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Short Communication

Some results about log-harmonic mappings

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Let A(α, β) be a subclass of certain analytic functions and H (D) is to be a linear space of all analytic functions defined on the open unit disc D = {z| |z| < 1}. A sense-preserving log-harmonic function is the solution of the non-linear elliptic partial differential equation; $\overline{f_{\bar{z}}} = w\frac{\bar{f}}{f}f_z,$ where w(z) is analytic, satisfies the condition |w(z)| < 1 for every z ∈ D and is called the second dilatation of f . It has been shown that if f is a non-vanishing log-harmonic mapping, then f can be represented by; $f(z) = h(z)\overline{g(z)},$ where h(z) and g(z) are analytic in D with h(0) = 0, g(0) = 1([1]). If f vanishes at z = 0, but it is not identically zero, then f admits the representation; $f(z) = z|z|^{2\beta}h(z)\overline{g(z)},$ where Reβ >-½, h(z) and g(z) are analytic in D with g(0) = 1 and h(0) = 0. The class of sense-preserving log-harmonic mappins is denoted by SLH. The aim of this paper is to give some distortion theorems of these classes.

Key words: Starlike, subordination, distortion.

INTRODUCTION

Let A(α, β) be the class of functions of the form:

$$h(z) = 1 - \sum_{1}^{\infty} c_n z^n (c_n \geq 0) \quad (1)$$

which are in the analytic , in the open unit disc D = **{**z||z| < 1**}** and satisfy

$$\operatorname{Re} h(z) + \alpha z h'(z) > 0, z \in D \quad (2)$$

where Re(α) > 0 and 0 ≤ β < 1. The class A(α, β) for real α > 0 was studied by Osman and Shigeyoshi (1992)

Let Ω be the family of functions φ(z) which are regular in D and satisfying the conditions φ(0) = 0, |φ(z)| < 1 for all z ∈ D.
Let S_1 (z) and S_2(z) be analytic functions in D with $S_1(0) = S_2(0)$. We say that S_1(z) subordinated to S_2(z) and denote by:

$$S_1(z) \prec S_2(z) \quad (3)$$

if $S_1(z) = S_2(\varphi(z))$ for some function φ(z) ∈ Ω and every z ∈ D. If $S_1(z) \prec S_2(z)$, then $S_1(D) \subset S_2(D)$ (Zayid and Abu, 1996).

Further, S_1 (z) is said to be quazi subordinate to S2 (z) if there exists an analytic function φ(z) Such that $S_1(z)$ is analytic in D, S_1(z):

$$\frac{S_1(z)}{\varphi(z)} \prec g(z), (z \in D) \quad (4)$$

and φ(z) ≤ 1, (z ∈ D). We also denote this quazi subordination () is equivalent to:

$$S_1(z) = \varphi(z).S_2(w(z)), \quad (5)$$

where |φ(z)| ≤ 1, (z ∈ D) and |w(z)| ≤ z (z ∈ D)studied by Zayid and Walter (1996).

In the quazi subordination if φ(z) ≡ 1 then, it becomes

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the subordination. For analytic functions $S_1(z)$ and $S_2(z)$ in D, we say that, $S_1(z)$ is majorized by $S_2(z)$ if there exists an analytic function $\varphi(z)$ in D satisfying $|\varphi(z)| \leq 1$ and $S_1(z) = \varphi(z).S_2(z)$ $(z \in D)$. We denote the majorization by;

$$S_1(z) << S_2(z), (z \in D) \tag{6}$$

If we take $w(z) = z$; then quazi subordination becomes the majorization (Yasunori and Shigeyoshi, 2008). Finally, let H (D) be the linear space of all analytic functions defined on the open unit disc D. A sense-preserving log-harmonic mapping is the solution of the non-linear elliptic partial differential equation:

$$\frac{\overline{f_{\bar{z}}}}{\overline{f}} = w(z)\frac{f_z}{f}, \tag{7}$$

where $w(z) \in H(D)$ is the second dilatation of **f** such that $|w(z)| < 1$ for every $z \in D$. It has been shown that if **f** is a non-vanishing log-harmonic mapping, then f can be expressed as:

$$f = h(z)\overline{g(z)} \tag{8}$$

where h(z) and g(z) are analytic functions in D.

On the other hand, if **f** vanishes at $z = 0$ and at no other point, then **f** admits the representation:

$$f = z\,|z|^{2\beta}\, h(z)\overline{g(z)}, \tag{9}$$

where $\mathrm{Re}\beta > -1/2$, h(z) and g(z) are analytic in D with $g(0) = 1$ and $h(0) = 0$. We note that the class of log-harmonic mappings is denoted by S_{LH} .

Let $f = zh(z)\overline{g(z)}$ be an element of S_{LH} was studied by Zayid and Daoud (1988).

RESULTS

Theorem 1

Let $f = zh(z)\overline{g(z)} = \phi(z).\overline{g(z)}$ be an element of $S^*_{LH}(A, B)$ $\phi(z) = zh(z)$, $g(z) = 1 + a_1z + a_2z^2 + \ldots$, $h(z) = b_0 + b_1z + b_2z^2 + \ldots$ And let $H(z) = \frac{z\phi^0(z)}{\phi(z)}$. Then:

$$H(z) \in A(\alpha, \beta) \Leftrightarrow \mathrm{Re}(H(z) + \alpha zH^0(z)) > \beta, (|z| < 1) \tag{10}$$

for some complex α, (α = 0) ; and for some β (0 ≤ β < 1).

Proof

Let $f \in S^*_{LH}(A, B)$. Then:

$$H(z) = \frac{z\phi^0(z)}{\phi(z)} = \frac{z(b_0 + 2b_1z + 3b_2z^2 + \ldots)}{b_0z + b_1z^2 + b_2z^3 + \ldots} \tag{11}$$

$$= (1 + 2\frac{b_1}{b_0}z + 3\frac{b_2}{b_0}z^2 + \ldots)(1 + \frac{b_1}{b_0}z + \frac{b_2}{b_0}z^2 + \ldots)^{-1} \tag{12}$$

$$= (1+2\frac{b_1}{b_0}z+3\frac{b_2}{b_0}z^2+\ldots)(1-(\frac{b_1}{b_0}z+\frac{b_2}{b_0}z^2+\ldots)+(\frac{b_1}{b_0}z+\frac{b_2}{b_0}z^2+\ldots)^2-(\frac{b_1}{b_0}z+\ldots)^3+\ldots) \tag{13}$$

$$= (1 + 2\frac{b_1}{b_0}z + 3\frac{b_2}{b_0}z^2 + \ldots)(1 - \frac{b_1}{b_0}z + (\frac{b_1^2}{b_0^2} - \frac{b_2}{b_0})z^2 + \ldots) \tag{14}$$

$$= 1 + \frac{b_1}{b_0}z + \frac{3b_1b_0 - b_1^2 - b_1b_2}{b_0^2}z_2 + \ldots \tag{15}$$

$$= 1 + h_1z + h_2z^2 + \ldots \tag{16}$$

satisfied easily.

Lemma 1: If H(z) is an element of A(α, β) then;

$$H(z) = 1 + \sum_{n=1}^{\infty} h_nz^n \in A(\alpha, \beta), h_n = |h_n|.e^{i(n\theta+\pi)} \Rightarrow \sum_{n=1}^{\infty}(1+n\mathrm{Re}(\alpha))|h_n| \leq 1-\beta. \tag{17}$$

Proof

$$\mathrm{Re}(H(z) + \alpha zH^0(z)) = \mathrm{Re}(1 + \sum_{n=1}^{\infty}(1 + n\alpha)h_nz^n) \tag{18}$$

$$= \mathrm{Re}(1 + \sum_{n=1}^{\infty}(1 + \alpha n)\ |h_n|\, e^{i(n\theta+\pi)}z^n) > \beta \tag{19}$$

for all $z \in D$ Let $z = |z|\,e^{-i\theta}$, then $z^n = |z|^n\, e^{-in\theta}$ and,

$$\mathrm{Re}(H(z) + \alpha zH^0(z)) = 1 - \sum_{n=1}^{\infty}(1 + n\mathrm{Re}(\alpha))\ |h_n|\,|z|^n > \beta \tag{20}$$

Telling $|z| \to 1^{-}$, we have;

$$\sum_{n=1} (1 + n\mathrm{Re}(\alpha))\ |h_n| \le 1 - \beta \tag{21}$$

Theorem 2

If H(z) is an element of A(α, β) then:

$$H(z) = 1 + \sum_{n=1} h_n z^n \in A(\alpha, \beta), h_n = |h_n|\, e^{-in\theta+\pi}, \mathrm{Re}(\alpha) \ge 0 \tag{22}$$

$$\Rightarrow 1 - \frac{1-\beta}{1+\mathrm{Re}\alpha}|z| \le \mathrm{Re}H(z) \le |H(z)| \le 1 + \frac{1-\beta}{1+\mathrm{Re}\alpha}|z|$$

for $|z| < 1$. (23)

Proof by Lemma 1,

$$\sum_{n=1} |h_n| \le \frac{1-\beta}{1+\mathrm{Re}(\alpha)} \tag{24}$$

Thus:

$$|H(z)| \le 1 + |z| \sum_{n=1} |h_n| \le 1 + \frac{1-\beta}{1+\mathrm{Re}\alpha}|z| \tag{25}$$

and,

$$\mathrm{Re}H(z) = 1 + \mathrm{Re}(\sum_{n=1} h_n z^n) \ge 1 - |z| \sum_{n=1} |h_n| \ge 1 - \frac{1-\beta}{1+\mathrm{Re}(\alpha)}|z| \tag{26}$$

Theorem 3

If H(z) is an element of A(α, β) then:

$$H(z) = 1 + \sum_{n=1} h_n z^n \in A(\alpha, \beta), h_n = |h_n|\, e^{-in\theta+\pi}, \mathrm{Re}(\alpha) \ge 0 \tag{27}$$

$$\Rightarrow \mathrm{Re}(h_1) - \frac{1-\beta-(1+\mathrm{Re}(\alpha_1)\ |h_1|)}{\mathrm{Re}(\alpha)}|z| \le \mathrm{Re}H^{0}(z) \tag{28}$$

$$\le |H^{0}(z)| \le |h_1| + \frac{1-\beta-(1+\mathrm{Re}(\alpha_1)\,|h_1|)}{\mathrm{Re}(\alpha)}|z|$$

for $|z| < 1$. (29)

Proof

Note that:

$$(1+\mathrm{Re}(\alpha))\,|h_1| + \sum_{n=1} (1+n\mathrm{Re}(\alpha))\,|h_n| \le \sum_{n=1} (1+n\mathrm{Re}(\alpha))\,|h_n| \le 1-\beta, \tag{30}$$

This gives that:

$$\sum_{n=2} n\,|h_n| \le \frac{1-\beta-(1+\mathrm{Re}(\alpha))\,|h_1|}{\mathrm{Re}(\alpha)}. \tag{31}$$

Thus we have that:

$$|H^{0}(z)| = h_1 + \sum_{n=2} n h_n z^{n-1} \le |h_1| + |z| + |z| \sum_{n=2} n\,|h_n| \tag{32}$$

$$\le |h_1| + \frac{1-\beta-(1+\mathrm{Re}(\alpha))\,|h_1|}{\mathrm{Re}(\alpha)}|z| \tag{33}$$

and,

$$\mathrm{Re}H^{0}(z) = \mathrm{Re}(h_1 + \sum n h_n z^{n-1}) \ge \mathrm{Re}(h_1) - |z| \sum n\,|h_n| \tag{34}$$

$$\ge \mathrm{Re}(h_1) - \frac{1-\beta-(1+\mathrm{Re}(\alpha))\,|h_1|}{\mathrm{Re}(\alpha)}|z|. \tag{35}$$

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