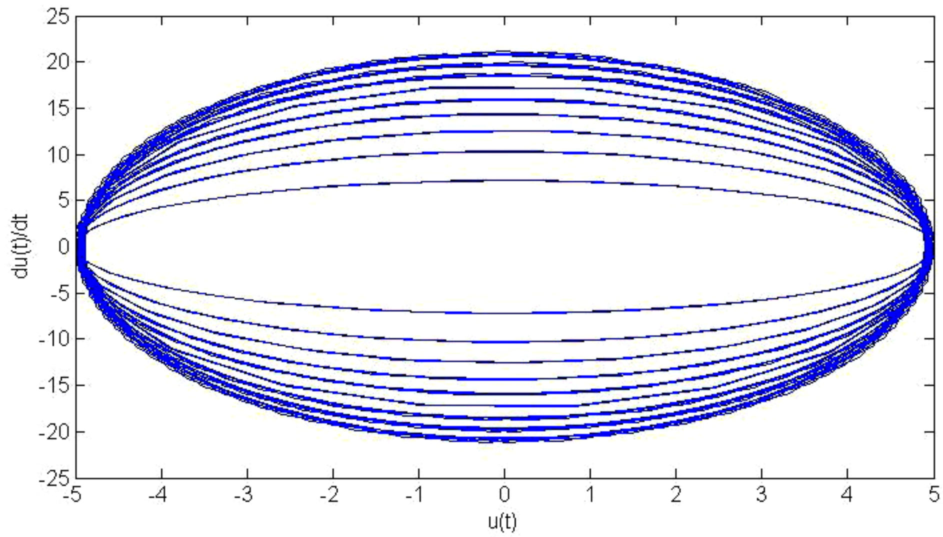

Figure 2. Phase plane maps showing the system stability, influence of σ in the stability $0.1 < \sigma < 0.9$.

applied for solving the Helmholtz-Duffing oscillator equation. Periodic solutions and natural frequencies are analytically obtained. Numerical simulations, for few typical points in the $\frac{du}{dt} - u$ plane, are carried out and their phase plane trajectories are presented graphically. This new approaches proves to be very rapid, effective and accurate and this is proved by comparing the solutions obtained through the proposed methods with the exact solution results. An excellent agreement between the present and exact solutions is achieved. The analysis given here further shows confidence on Hamiltonian approach and coupled homotopy-variational formulation.

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