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Full Length Research Paper

On the solution of singular initial value problems in ordinary differential equations using a new third order inverse Runge-Kutta method

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A new third order rational Runge-Kutta formula is derived and implemented. Results obtained are compared with the existing formulae to determine the level of its performance, consistency and accuracy. It was discovered from the computation carried out that errors are minimized while using the new scheme.

Key words: Inverse Runge-Kutta method, rational mesh size, rate of convergence, consistence computer time and error analysis.

INTRODUCTION

In this paper, we developed a third–stage inverse Rgunge-Kutta method to solve initial value problems (IVPs) of the form

$$y' = f(x, y), y(x_0) = y_0, a \le x \le b, \tag{1.1}$$

aimed at finding out its level of consistency and the rate of convergence in the solution of first order differential equations.

Many researchers have done a great deal of work in this area. Worthy of note are those of Ademiluyi (2002, 2005); Okunbor (1985); Ademiluyi and Babatola (2000); Burden et al. (2005); Verner (1990, 1991) and a host of others. In their different approaches, they showed that a two-stage explicit inverse Runge-Kutta method posses the potential of improving results if the parameters are varied.

This idea led us to evolve the new third-stage explicit inverse Runge-Kutta method with the belief that it rate of convergence will be faster with improved results.

According to Lambert (1973), the philosophy behind the Runge-Kutta method is to retain the advantages of one - step methods and to improve on the performance of Euler method. Due to loss of linearity in the one-step methods, error analysis is considerably more difficult than the case of linear multi-step methods. Traditionally, Runge-Kutta methods are all explicit, although, recently, implicit Runge-Kutta methods have extensively been used to improved weak stability characteristics. Thus, a Runge-Kutta method may be regarded as a particular case of

$$y_{n+1} - y_n = h\phi(x_n, y_n; h) \tag{1.2}$$

According to Shepley and Ross (1989); "the fact that the general method (2) makes no mention of the function $f(x, y)$, which defined the differential equation, makes it impossible to define the order of the methods independently of the differential equation, as was the case with linear multi- step methods".

As a matter of fact, when we derived a particular order of Runge-Kutta method there are, in general, a number of free parameters that cannot be used to increase the order. Shepley and Ross (1989); continued by saying that "we choose these parameters in such a way that the resulting methods have simple coefficients convenient for desk computation". When these parameters are not carefully selected, it may lead to an increase of the local truncation error.

Perhaps, the most important task is to apply these parameters for the reduction of the local truncation error. This error depends on the complicated nature of the function $f(x, y)$.

These iterative methods (the Runge-Kutta methods)

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can be used to solve singular and non-singular initial value problems as will be seen later in this work. Agbeboh (2006) reaffirmed that "singular initial-value problems are problems that have points of discontinuities in a differential system. While non -singular initial -value problems do not have points of discontinuities". We are therefore, interested in solving the general initial - value problem in (1.1), (whose gradient function $f(x,y)$ may have points of discontinuities), using our third-stage explicit inverse Runge-Kutta formula.

Arising from the above stated problems, we:

(a) Developed the new method by increasing the order of the two stage to three and derived a new formula that gives improved results in the solution of (1.1),
(b) Implement the new rational third order Runge-Kutta method for better result by use of appropriate FORTRAN package,
(c) Compared the performance of the new method with Ademiluyi (2005); Agbeboh et al. (2009) and the classical 4^{th} order RKF with a view to finding out which of the methods has a faster rate of convergence with minimal error bound.

METHOD OF RESEARCH

To carry out our assignment satisfactorily we modified the Runge-Kutta methods of order three to produce our new version of third order inverse rational Runge-Kutta method by using Taylor series and binomial algorithm to develop our new Rational R-K scheme, and use the FORTRAN algorithms, to solve some tested singular ivps with known results. The results obtained from our formula compared favourably well with existing formulae in the works of Agbeboh et al. (2009); Ademiluyi (2005); Butcher (1987); Lambert (1973,1995) and Okunbor (1985) some of which are shown.

However, these methods vary in performance and accuracy hence, Butcher (1987), specified, that "not all such method have the capacity to find solutions to these initial value problems". This gave us the motivation to develop a one-step explicit Runge-Kutta formula that can provide solutions to singular and non -singular initial value problems. Our knowledge of complex analysis exposed us to a wide range of problem with singularities.

Before designing our formula, we considered the works of Agbeboh (2006); Ademiluyi (2005); Okunbor (1985); Ademiluyi and Babalola (2000); Aashikpelokhai (1991) and Fatunla (1980, 1988) and were motivated by their striking proposals to study the third - order inverse Runge-Kutta formula.

DERIVATION OF THE SCHEME

Consider the case where R = 3, so that

$$y_{n+1} = \frac{y_n}{1 + y_n \sum_{i=1}^{3} w_i k_i} \quad (3.1)$$

$$k_1 = hg(x_n, z_n) = hg \quad (3.1a)$$

$$k_2 = hg(x_n + a_2 h, z_n + b_{21} k_1) \quad (3.1b)$$

$$k_3 = hg(x_n + a_3 h, z_n + b_{31} k_1 + b_{32} k_2) \quad (3.1c)$$

$$g(x_n, z_n) = -zf(x_n, z_n), z_n = \frac{1}{y_n} \quad (3.1d)$$

With constraints

$$a_{11} = b_{22} = b_{33} = 0, a_2 = b_{21},$$
$$a_3 = b_{31} + b_{32} \Rightarrow b_{31} = a_3 - b_{32} \quad (3.1e)$$

Thus, a 3- stage explicit inverse formula of the form (3.1) is a formula of the form.

$$y_{n+1} = \frac{y_n}{1 + y_n (w_1 k_1 + w_2 k_2 w_3 k_3)} \quad (3.2)$$

In developing the order we assume that the differential equation (3.1b) has a unique solution y(x) on [a, b] when expanded in Taylor series about any point x_n

as

$$y(x) = y(x_n) + \frac{(x-x_n)}{1!} y'(x_n) + \frac{(x-x_n)^2}{2!} y''(x_n) + \frac{(x-x_n)^3}{3!} y'''(x_n) + \ldots + \frac{h^{p+1}}{(p+1)!} y^{(p+1)} + O(h^{p+2}) \quad (3.3)$$

This expansion holds for all $x_n, x_{n+1} \in (a,b); x_n < x_{n+1} < x.$ since the step length $h = x_{n+1} - x_n$, then, substituting $x = x_n, x_{n+1} = x_n + h$ in (3.3), yields:

$$y'(x_n + h) = y(x_n) + \frac{h}{1!} y'(x_n) + \frac{h^2}{2!} y''(x_n) + \frac{h^3}{3!} y'''(x_n) + 0(h^4) \quad (3.4)$$

Replacing $y(x_n)$ by y_n, $y(x_n + h)$ by y_{n+1}, we have,

$$y_{n+1} = y_n + hy' + \frac{h^2}{2!} y'' + \frac{h^3}{3!} y''' + 0(h^4) \quad (3.5)$$

Where,

$$f(x,y) = f, \frac{\partial f(x,y)}{\partial x} = f_x, \frac{\partial^2 f(x,y)}{\partial x^2} = f_{xx},$$

$$\frac{\partial f(x,y)}{\partial y} = f_y, \frac{\partial^2 f(x,y)}{\partial y^2} = f_{yy}, \frac{\partial^2 f(x,y)}{\partial x \partial y} = f_{xy}, \tag{3.6}$$

Adopting the differential notations in (3.4), we have

$$y' = f(x,y) = f, \quad y'' = f'(x,y) = f_x + ff_y \ \textit{and}$$
$$y''' = f''(x,y) = f_{xx} + 2ff_{xy} + f^2 f_{yy} + f_y(fx + ff_y) \tag{3.7}$$

Where f_x, f_y represents the derivatives of f with respect to x and y respectively.
Substituting (3.5) in (3.3), we have

$$y_{n+1} = y_n + h f(x,y) + \frac{h^2}{2!}(f_x + ff_y) + \frac{h^3}{3!}\left\{f_{xx} + 2ff_{xy} + f^2 f_{yy} + f_y(f_x + ff_y) + 0(h)^4\right\} \tag{3.8}$$

$$F = f_x + ff_y, \; G = f_{xx} + 2ff_{xy} + f^2 f_{yy} \tag{3.9}$$

Substituting (3.9) into (3.8) we obtain

$$y_{n+1} = y_n + hf(x,y) + \frac{h^2}{2!}F + \frac{h^3}{3!}(G + Ff_y) + 0(h^4) \tag{3.10}$$

Thus

$$y_{n+1} = y_n + hf + \frac{h^2}{2!}F + \frac{h^3}{3!}(G + Ff_y) + 0(h^4) \tag{3.11}$$

From (3.1c) we have that an R–stage explicit inverse Runge-Kutta method is

$$y_{n+1} = \frac{y_n}{1 + y_n \sum_{i=1}^{R} w_i k_i} \tag{3.12}$$

Where $K_1 = hg(x_n, z_n) = hg$ (3.13)

$$K_i = hg\left(x_n + a_i h, \; z_n + \sum_{i=1}^{n} b_{ij} k_j\right) \; i = 1(1) R \tag{3.14}$$

$g(x_n, z_n) = -z_n^2 f(x_n, y_n), \; z_n = \frac{1}{y_n}$, with constraints $a_i = \sum_{i=1}^{i} b_{ij}, \sum_{i=1}^{R} w_i = 1$, $b_{ij} = 0$ if $j \geq i$ and h is the step length or grid – spacing, we obtain k_i's$, i = 2,3$ in (3.12) about (x_n, z_n) such that

$$K_2 = h\{g + (a_2 hg_x + (b_{21}k_1)g_z + \frac{h}{2!}(a_2^2 hg_{xx} + 2a_2 b_{21} k_1 g_{xz} + (b_{21}k_1)^2 g_{zz} + \frac{h}{3!}(a_2^3 hg_{xxx} + 3a_2^2 h(b_{21}k_1)g_{xxz} + 3a_2(b_{21}k_1)^2 g_{xzz} + (b_{21}k_1)^3 g_{zzz} + 0(h^4)\} \tag{3.16}$$

Substituting (3.12) in (3.16), we obtain

$$K_2 = h(g + h(a_2 g_x + (b_{21}hggz) + \frac{h^2}{2!}(a^2 g_{xx} + 2a_2 b_{21} gg_x + (b_{21}^2 g^2 g_{zz}) + \frac{h^3}{3!}\{a_2^3 g_{xxz} + 3a_2^2 gg_{xxz} + 3a_2 b_{21}^2 g^2 g_{xzz} + a_{21}{}^2 g^3 g_{zzz} + O(h^4)\} \tag{3.17}$$

$\Rightarrow$

$$k_2 = hg + h^2(a_2 g_x + b_{21} gg_z) + \frac{h^3}{2!}(a_2^2 g_{xx} + 2a_2 b_{21} gg_{xz} + b_{21}^2 g^2 g_{zz}) + 0(h^4) \tag{3.18}$$

$$+ (b_{21}^2 g^2 g_{zz}) + \frac{h^3}{3!}\{a_2^3 g_{xxz} + 3a_2^2 gg_{xxz} + 3a_2 b_{21}^2 g^2 g_{xzz} + a_{21}{}^2 g^3 g_{zzz} + O(h^4)\} \tag{3.19}$$

Since $a_2 = b_{21}, a_2^2 = a_2 b_{21} = b_{21}^2$

From (3.1.4), we know that

$$g(x,z) = -z^2 f(x, \frac{1}{z}) = -\frac{f(x,y)}{y^2}, z = \frac{1}{y} \Rightarrow g = \frac{-f}{y^2} \tag{3.20}$$

By sufficiently differentiating (3.20), we can express the differentials involving g and its partial derivatives in terms of f and its partial derivatives. To facilitate the comparison of coefficients in terms of f and its partial derivatives only, we make this following substitution:

$$g_x = \frac{-f_x}{y^2}, g_{xx} = \frac{-f_{xx}}{y^2}, g_{xxx} = \frac{-f_{xxx}}{y^2}, g_z = \frac{-2f}{y} + fy,$$

$$g_x = \frac{-f_x}{y^2}, g_{xx} = \frac{-f_{xx}}{y^2}, g_{xxx} = \frac{-f_{xxx}}{y^2}, g_z = \frac{-2f}{y} + fy,$$

$$g_x = \frac{-f_x}{y^2}, g_{xx} = \frac{-f_{xx}}{y^2}, g_{xxx} = \frac{-f_{xxx}}{y^2}, g_z = \frac{-2f}{y} + fy, \quad (3.21)$$

From (3.19), we have that

$$K_2 = hg + h^2 a_2 F_1 + \frac{1}{2} h^3 a_2^2 G_1 + 0\left(h^4\right) \quad (3.22)$$

where

$$F_1 = g_x + g g_z, \qquad G_1 = g_{xx} + 2 g g_{xz} + g^2 g_{zz}$$

In a similar manner, expanding K_3 about (x_n, z_n), we have

$$K_3 = h\left(g + h\left(a_3 g_x + \{(a_3 - b_{32}) k_1 + b_{32} k_2\} g_z\right\} + \frac{1}{2}\{a^2 g_{xx} + 2a_3 \left((a_3 - b_{32}) k_1 + b_{32} k_2\right) g_{xz} + \{(a_3 - b_{32}) k_1 + b_{32} k_2\}^2 g_{zz}\} + 0\left(h^3\right) \quad (3.23)$$

On substituting for k_1 and k_2 and deduce by comparison, we obtain from $K_2 = hg + h^2 a_2 F_1 + \frac{1}{2} h^3 a_2^2 G_1 + 0\left(h^4\right)$ (3.24)

We have

$$K_3 = hg + h^2 a_3 F_1 + \frac{1}{2} h^3 a_3^2 G_1 + h^3 a_2 b_{32} F_1 f_y + 0\left(h^4\right) \quad (3.25)$$

$$\therefore K_3 = hg + h^2 a_3 F_1 + h^3 (a_2 b_{32} F_1 f_y + \frac{1}{2} a_3^2 G_1) + 0\left(h^4\right) \quad (3.26)$$

Adopting the binomial expansion theorem on the right hand side of (3.4) and ignoring higher terms we obtain

$$y_{n+1} = y_n - y_n^2 \sum_{i=1}^{3} w_i k_i \text{ + (higher order terms)} \quad (3.27)$$

Simplifying further for i=1, 2, 3, (3.27) becomes

$$y_{n+1} = y_n - y_n^2 \left(w_1 k_1 + w_2 k_2 + w_3 k_3\right) \quad (3.28)$$

Substituting for k_1, k_2 and k_3 in (3.28), we obtain

$$y_{n+1} = y_n - y_n^2 (w_1 (hg) + w_2 (hg + h^2 a_2 F_1 + \frac{1}{2} h^3 a^2 G_1) + \quad (3.29)$$

$$y_{n+1} = y_n - y_n^2 (h w_1 g + h w_2 g + h^2 a_2 w_2 F_1 + \frac{1}{2} h^3 a_2^2 w_2 G_1 + h w_3 g \quad (3.30)$$

Simplifying and arranging in powers of h, we have

$$y_{n+1} = y_n - y_n^2 \left(h\left(w_1 + w_{2+} w_3\right) g + h^2 (a_2 w_2 + a_3 w_3) F_1 + \frac{1}{2} h^3 ((a^2 w_2 + a_3^2 w_3) G_1 + 2 a_2 b_{32} w_3 F_1 f_y) + 0\left(h^4\right)\right) \quad (3.31)$$

Recalling from (3.11), we have that

$$y_{n+1} = y_n + hf + \frac{h^2}{2!} F + \frac{h^3}{3!}\left(G + F f_y\right) + 0\left(h^4\right) \quad (3.32)$$

Comparing the coefficients of h, h^2 and h^3 in (3.30) and (3.31) using (3.5.1) and (3.5.2) where condition

$$T_{n+1} = 0\left(h^4\right) \quad (3.33)$$

is imposed. We obtained the following system of equations for the family of three – stage schemes of order three.

$$w_1 + w_2 + w_3 = 1, a_2 w_2 + a_3 w_3 = ½,$$
$$a^2 w_2 + a_3^2 w_3 = \frac{1}{3}, a_2 b_{32} w_3 = \frac{1}{6} \quad (3.34)$$

While R = 1 and R = 2 give $y_{n+1} = \dfrac{y_n^2}{y_n - h f_n}$ (3.35)

And $y_{n+1} = \dfrac{y_n}{1 + \frac{y_n}{4}\left(k_1 + 3k_2\right)}$ (3.36)

Where, $k_1 = hg\left(x_n, z_n\right)$

$$K_2 = hg\left(x_n + 2/3h, z_n + 2/3k_1\right),$$
$$g\left(x_n, z_n\right) = -z_n^2 f\left(x_n, y_n\right), z_n = 1/y_n \quad (3.37)$$

respectively. By putting R =3, we obtain (3.2) a scheme of order three with $T_{n+1} = 0\left(h^4\right)$, whose parameters must satisfy the following system of equations

$$w_1 + w_2 + w_3 = 1, a_2 w_2 + a_3 w_3 = 1/2,$$
$$a^2 w_2 + a_3^2 w_3 = {}^1/_3 \text{ and } a_2 b_{32} w_3 = 1/6 \quad (3.38)$$

There are now four equations in six unknown and there exist two parameters of families of curves. Ignoring terms of order h^4, in this derivation, shows that no solution of (3.38), cause the expansion to differ by a term of order higher than h^4.

Thus, there exist a doubly infinite family of three–stage inverse Runge-Kutta methods of order three, and none of order greater than three. Two particular solutions of (3.38) lead to well know third-order Runge- Kutta methods.

Because the 6 unknowns have to satisfy only four equations, the values of the two of them can be chosen arbitrarily provided the equations have solution.

By setting $a_2 = ½$ and $a_3 = 1$, equation (3.38) becomes

$$W_1 + w_2 + w_3 = 1 \quad (3.38a)$$

$${}^1/_2 w_2 + w_3 = ½ \quad (3.38b)$$

$$^1/_4 w_2 + w_3 = {}^1/_3 \quad (3.38c)$$

$$^1/_2 b_{32} w_3 = {}^1/_6 \quad (3.38d)$$

Solving these equations non-linearly we have,

$$w_1 = \frac{1}{6}, w_2 = \frac{2}{3}, w_3 = \frac{1}{6}, a_2 = b_{21} = \frac{1}{2}, \ a_3 = 1, \text{ and } \ b_{32} = 2.$$

Thus the resulting method is

$$y_{n+1} = \frac{y_n}{1 + \frac{y_n}{6}(k_1 + 4k_2 + k_3)} \quad (3.39)$$

where $K_1 = hg\left(x_n, z_n\right)$

$$K_2 = hg\left(x_n + ½\, h,\ z_n + ½\, k_1\right)$$

$$K_3 = hg\left(x_n + h,\ z_n - k_1 + 2k_2\right)$$

$$g(x_n, z_n) = -z_n^2 f(x_n, y_n),\ z_n = 1/y_n$$

ERROR, ORDER AND CONVERGENCE OF THE METHOD

According to Shephey and Ross (1989) numerical methods are employed in the solution of the differential equations of type $y' = f(x, y)$ with the initial condition $y(x_0) = y_0$ to obtain approximate solution at various selected values of x with the aim of having exact solution. To do this, we set ϕ as the exact solution of the problem, and let h denote a small positive increment in x, such that $x_1 = x_0 + h$, $x_2 = x_1 + h$, …., $x_n = x_{n-1} + h$, and consider $\phi(x_1)$, $\phi(x_2)$…..$\phi(x_n)$ as the solution set of the d. e. Let y_1, y_2, …,y_n be the approximations to $\phi(x_1)$, $\phi(x_2)$……..$\phi(x_n)$ respectively, so that finding y_n and finding an approximation to $\phi(x_n)$ mean the same thing. In finding the approximation y_1, y_2, y_3,…,y_n, we proceed in the following way;

First, we find the approximation y_1 using the method of interest to solve the differential equation y = f(x,y) with the initial value y_0. Then y_2 is estimated using the estimate y_1, y_3 is estimated using the estimate y_2, and so on, so that in general, y_{n+1} are estimated using the estimate y_n. After the derivation of our method, we proceeded in the above manner, to generate results using our method.

Shepley and Ross (1989), continued by saying that given an approximation y_n to $\phi(x_n)$, the absolute error or simply the error defined as $\left|y_n - \phi(x_n)\right|$; is the error which measures how far away the approximation y_n is from the exact value $\phi(x_n)$. Naturally we hope that any given numerical method will keep the error small, that is, the method should have some level of accuracy. However, the size of this absolute error alone should not be used to judge the accuracy of a method, for the size of the error must be considered in the light of the size of what is being approximated. At this stage we introduce a better measurement of accuracy which is called the percentage relative error, defined as the difference between the exact solution and its approximation divided by the exact solution itself and then multiplied by 100 given by:

$$\text{Percentage relative } Error = 100x\frac{error}{\left|\phi(x_n)\right|} = \left|\frac{y_n - \phi(x_n)}{\phi(x_n)}\right|$$

Notice that the Percentage Relative Error could be very large when $\phi(x_n)$ itself is near zero; in fact if $\phi(x_n) = 0$, the percentage relative error is undefined. Now, when one method provides more accuracy than another, there is usually a corresponding increase in its computational complexity as well. Furthermore, the sizes of the errors often increase in succession when calculating y_1, y_2,…, y_n, and there are two reasons why this may not be out of place. First, since at each stage finding y_{n+1} involve using previous approximations, chances are that y_{n+1} are much less accurate. Secondly, although a computer or calculator may be able to store numbers with, say, 10 or 11digit level of accuracy, any error introduced because such a machine cannot perfectly and accurately store (most) real numbers is less significant in size after thousand of (or may have far fewer) computations.

When the exact solution $\phi(x_1)$, $\phi(x_2)$, ..., $\phi(x_n)$, of the given initial – value problems have been found, we can compare the approximations y_1, y_2, y_3,…,y_n, found by a given method to the exact values and thereby gain some insight into the accuracy of the method. Finally, a numerical method involves doing numerical calculations, and it is this kind of computational work that computers are designed to carry out. One of the major and very significant aspects of numerical scheme is its ability to reliably control the global error

$$e_{n+1} = y_{n+1} - y\left(x_{n+1}\right) \quad (4.1)$$

Where y_{n+1} is the numerical solution at step x_{n+1} and y (x_{n+1}) is the theoretical solution. A general requirement is that this error should be made as small as possible by making h sufficiently close to zero. Adopting the above procedure, we tested our method on problems below.

COMPARISON OF RESULTS OF SOME SOLVED (VIPS)

Problem 1: $y^1 = -y$, $y(0)=1$, $0 \le x \le 1$

New Inverse Runge-Kutta method of order 3 $y^1 = -y$, $y(0)=1$, $0 \le x \le 1$.

N	Xn	Y(Xn)	Yn	En
1	0.1000	0.913242008	0.904837419	-0.8405D-02
2	0.2000	0.834769593	0.818730756	-0.1604D-01
3	0.3000	0.763864099	0.740818219	-0.2305D-01
4	0.4000	0.699874019	0.670320050	-0.2955D-01
5	0.5000	0.642208660	0.606530669	-0.3568D-01

Problem 1. Continued.

6	0.6000	0.590332164	0.548811633	-0.4152D-01
7	0.7000	0.543757951	0.496585291	-0.4717D-01
8	0.8000	0.502043456	0.449328943	-0.5271D-01
9	0.9000	0.464785127	0.406569632	-0.5822D-01
10	1.0000	0.431613653	0.367879408	-0.6373D-01

Inverse Runge-Kutta method of order 2 $y^1= -y$, $y(0)=1$, $0 \le x \le 1$, (Ademiluyi, 2005).

N	Xn	Y(Xn)	Yn	En
1	0.1000	0.904837419	0.913103027	-0.8266D-02
2	0.2000	0.818730756	0.834461076	-0.1573D-01
3	0.3000	0.740818219	0.763345389	-0.2253D-01
4	0.4000	0.670320050	0.699091815	-0.2877D-01
5	0.5000	0.606530669	0.641093872	-0.3456D-01
6	0.6000	0.548811633	0.588796260	-0.3998D-01
7	0.7000	0.496585291	0.541688712	-0.4510D-01
8	0.8000	0.449328943	0.499300257	-0.4997D-01
9	0.9000	0.406569632	0.461193861	-0.5462D-01
10	1.0000	0.367879408	0.426961516	-0.5908D-01

Sixth Order R-KF $y^1= -y$, $y(0)=1$, $0 \le x \le 1$ (Agbeboh et al., 2009).

X_N	Y_N	TSOL error
.100	0.9048374417318D+00	0.9048374180360D+00 - .2369581120210D-07
.200	0.8187307959597D+00	0.8187307530780D+00 - .4288171395750D-07
.300	0.7408182788832D+00	0.7408182206817D+00 -.5820146953273D-07
.400	0.6703201162528D+00	0.6703200460356D+00 -.7021715764388D-07
.500	0.6065307391315D+00	0.6065306597126D+00 -.7941889079710D-07
.600	0.5488117223274D+00	0.5488116360940D+00 - .8623342206970D-07
.700	0.4965853948232D+00	0.4965853037914D+00 -.9103176595859D-07
.800	0.4493290582532D+00	0.4493289641172D+00 -.9413594187491D-07
.900	0.4065697555655D+00	0.4065696597406D+00 -.9582493909477D-07

Sixth Order R-KF $y^1= -y$, $y(0)=1$, $0 \le x \le 1$ (Agbeboh et al., 2009). Continued.

1.00	0.3678795375114D+00	0.3678794411714D+00 -.9633999070724D-07

Classical 4th order RKF $y^1= -y$, $y(0)=1$, $0 \le x \le 1$.

N	Xn	Y(Xn)	Yn	En
1	0.1000	0.904837419	0.904837500	-0.8056D-07
2	0.2000	0.818730756	0.818730901	-0.1458D-06
3	0.3000	0.740818219	0.740818422	-0.2034D-06
4	0.4000	0.670320050	0.670320289	-0.2387D-06
5	0.5000	0.606530669	0.606530934	-0.2655D-06
6	0.6000	0.548811633	0.548811934	-0.3014D-06
7	0.7000	0.496585291	0.496585619	-0.3280D-06
8	0.8000	0.449328943	0.449329290	-0.3468D-06
9	0.9000	0.406569632	0.406569991	-0.3591D-06
10	1.0000	0.367879408	0.367879774	-0.3659D-06

Problem 2. $y^1=y$, $y(0)=1$, $0 \le x \le 1$.
New Inverse Runge-Kutta method of order 3.

N	Xn	Y (Xn)	Yn	En
1	0.1000	1.105175909	1.105170916	-0.4992D-05
2	0.2000	1.222564214	1.221402754	-0.1161D-02
3	0.3000	1.353486066	1.349858811	-0.3627D-02
4	0.4000	1.499408987	1.491824688	-0.7584D-02
5	0.5000	1.661964030	1.648721246	-0.1324D-01
6	0.6000	1.842964620	1.822118811	-0.2085D-01
7	0.7000	2.044427293	2.013752761	-0.3067D-01
8	0.8000	2.268594576	2.225541034	-0.4305D-01
9	0.9000	2.517960158	2.459603278	-0.5836D-01
10	1.0000	2.795297009	2.718282070	-0.7701D-01

Problem 2: $y^1=y$, $y(0)=1$, $0 \le x \le 1$ (Ademiluyi, 2005).
Inverse Runge-Kutta method of order 2.

N	Xn	Y(Xn)	Yn	En
1	0.1000	1.108033243	1.105170916	-0.2862D-02
2	0.2000	1.228368844	1.221402754	-0.6966D-02
3	0.3000	1.362346272	1.349858811	-0.1249D-01
4	0.4000	1.511455951	1.491824688	-0.1963D-01
5	0.5000	1.677355960	1.648721246	-0.2863D-01
6	0.6000	1.861890601	1.822118811	-0.3977D-01
7	0.7000	2.067110979	2.013752761	-0.5336D-01
8	0.8000	2.295297850	2.225541034	-0.6976D-01
9	0.9000	2.548986990	2.459603278	-0.8938D-01
10	1.0000	2.830997453	2.718282070	-0.1127D+00

Problem 2. $y^1=y$, $y(0)=1$, $0\le x\le 1$ (Agbeboh et al., 2009).
Sixth order R-K method.

XN	YN	TSOL	Error
.1D+00	0.1105170892253D+01	0.1105170918076D+01	0.2582304325927D-07
.2D+00	0.1221402701082D+01	0.1221402758160D+01	0.5707775252439D-07
.3D+00	0.1349858712955D+01	0.1349858807576D+01	0.9462100680757D-07
.4D+00	0.1491824558211D+01	0.1491824697641D+01	0.1394298454471D-06
.5D+00	0.1648721078083D+01	0.1648721270700D+01	0.1926172610300D-06
.6D+00	0.1822118544941D+01	0.1822118800391D+01	0.2554499913821D-06
.7D+00	0.2013752378102D+01	0.2013752707470D+01	0.3293685479910D-06
.8D+00	0.2225540512483D+01	0.2225540928492D+01	0.4160097559769D-06
.9D+00	0.2459602593925D+01	0.2459603111157D+01	0.5172321126956D-06
.1D+01	0.2718281193315D+01	0.2718281828459D+01	0.6351443135877D-06

Problem 2. $y^1=y$, $y(0)=1$, $0\le x\le 1$
Classical 4th order RKF.

XN	YN	TSOL	Error
.1D+00	0.9048375000000D+00	0.9048374180360D+00	.8196404044369D-07
.2D+00	0.8187309014063D+00	0.8187307530780D+00	-.1483282683346D-06
.3D+00	0.7408184220012D+00	0.7408182206817D+00	-.2013194597694D-06
.4D+00	0.6703202889175D+00	0.6703200460356D+00	-.2428818514089D-06
.5D+00	0.6065309344234D+00	0.6065306597126D+00	-.2747107467060D-06
.6D+00	0.5488119343763D+00	0.5488116360940D+00	-.2982822888686D-06
.7D+00	0.4965856186712D+00	0.4965853037914D+00	-.3148798197183D-06
.8D+00	0.4493292897344D+00	0.4493289641172D+00	-.3256172068089D-06
.9D+00	0.4065699912001D+00	0.4065696597406D+00	-.3314594766990D-06
.1D+01	0.3678797744125D+00	0.3678794411714D+00	-.3332410563051D-06

Problem 3. $y^1=1+y^2$, $y(0)=1$, $0\le x\le 1$.
New Inverse Runge-Kutta method of order 3.

N	Xn	Y(Xn)	Yn	En
1	0.1000	1.223048884	1.235076167	-0.1203D-01
2	0.2000	1.508497657	1.539838714	-0.3134D-01
3	0.3000	1.895765178	1.960015192	-0.6425D-01
4	0.4000	2.464962799	2.592044896	-0.1271D+00
5	0.5000	3.408223442	3.679681411	-0.2715D+00
6	0.6000	5.331855925	6.068965225	-0.7371D+00
7	0.7000	11.681380355	16.077934500	0.4397D+01
8	0.8000	68.479332859	26.198316356	0.4228D+02
9	0.9000	8.687622253	7.230198297	0.1457D+01
10	1.0000	4.588035196	4.164509569	0.4235D+00

Conclusion

Having derived and implemented the formula we compared our results with those of the existing formulae already considered in the works of Ademiluyi (2005), Agbeboh et al. (2009) and the classical 4th order RKF and discovered that the errors committed in using this new method to solve ivps is minimal. We also find out that the computer time required was smaller in ours than other methods considered. Furthermore, the parameters used represent constraints that ensure the consistency of the scheme; These parameters were chosen to ensure that

Problem 3. $y^1=1+y^2$, $y(0)=1$, $0 \le x \le 1$ (Ademiluyi, 2005).
Inverse Runge-Kutta Method of Order 2.

N	Xn	Y(Xn)	Yn	En
1	0.1000	1.223048884	1.234567905	-0.1152D-01
2	0.2000	1.508497657	1.538880977	-0.3038D-01
3	0.3000	1.895765178	1.958484913	-0.6272D-01
4	0.4000	2.464962799	2.589525967	-0.1246D+00
5	0.5000	3.408223442	3.674915625	-0.2667D+00
6	0.6000	5.331855925	6.056638867	-0.7248D+00
7	0.7000	11.681380355	15.994363155	-0.4313D+01
8	0.8000	-68.479332859	-26.420600456	-0.4206D+02
9	0.9000	-8.687622253	-7.247147091	-0.1440D+01
10	1.0000	-4.588035196	-4.170275448	-0.4178D+00

Problem 3. $y^1=1+y^2$, $y(0)=1$, $0 \le x \le 1$ (Agbeboh et al., 2009).
Sixth Order R-K method.

XN	YN	TSOL	Error
.1D+00	0.1223027752889D+01	0.1223048880450D+01	0.2112756091166D-04
.2D+00	0.1508427566955D+01	0.1508497647121D+01	0.7008016635046D-04
.3D+00	0.1895570024855D+01	0.1895765122854D+01	0.1950979986147D-03
.4D+00	0.2464395431841D+01	0.2464962756723D+01	0.5673248815250D-03
.5D+00	0.3406231293883D+01	0.3408223442336D+01	0.1992148453307D-02
.6D+00	0.5321220102716D+01	0.5331855223459D+01	0.1063512074226D-01
.7D+00	0.1151231991096D+02	0.1168137380031D+02	0.1690538893523D+00
.8D+00	0.3064002024514D+03	-.6847966834558D+02	-.3748798707970D+03
.9D+00	0.3614386869214D+69	-.8687629546482D+01	-.3614386869214D+69

Problem 3. $y^1=1+y^2$, $y(0)=1$, $0 \le x \le 1$.
Classical 4th order RKF.

XN	YN	TSOL	Error
.1D+00	0.1223048913837D+01	0.1223048880450D+01	-.3338691878518D-07
.2D+00	0.1508496167191D+01	0.1508497647121D+01	0.1479930124670D-05
.3D+00	0.1895754160233D+01	0.1895765122854D+01	0.1096262057532D-04
.4D+00	0.2464899686958D+01	0.2464962756723D+01	0.6306976472326D-04
.5D+00	0.3407820425152D+01	0.3408223442336D+01	0.4030171840324D-03
.6D+00	0.5327896816591D+01	0.5331855223459D+01	0.3958406868224D-02
.7D+00	0.1155393207572D+02	0.1168137380031D+02	0.1274417245901D+00
.8D+00	0.1921699249630D+03	-.6847966834558D+02	-.2606495933085D+03
.9D+00	0.3119852767022D+18	-.8687629546482D+01	-.3119852767022D+18
.1D+01	0.3278192015012+261	-.4588037824984D+01	-.3278192015012+261

the resultant method, has; (a) Adequate and higher level of accuracy; (b) Minimum bound of local truncation error; (c) Large maximum interval of absolute stability, (d) Minimum computer storage facility and (e) Faster rate of convergence. In subsequent publications we shall focus attention on the stability of the method.

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