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On enhanced control charting for process monitoring

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The information on auxiliary characteristics helps significantly in increasing the efficiency of control charts for detecting shifts in process parameters. In this study we proposed Shewhart type control charts, namely the A_r chart, the D_r chart and the K_r chart, which utilizes information on two auxiliary characteristics (*X* and *Z*) for improved monitoring of process location parameter with respect to a single quality characteristic of interest (*Y*). Assuming trivariate normality of (*Y, X, Z*), a general control chart structure is developed in the form of the three sigma and the probability limits. The performance of the proposed charts is compared with the usual Shewhart $\bar{Y}$ chart, the M_r chart of Riaz (2008a), the control limits of Zhang (1984) and Wade and Woodall (1993). It has been observed that the proposed charts perform superior, in terms of discriminatory power, as compared to the above mentioned counterparts, depending upon the correlation structure among the auxiliary characteristics and the quality characteristic of interest. The said superiority zone of the correlation structures, favoring the proposed charts, needs to be identified very carefully to apply it in a given situation.

Key words: Auxiliary characteristics mean control charts, location parameter, normality, power curves, quality characteristics.

INTRODUCTION

A process is generally described by its characteristics and out of these some are of main concern and others are of a supplementary nature. The characteristics of main concern are termed as quality characteristics of interest while the other characteristics are termed as auxiliary characteristics. The auxiliary characteristics need to be identified very carefully along with the characteristics of interest. An auxiliary characteristic may be an early measurement in a process, crude but simple to obtain measurement, a property that would be monitored etc. The quality characteristic of interest may be any current variable of major interest which needs to be monitored for example, weight of a machine component, diameter of a shaft, spinning speed of wheel etc.

It is a common practice to take benefit out of the information available on auxiliary variable(s), along with the main study variable(s) of interest, in order to improve the efficiency in statistical terms. There is a variety of literature available in this regard for example, Kiregyera (1984), Mukerjee et al. (1987), Singh (2001), Singh et al. (2004), Kadilar and Cingi (2005), Joarder (2009), (2011) and Omar and Joarder (2011). This idea of using information on some additional characteristic(s) (for example, auxiliary characteristic(s)) which are associated with the main quality characteristic of interest has also been used in quality control literature. Particularly in control charting methodology, it has been used in the form of cause-selecting and regression-adjusted control charts (for example, Mandel (1969); Zhang (1984); Hawkins (1991, 1993); Wade and Woodall (1993); Shu et al. (2005) and auxiliary information based control charts

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(for example, Riaz (2008a, b); Riaz and Does (2009)) for the sake of an improved process monitoring with respect to the quality characteristic of interest.

Riaz (2008a) proposed a location control chart, namely the M_r chart, in which he used the information of single auxiliary characteristic on regression pattern. Riaz (2008b) and Riaz and Does (2009) gave proposals for variability control charts based on a single auxiliary characteristic in which they used the auxiliary informationon regression and ratio patterns respectively. There may be many real situations when more than one auxiliary characteristics are available to be used e.g. to monitor the inner diameter of a shaft its outer diameter and weight may be the two possible auxiliary characteristics. For the sake of simplicity we assume that there are two auxiliary characteristics available along with the quality characteristics of interest.

The rest of this article is organized as follows: The design structure of the proposed charts, performance evaluation measures and a comparison of the proposed charts with the usual Shewhart $\bar{Y}$ chart and the M_r chart of Riaz (2008a), the steps for using the proposed chart with help of simulated examples and finally, the paper ends with conclusions.

THE PROPOSED CHARTS

Suppose that the quality characteristic of interest is denoted by Y and the two auxiliary characteristics by X and Z. We write these three variables of this study in the form of a triplet as

(Y, X, Z) . Let $(Y, X, Z) \sim N_3(\underline{\mu}, \Sigma)$ where

$$\underline{\mu} = \begin{pmatrix} \mu_y \\ \mu_x \\ \mu_z \end{pmatrix} \text{ and } \Sigma = \begin{bmatrix} \sigma_{yy}\, \sigma_{yx}\, \sigma_{yz} \\ \sigma_{xy}\, \sigma_{xx}\, \sigma_{xz} \\ \sigma_{zy}\, \sigma_{zx}\, \sigma_{zz} \end{bmatrix}.$$

Here N_3 represents the trivariate normal distribution, μ_y, μ_x and μ_z represent the means of Y, X and Z, respectively; σ_{yy}, σ_{xx} and σ_{zz} represent the variances of Y, X and Z, respectively; σ_{yx}, σ_{yz} and σ_{xz} represent the co-variances between Y, X; Y, Z and X, Z; respectively.

Now the objective is to monitor the variations in Y by exploiting the information on X and Z. We consider here the case when the auxiliary characteristics X and Z remain stable over time. Also we assume that the spread parameters of Y is in- control so the only parameter we are concerned with is the location parameter of Y, that is, μ_y. In a broader sense we can consider any process without cascade property (cf. Hawkins (1993)) where each variable may undergo a distributional change without affecting the other variables of the process.

To monitor the behaviour of Y, in terms of μ_y, taking into account the information on X and Z we need an estimator for μ_y which capitalizes the information on X and Z as well along with the information of Y. Let (y_k, x_k, z_k) *where* $k = 1, 2, ...n$ be a trivariate random sample of size n drawn from (Y, X, Z). We considered three estimators of process location which uses the information of auxiliary characteristics X and Z, given as:

The regression type estimator given in Kadilar and Cingi (2005), defined as:

$$A_r = \bar{y} + b_{yx}(\mu_x - \bar{x}) + b_{yz}(\mu_z - \bar{z}) \tag{1}$$

The ratio type estimator proposed by Abu-Dayyeh et al. (2003), defined as:

$$D_r = \bar{y}\left(\frac{\mu_x}{\bar{x}}\right)^{\alpha_1}\left(\frac{\mu_z}{\bar{z}}\right)^{\alpha_2} \tag{2}$$

The ratio type estimator proposed by Kadilar and Cingi (2003), defined as:

$$K_r = \bar{y}\left(\frac{\mu_x}{\bar{x}}\right)^{\beta_1}\left(\frac{\mu_z}{\bar{z}}\right)^{\beta_2} + b_{yx}(\mu_x - \bar{x}) + b_{yz}(\mu_z - \bar{z}) \tag{3}$$

where $\bar{y}$, $\bar{x}$ and $\bar{z}$ are the sample means of Y, X and Z respectively. The quantities b_{yx} and b_{yz} are defined as:

$$b_{yx} = s_{yx} / s_{xx} \quad \text{and} \quad b_{yz} = s_{yz} / s_{zz}$$

where s_{xx} and s_{zz} are the sample variances of X and Z, respectively, s_{yx} and s_{yz} are the sample co-variances between Y, X and Y, Z, respectively. We used the optimal choices of $\alpha_1, \alpha_2, \beta_1$ and β_2 as reported in Abu-Dayyehh et al. (2003) and Kadilar and Cingi (2005), respectively, that is:

$$\alpha_1 = \frac{c_y b_{yx.z}}{c_x},\ \alpha_2 = \frac{c_y b_{yz.x}}{c_z}, \beta_1 = \frac{s_y}{r_1 s_x}\rho_1^* \text{ and } \beta_2 = \frac{s_y}{r_2 s_z}\rho_2^*$$

Where

$$c_y = \frac{s_y}{\bar{y}},\ c_x = \frac{s_x}{\bar{x}},\ c_z = \frac{s_z}{\bar{z}},\ b_{yx.z} = \frac{\rho_{yx} - \rho_{yz}\rho_{xz}}{1-\rho_{xz}^2},\ b_{yz.x} = \frac{\rho_{yz} - \rho_{yx}\rho_{xz}}{1-\rho_{xz}^2}$$

$$\rho_1^* = \frac{\rho_{xz}(\rho_{yx}\rho_{xz} - \rho_{yz})}{1-\rho_{xz}^2} \text{ and } \rho_2^* = \frac{\rho_{xz}(\rho_{yz}\rho_{xz} - \rho_{yx})}{1-\rho_{xz}^2}$$

The population means of the two auxiliary characteristics that is, μ_x and μ_z are assumed to be known in Equation 1 to 3. We will refer to the charts based on A_r, D_r and K_r estimators as the A_r chart, the D_r chart and the K_r chart, respectively.

Some distributional results

A general control chart structure that can be used with any of the three estimators will develop here.
First we define a pivotal quantity G, which is based on the sample statistic, say T, given as:

$$G = \sqrt{n}(T_r - \mu_y)/\sigma_y \tag{4}$$

Here T represents any of the sample statistic given in Equation 1 to 3, that is, A_r, D_r or K_r. The distributional behaviour of G entirely depends on the four quantities namely n, ρ_{yx}, ρ_{yz} and ρ_{xz}. Here ρ_{yx}, ρ_{yz} and ρ_{xz} represent the correlations between Y, X; Y, Z and X, Z; respectively, where $\rho_{ij} = \sigma_{ij}/\sqrt{\sigma_{ii}\sigma_{jj}}$. Now we let the following:

$$\left.\begin{array}{l} \mu_G = g_2, \\ \sigma_G = g_3 \text{ and} \\ \alpha^{th} \text{ quantile of } G = G_\alpha \end{array}\right\}, \tag{5}$$

where μ_G, σ_G and G_α are the mean, standard deviation and αth quartile of the distribution of G. The asymptotic results for coefficients g_2 and g_3 considering different choices of T are provided in the Appendix.

The results given in Appendix can be used satisfactorily for larger values of n or $\rho_{ij}\ (i = y, x, z = j \text{ where } i \neq j)$. For other choices of n and ρ_{ij} s, we need the true results for g_2, g_3 and G_α which may be obtained either analytically or through Monte Carlo simulations. To avoid the analytical complications of dealing with the distributional properties of G in the form of g_2, g_3 and G_α, we have written a code in R language to obtain the simulation results for these quantities. For some representative values of n and ρ_{ij} s the simulation results of g_2 and g_3 are given in Table 1 while the lower and upper quartiles of G at $\alpha = 0.01, 0.005$ and 0.0027 are given in Tables 2 to 4, respectively for the A_r chart. The standard errors for the simulated results (reported in Tables 1 to 4) are less than 1%, which is acceptable in control chart studies (Schaffer and Kim, 2007). These simulation results are based on 1000 repetitions of 10,000 Monte Carlo simulations. Similar results can be easily obtained for D_r and K_r charts.

Control chart design

Based on the results given above, we are now in a position to define the design structure of a general control chart namely the T_r Chart. From the expressions given in (4) and (5) above, we have the following:

$$\left.\begin{array}{ll} \mu_G = g_2 & \Rightarrow \mu_{T_r} = \mu_y + g_2\sigma_y/\sqrt{n} \\ \sigma_G = g_3 & \Rightarrow \sigma_{T_r} = g_3\sigma_y/\sqrt{n} \\ G_\alpha = \alpha^{th} \text{ quantile of } G & \Rightarrow T_{r_\alpha} = \mu_y + G_\alpha\sigma_y/\sqrt{n} \end{array}\right\} \tag{6}$$

where μ_{A_r}, σ_{A_r} and T_{r_α} are the mean, standard deviation and αth quartile of the distribution of T_r, respectively. Based on these results for μ_{T_r}, σ_{T_r} and T_{r_α}, we can define the control limits of the T_r charts, using the three sigma limit and the probability limit approaches, as:

Three Sigma Limits

$$CL = \mu_{T_r},\ LCL = \mu_{T_r} - 3\sigma_{T_r} \text{ and } UCL = \mu_{T_r} + 3\sigma_{T_r} \tag{7}$$

where CL, LCL and UCL refer to the Central Line, Lower Control Limit and Upper Control Limit on the T_r chart and the results for μ_{T_r} and σ_{T_r} are given in (6).

Probability Limits

$$ML = T_{r_{0.50}},\ LPL = T_{r_{\alpha/2}} \text{ and } UPL = T_{r_{(1-\alpha/2)}} \tag{8}$$

Table 1. Control chart coefficients g_2 and g_3 for the proposed A_r chart.

n	ρ_{xz}	ρ_{yx}	ρ_{yz}							
			0.2	**0.5**	**0.6**	**0.7**	**0.2**	**0.5**	**0.6**	**0.7**
			g_2				g_3			
5	0.1	0.2	0.02528	0.01118	0.00286	-0.02103	1.37879	1.25962	1.19871	1.11642
		0.5	0.02124	0.00101	0.00559	0.01226	1.25305	1.12344	1.09129	1.00844
		0.7	-0.00531	0.01388	-0.00072	0.00301	1.10444	0.98864	0.88997	0.7833
	0.4	0.2	0.00062	-0.00609	0.0072	0.01235	1.38015	1.33266	1.22746	1.1622
		0.5	-0.02645	-0.00976	-0.002	-0.00839	1.32587	1.22778	1.18168	1.10548
		0.7	-0.00705	0.00271	-0.0174	0.01366	1.15595	1.10066	1.06812	0.99845
	0.6	0.2	-0.00197	0.00907	0.01026	0.00406	1.46208	1.33059	1.27938	1.18668
		0.5	-0.00711	-0.02869	-0.00815	-0.00565	1.35712	1.33152	1.25687	1.16798
		0.7	-0.01363	-0.0016	-0.01275	0.00543	1.14878	1.19654	1.14881	1.10673
10	0.1	0.2	-0.00563	-0.00288	-0.00945	0.00688	1.08122	0.98617	0.92521	0.84061
		0.5	-0.00461	0.01465	0.00944	-0.00743	0.98483	0.8928	0.81227	0.71503
		0.7	-0.00273	0.00824	-2e-04	0.00255	0.83997	0.72319	0.63659	0.51959
	0.4	0.2	0.02833	0.01165	0.01507	-0.01221	1.11388	1.02812	0.96857	0.90049
		0.5	0.0087	-0.0062	0.00074	-0.01407	1.03116	0.98071	0.92882	0.86788
		0.7	0.00306	0.00767	0.00337	-0.00634	0.8959	0.87357	0.81322	0.75689
	0.6	0.2	-0.02319	-0.01109	0.00263	0.02549	1.15461	1.0691	1.02283	0.92831
		0.5	0.01838	0.01306	-0.01017	-0.00651	1.06936	1.04593	1.00541	0.95394
		0.7	0.00656	0.01254	0.00046	0.01008	0.93079	0.94729	0.92794	0.89932
15	0.1	0.2	0.001	-0.00371	-0.00448	-0.00194	1.04659	0.93876	0.85827	0.78882
		0.5	0.01472	-0.00889	-0.00796	0.01011	0.93186	0.82375	0.76034	0.66147
		0.7	-0.00161	-0.0061	0.01524	-0.00027	0.79915	0.67325	0.58489	0.45589
	0.4	0.2	-0.01385	0.00744	-0.00434	0.01175	1.06876	0.96603	0.92541	0.85329
		0.5	-0.0112	0.0072	0.00733	0.00566	0.96556	0.91324	0.88048	0.80839
		0.7	-0.00789	-0.00359	0.02181	-0.01146	0.84572	0.81578	0.7686	0.71474
	0.6	0.2	0.00597	0.00838	-0.00581	0.01725	1.08526	0.99464	0.94147	0.88688
		0.5	-0.00031	0.01003	-0.00629	0.01038	1.00483	0.97996	0.95624	0.89657
		0.7	0.00443	0.00981	0.01118	-0.00304	0.87396	0.91137	0.88681	0.84288

where ML, LPL and UPL refer to the Median Line, Lower Probability Limit and Upper Probability Limit on the T_r chart and the results for T_{r_α} s are given in (6). Here α is a pre-specified false alarm rate which is equally divided on both the tails to define the probability limits.

In probability limits approach it is preferable to replace the CL by ML as we did in (8). The control limits given in (7) and (8) are the specified parameters versions of the three sigma and probability limits structures of the T_r chart. In case of unspecified parameters the estimated versions may used by replacing μ_y and σ_y by their estimators $\hat{\mu}_y$ and $\hat{\sigma}_y$, where $\hat{\sigma}_y$ may be obtained from some spread control chart e.g. the R chart and $\hat{\mu}_y = \overline{T_r}$ based on an initial set of say *m* samples from the process in a stable situation. The choices of *T* as A_r, D_r and K_r will be referred as the proposed A_r chart, D_r chart and K_r chart, respectively.

PERFORMANCE EVALUATION AND COMPARISONS

To evaluate the performance of a control chart discriminatory power is a very popular measure. We evaluate here the performance of our proposed control charts using the same performance measure. Let the in control value of μ_y be denoted by μ_y^0 and the shifted value by μ_y^1. We define here μ_y^1 in terms of μ_y^0 and

Table 2. Quantile points of the distribution of G when $\alpha = 0.01$ for the proposed A_r chart.

			ρ_{yz}							
			0.2	0.5	0.6	0.7	0.2	0.5	0.6	0.7
n	ρ_{xz}	ρ_{yx}	Lower quantile points				Upper quantile points			
5	0.1	0.2	-3.97102	-3.78103	-3.6426	-3.65317	4.14905	3.73057	3.8889	3.43684
		0.5	-3.62557	-3.27497	-3.16787	-2.92716	3.78079	3.31657	3.38637	3.01347
		0.7	-3.44537	-3.12252	-2.82657	-2.58031	3.37663	3.07759	2.69982	2.61299
	0.4	0.2	-4.11569	-4.19893	-3.73861	-3.59456	4.30716	4.07241	3.77613	3.56266
		0.5	-4.41764	-3.683	-3.54701	-3.39483	3.71978	3.78464	3.80811	3.46128
		0.7	-3.60466	-3.36004	-3.34861	-2.9802	3.65047	3.5377	3.35836	3.04366
	0.6	0.2	-4.50682	-4.23655	-3.98565	-3.68707	4.41788	3.85878	3.88798	3.76618
		0.5	-4.23524	-4.55563	-3.95808	-3.67121	3.94551	3.86092	3.88779	3.40115
		0.7	-3.61571	-3.88482	-3.75627	-3.44119	3.34483	3.52922	3.48809	3.41596
10	0.1	0.2	-2.86258	-2.59285	-2.46205	-2.24404	2.82898	2.68028	2.42749	2.30314
		0.5	-2.64839	-2.3952	-2.22006	-1.98339	2.67217	2.39217	2.23577	1.94372
		0.7	-2.25481	-2.03972	-1.80749	-1.58932	2.15949	2.01814	1.77604	1.52854
	0.4	0.2	-2.96395	-2.83776	-2.54074	-2.4675	3.06357	2.85225	2.72894	2.49768
		0.5	-2.83123	-2.7103	-2.51405	-2.51962	2.72242	2.70332	2.5911	2.36777
		0.7	-2.47889	-2.46715	-2.23234	-2.20898	2.3918	2.45258	2.30924	2.17702
	0.6	0.2	-3.14441	-2.84443	-2.87916	-2.55466	3.09088	2.95342	2.7877	2.66518
		0.5	-2.9006	-2.81961	-2.71347	-2.78393	2.88504	2.96454	2.7522	2.55112
		0.7	-2.54317	-2.57074	-2.64255	-2.57708	2.65917	2.711	2.67683	2.61477
15	0.1	0.2	-2.72483	-2.36326	-2.27587	-2.0561	2.70286	2.53425	2.19781	2.01474
		0.5	-2.36822	-2.19739	-1.99993	-1.75113	2.43056	2.18883	2.00273	1.75705
		0.7	-2.06945	-1.82418	-1.59031	-1.29214	2.15204	1.78648	1.60332	1.33872
	0.4	0.2	-2.82178	-2.58514	-2.53264	-2.22029	2.78592	2.54471	2.42061	2.33632
		0.5	-2.46588	-2.28677	-2.30213	-2.20105	2.48775	2.38369	2.40907	2.17542
		0.7	-2.24404	-2.23402	-1.9964	-1.97943	2.26355	2.15898	2.0529	1.89697
	0.6	0.2	-2.7444	-2.60271	-2.45856	-2.31174	2.86499	2.56944	2.43297	2.47822
		0.5	-2.64485	-2.53601	-2.70039	-2.52149	2.66358	2.54752	2.57133	2.51461
		0.7	-2.40357	-2.45853	-2.31856	-2.38694	2.33469	2.44131	2.47169	2.38482

$\delta\sigma_y$ as:

$$\mu_y^1 = \mu_y^0 + \delta\sigma_y \tag{9}$$

Now the ability of the three sigma and the probability limits based design structures of the proposed charts is given by the following power expressions:

$$Power_{T\,Chart} = Pr((T_r < LCL \ or \ T_r > UCL) / \mu_y^1 = \mu_y^0 + \delta\sigma_y) \quad (for \ three \ sigma \ limits) \tag{10}$$

$$Power_{T\,Chart} = Pr((T_r < LPL \ or \ T_r > UPL) / \mu_y^1 = \mu_y^0 + \delta\sigma_y) \quad (for \ probability \ limits) \tag{11}$$

These power expressions for the proposed A_r chart may be evaluated using the distributional results given above. It is a common practice to evaluate such expressions by varying the values of δ and we do the same here in this article. By varying the values of δ from 0.0 to 3.0, we have evaluated the power expressions for different n by fixing the false rate at α level. It is to be mentioned that for power evaluation, the shifts are taken in μ_y in terms of σ_y units. The resulting powers/power curves are provided in the following sub-sections along with the comparison with other charts. We expect that the proposed control charts should detect any shift in the parameter of interest with a high probability, keeping the false alarm rate at fixed low level. We compare the

Table 3. Quantile points of the distribution of G when $\alpha = 0.005$ for the proposed A_r chart.

			ρ_{yz}							
			0.2	0.5	0.6	0.7	0.2	0.5	0.6	0.7
n	ρ_{xz}	ρ_{yx}	Lower quartile points				Upper quartile points			
5	0.1	0.2	-4.54539	-4.33528	-3.97429	-3.97104	4.74617	4.56065	4.17105	4.07236
		0.5	-4.47279	-4.31902	-4.18939	-3.77039	4.80639	3.96592	3.64165	3.71793
		0.7	-3.99686	-3.69871	-3.27888	-3.15671	4.04841	3.72148	3.69719	3.13192
	0.4	0.2	-5.2083	-4.82689	-4.35783	-4.14606	5.23378	4.41552	4.34827	4.09887
		0.5	-4.94663	-4.46005	-4.30744	-4.25492	4.75222	4.42103	4.13466	3.88508
		0.7	-4.17621	-3.93938	-3.87112	-3.97246	3.89642	3.93888	3.71839	3.65786
	0.6	0.2	-5.38393	-4.99306	-4.47041	-4.14798	5.42804	4.92632	4.58492	4.3932
		0.5	-4.81035	-4.49234	-5.05762	-4.555	4.66916	4.44825	4.45817	4.33252
		0.7	-4.31651	-4.25503	-4.01866	-4.20301	4.46666	4.15392	4.31538	4.16839
10	0.1	0.2	-3.15883	-2.94136	-2.71138	-2.5647	3.30012	2.89895	2.67749	2.70204
		0.5	-2.91722	-2.67483	-2.58528	-2.07595	2.80245	2.55839	2.48538	2.34822
		0.7	-2.59091	-2.22275	-1.98301	-1.86076	2.64974	2.22036	2.08953	1.71718
	0.4	0.2	-3.29782	-2.96312	-2.86768	-2.75342	3.22129	2.8768	2.91972	2.81218
		0.5	-3.09865	-3.09822	-2.8358	-2.79805	3.09848	3.01829	2.89781	2.71626
		0.7	-2.7672	-2.69623	-2.62449	-2.39855	2.66776	2.70528	2.67277	2.40624
	0.6	0.2	-3.48474	-3.14002	-2.9252	-2.94404	3.58448	3.12041	3.09335	2.92391
		0.5	-3.18597	-3.34953	-3.35038	-2.84796	3.42229	3.31241	2.99185	2.96006
		0.7	-2.81364	-2.92902	-2.91798	-2.77572	3.08454	3.055	2.91007	2.8502
15	0.1	0.2	-2.85949	-2.75897	-2.51351	-2.15714	3.04247	2.68486	2.51987	2.34164
		0.5	-2.77216	-2.46103	-2.1907	-1.90911	2.64823	2.22662	2.21324	1.87076
		0.7	-2.31086	-1.87003	-1.75189	-1.51016	2.39498	2.04127	1.70571	1.51232
	0.4	0.2	-3.09685	-2.83138	-2.49248	-2.53114	3.14351	2.94631	2.59407	2.51537
		0.5	-2.71765	-2.62582	-2.54215	-2.52317	2.66559	2.70403	2.57528	2.39982
		0.7	-2.37308	-2.45781	-2.36547	-2.33688	2.51149	2.49198	2.44836	2.23488
	0.6	0.2	-3.19168	-3.01762	-2.8691	-2.55438	3.23773	2.99301	2.87793	2.66199
		0.5	-2.94557	-3.18495	-2.72081	-2.79505	3.04275	3.07343	2.89973	2.75213
		0.7	-2.67956	-2.8704	-2.69916	-2.6941	2.69604	2.72166	2.69453	2.61663

performance of the proposed A_r, D_r and K_r charts, in terms of discriminatory power, with the existing counterparts serving the same purpose. These exiting counterparts are the M_r chart of Riaz (2008a) and the usual Shewhart $\bar{Y}$ chart. An indirect comparison with the control limits of Zhang (1984) and Wade and Woodall (1993) is also provided.

Comparison of proposed A_r, D_r and K_r charts with M_r and $\bar{Y}$ charts. We have evaluated the power expression (11) of the proposed A_r, D_r and K_r charts for $n=15$, $\alpha=0.01$ and $\delta=0.0-0.3$ for different representative combinations of ρ_{yx}, ρ_{yz} and ρ_{xz}. These powers are plotted against δ and the resulting power curves are shown in the Figures 1 to 9 whereas Figure 10 presents power curves for a specific case when $n=25$, to investigate the effect on sample size on the performance of the proposed chart. We have varied the values of ρ_{yx} and ρ_{xz} between Figures 1 to 10 and within each figure we have taken different choices of ρ_{yz}. The similar power curves for the M_r chart of Riaz (2008a) and the usual Shewhart $\bar{Y}$ chart are also provided in Figures 1 to 10.

The symbols used in Figures 1 to 10 are defined as:

$\bar{Y}$ refers to the power curve of the usual $\bar{Y}$ chart; $M_{r_{0.5}}$ refers to the power curve of the M_r chart when $\rho_{yx}=0.5$;

Table 4. Quantile points of the distribution of G when $\alpha = 0.0027$ for the proposed A_r chart.

			ρ_{yz}							
			0.2	0.5	0.6	0.7	0.2	0.5	0.6	0.7
n	ρ_{xz}	ρ_{yx}	Lower quartile points				Upper quartile points			
5	0.1	0.2	-5.39974	-5.00368	-5.43807	-4.58529	5.01409	5.1413	5.01839	5.07115
		0.5	-4.52258	-5.14223	-5.18399	-4.04945	4.89941	4.66119	4.86417	4.38703
		0.7	-4.90532	-4.25886	-4.17131	-3.94228	4.7847	3.87925	4.36303	4.01529
	0.4	0.2	-5.50985	-5.12604	-4.94594	-4.99818	5.24367	5.46468	4.72265	4.79616
		0.5	-5.34707	-5.2568	-5.02791	-4.46698	5.13957	4.94856	4.7503	4.52779
		0.7	-5.14314	-4.12201	-4.77943	-4.28963	4.97546	4.12672	4.25718	4.49008
	0.6	0.2	-5.52532	-5.81717	-6.24932	-5.17663	7.13212	5.40913	5.56254	5.39732
		0.5	-6.33033	-5.10441	-4.90417	-5.89414	5.52745	5.18017	5.71029	5.04137
		0.7	-5.65621	-5.24406	-4.69353	-5.63756	5.52347	5.35344	5.33113	5.1099
10	0.1	0.2	-3.27143	-3.22835	-3.01715	-2.63105	3.22923	3.13473	3.13884	2.78724
		0.5	-3.05905	-2.8379	-2.69147	-2.37552	3.09342	2.98674	2.7367	2.38318
		0.7	-2.56669	-2.35595	-2.17836	-2.04356	2.85726	2.40829	2.21894	2.09169
	0.4	0.2	-3.58836	-3.79965	-3.46405	-3.03036	3.62924	3.27544	3.25358	3.03745
		0.5	-3.32176	-3.47487	-2.9817	-3.1034	3.23131	3.2391	3.28229	2.88333
		0.7	-3.03735	-2.76057	-2.95401	-2.63614	3.11502	3.12867	2.70669	2.59237
	0.6	0.2	-3.67768	-3.61903	-3.42601	-3.4583	3.93277	3.83308	3.446	3.25822
		0.5	-3.68748	-3.52476	-3.64232	-3.23163	3.3837	3.40624	3.4675	3.44901
		0.7	-3.06349	-3.36299	-3.16347	-3.55088	3.14006	3.3475	3.27739	3.26034
15	0.1	0.2	-3.0439	-2.90463	-2.71473	-2.37663	3.12921	2.96994	2.66318	2.40894
		0.5	-2.98606	-2.55459	-2.40091	-2.23551	2.95396	2.55838	2.31413	2.09056
		0.7	-2.58221	-2.10444	-1.76171	-1.54547	2.53261	2.06038	1.87562	1.68233
	0.4	0.2	-3.41324	-3.14535	-2.78668	-2.70091	3.23811	3.00637	2.80955	2.79875
		0.5	-3.053	-3.02028	-2.72883	-2.6213	3.25452	2.98602	2.58218	2.54357
		0.7	-2.73916	-2.66978	-2.4749	-2.5905	2.73159	2.57113	2.53498	2.42894
	0.6	0.2	-3.54761	-3.02662	-3.16646	-2.99985	3.39046	3.18865	3.32626	2.83037
		0.5	-3.10087	-3.02705	-3.01128	-2.90399	3.34941	3.10859	3.08018	2.92046
		0.7	-2.87782	-2.84046	-2.96856	-2.81365	2.6707	3.05733	2.96639	2.76807

$A_{r_{0.5}}$ refers to the power curve of the A_r chart when $\rho_{yz} = 0.5$; $D_{r_{0.5}}$ refers to the power curve of the D_r chart when $\rho_{yz} = 0.5$; $K_{r_{0.5}}$ refers to the power curve of the K_r chart when $\rho_{yz} = 0.5$;

Similarly the other symbols are defined.

The power curve analysis advocates the following for the proposed charts:

The D_r chart is performing better than all the other competing charts followed by the K_r chart. A_r chart although less efficient then D_r and K_r charts, performs better then M_r and $\bar{Y}$ charts.For a fixed value of ρ_{xz}, power of the proposed A_r, D_r and K_r charts increases with an increase in ρ_{yx} and ρ_{yz} (Figures 1 to 3). Power of the proposed charts decreases with an increase in ρ_{xz} and vice versa (Figures 1 and 4).

The proposed charts performs better than both the $\bar{Y}$ and M_r charts for low and moderate values of ρ_{xz} irrespective of the values of ρ_{yx} and ρ_{yz} (that is, for low, moderate and high values of ρ_{yx} and ρ_{yz}, the proposed

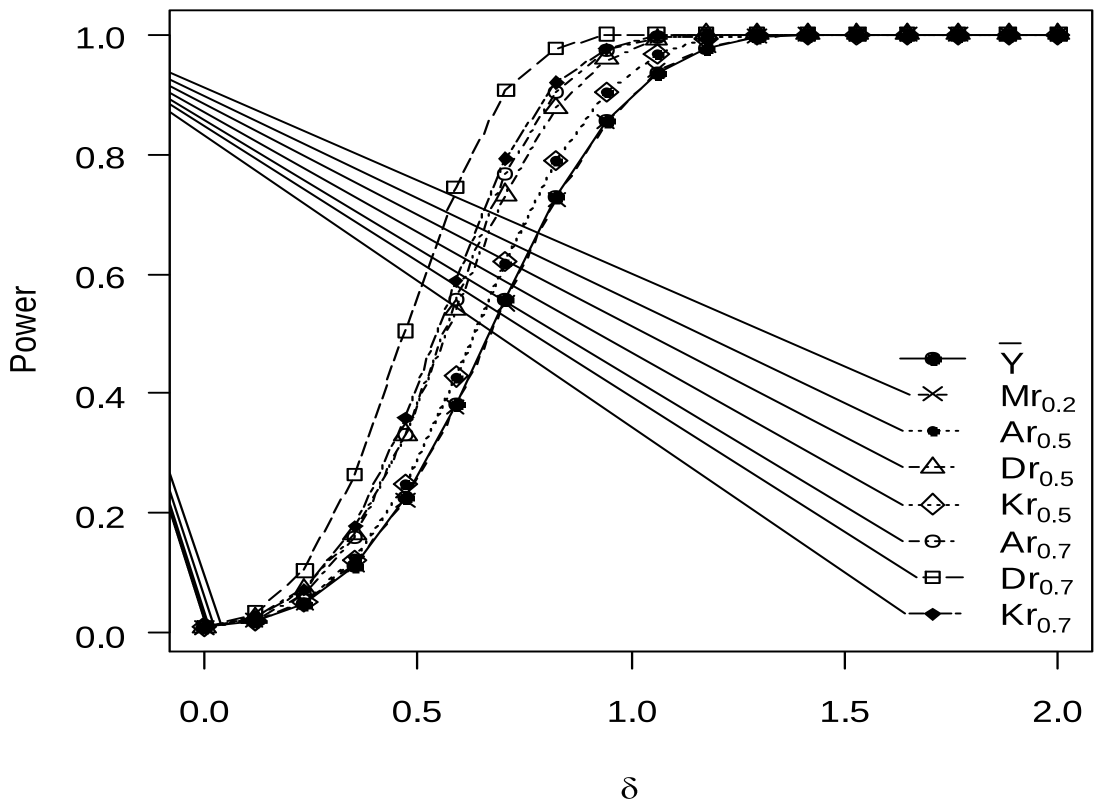

Figure 1. Power curves of $\overline{Y}, M_r, A_r, D_r$ and K_r charts for $\rho_{xz} = 0.1, \rho_{yx} = 0.2$ and $\rho_{yz} = 0.5$ and 0.7.

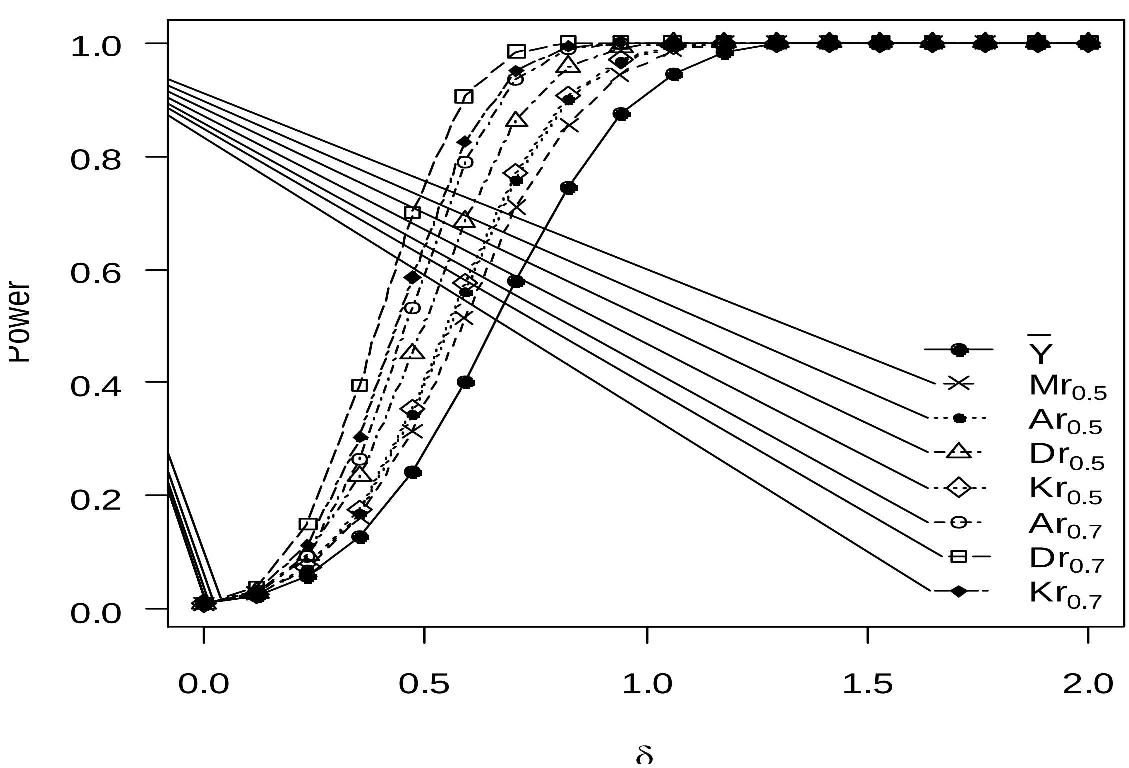

Figure 2. Power curves of $\overline{Y}, M_r, A_r, D_r$ and K_r charts for $\rho_{xz} = 0.1, \rho_{yx} = 0.5$ and $\rho_{yz} = 0.5$ and 0.7.

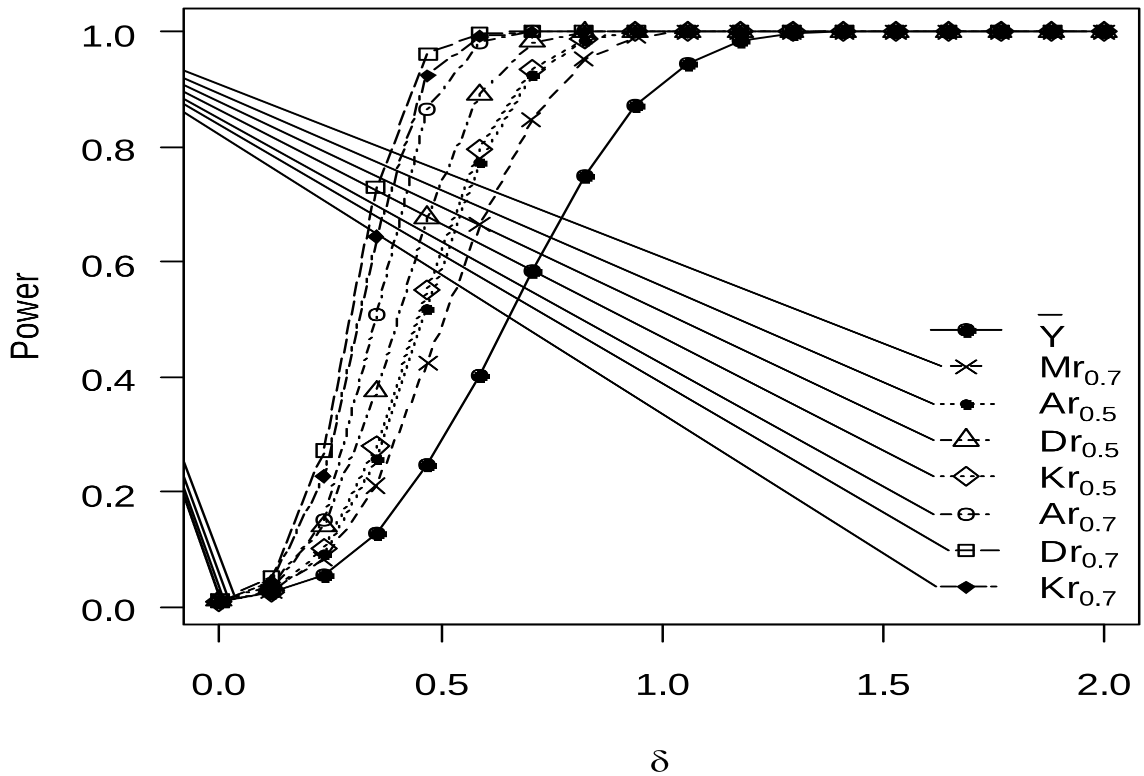

Figure 3. Power curves of $\bar{Y}, M_r, A_r, D_r$ and K_r charts for $\rho_{xz} = 0.1, \rho_{yx} = 0.7$ and $\rho_{yz} = 0.5$ and 0.7.

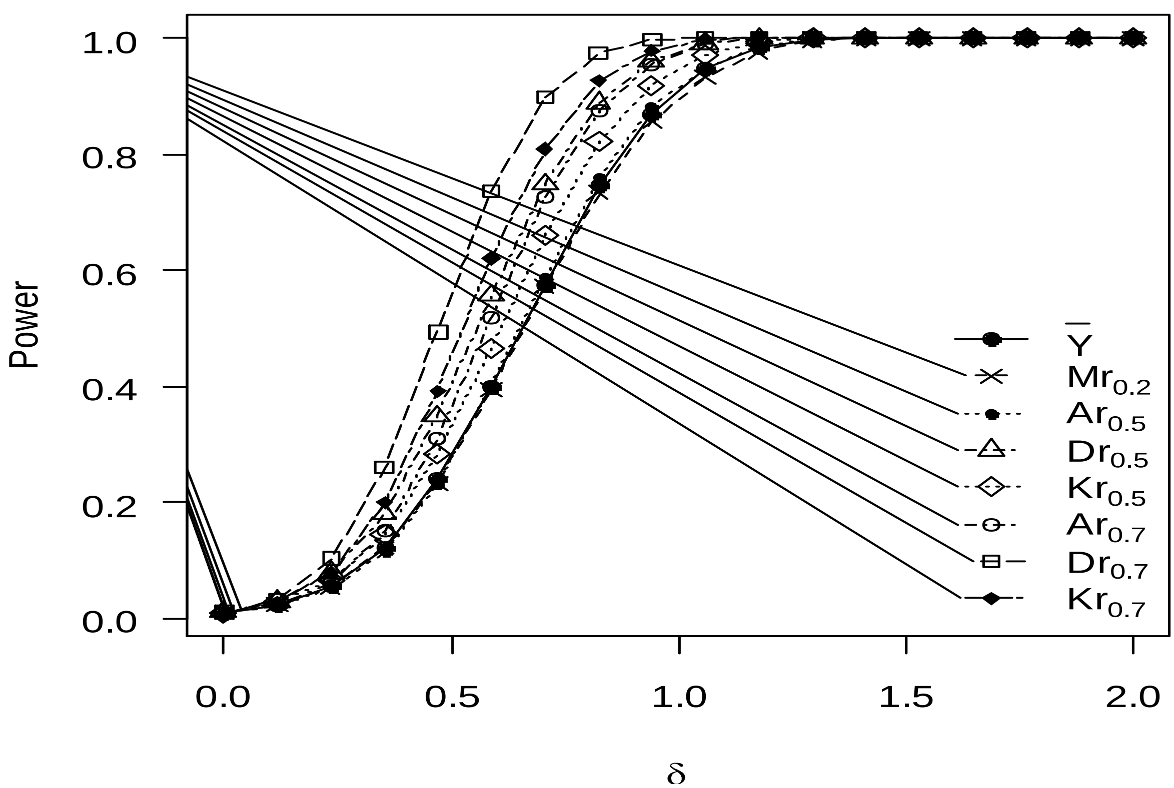

Figure 4. Power curves of $\bar{Y}, M_r, A_r, D_r$ and K_r charts for $\rho_{xz} = 0.4, \rho_{yx} = 0.2$ and $\rho_{yz} = 0.5$ and 0.7.

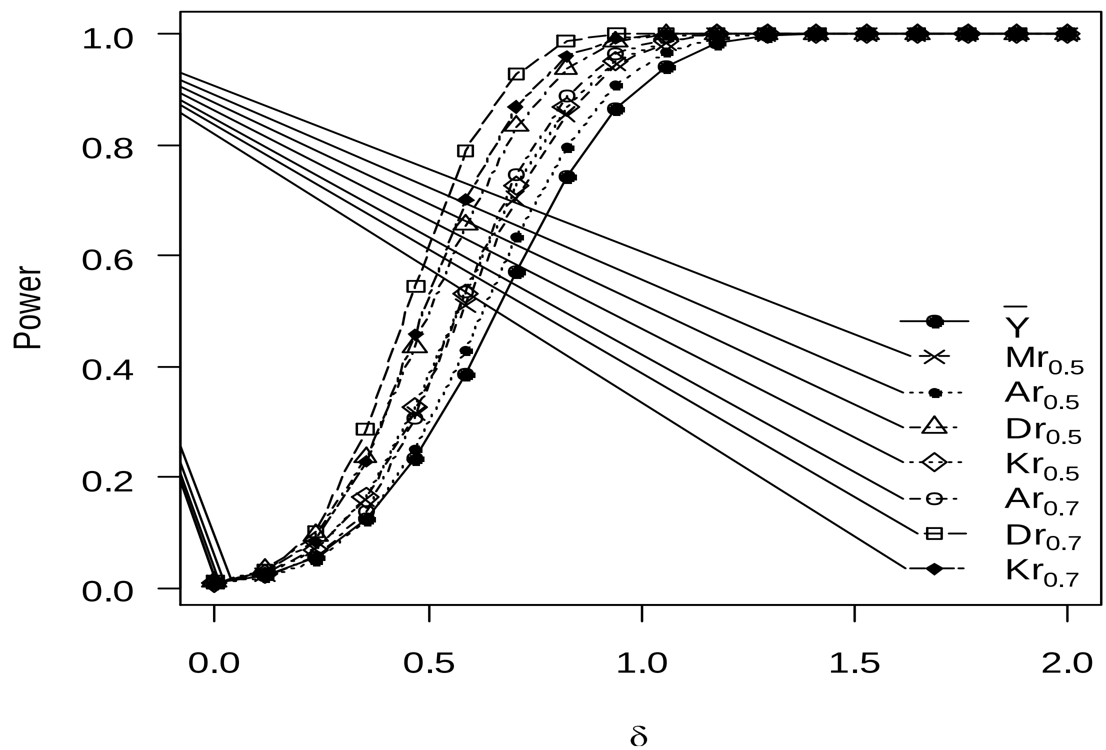

Figure 5. Power curves of $\bar{Y}, M_r, A_r, D_r$ and K_r charts for $\rho_{xz} = 0.4, \rho_{yx} = 0.5$ and $\rho_{yz} = 0.5$ and 0.7

provided ρ_{xz} is not high (Figures 1 to 6)). The M_r chart unconditionally performs better than the $\bar{Y}$ chart (Figures 1 to 10).

In brief, the proposed charts has shown better performance for the case when the auxiliary variables (X and Z) are not much correlated with each other but strongly/moderately correlated with main variable (Y). To be specific, the correlation between X and Z (that is, ρ_{xz}) should not exceed 0.50 while the correlations of X and Z with Y (that is, ρ_{yx} and ρ_{yz}) should be at least 0.30 in order to have a superior performance from the proposed charts. An important point to be noted is that these constraints on ρ_{xz} , ρ_{yx} and ρ_{yz} keeps relaxing with an increase in the value of n. This can be examined by looking at Figures 5 and 10. One can easily see the gaps among the curves in Figures 10 are more than those of in Figures 5. It means that the conditions on the correlation structures ρ_{xz} , ρ_{yx} and ρ_{yz} have been relaxed with the increase in sample size n.

A_r Chart vs. Zhang (1984); Wade and Woodall (1993)

Zhang (1984) proposed an improvement over the separate use of the usual Shewhart's $\bar{Y}$ charts. Later Wade and Woodall (1993) gave an improvement over Zhang (1984) by proposing their prediction limits. Riaz (2008a) proved superiority of the M_r chart over the control limits of Zhang (1984); Wade and Woodall (1993). We have shown from above that the charts proposed in this study perform better than the M_r chart. Hence we can indirectly draw the conclusion that the proposed A_r, D_r and K_r charts will perform better than the control limits of Zhang (1984); Wade and Woodall (1993).

ILLUSTRATIVE EXAMPLES

In this section we will illustrate the application procedure for one of the three proposed charts that is, the A_r chart. Let us consider the data in which we have two auxiliary characteristics (X and Z) and one quality characteristic of interest (Y). Suppose we have samples of size $n=10$ available on 30 time points and the sample on each time point comes from one of the following two mechanisms:

Example 1: In all the 30 samples as a whole, 90% of the observations come from $N_3\left(\begin{pmatrix}0.0\\0.0\\0.0\end{pmatrix},\begin{bmatrix}1.0 & 0.5 & 0.6\\0.5 & 1.0 & 0.1\\0.6 & 0.1 & 1.0\end{bmatrix}\right)$ and

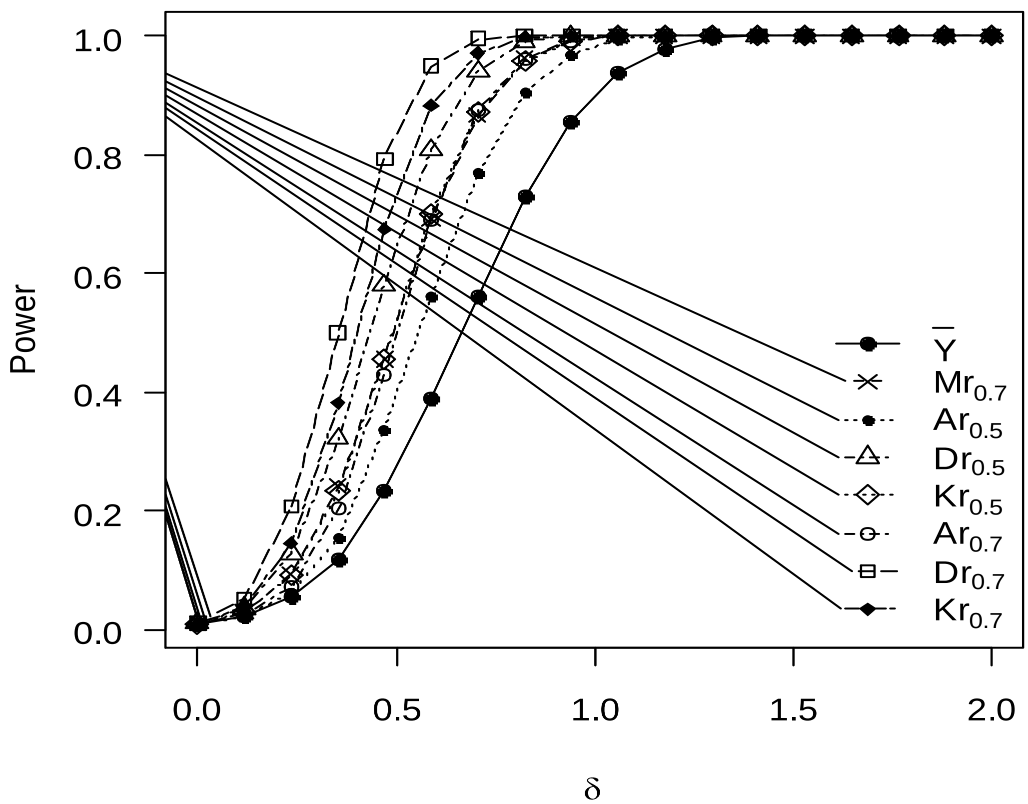

Figure 6. Power curves of $\overline{Y}, M_r, A_r, D_r$ and K_r charts for $\rho_{xz} = 0.4, \rho_{yx} = 0.7$ and $\rho_{yz} = 0.5 \text{ and } 0.7$.

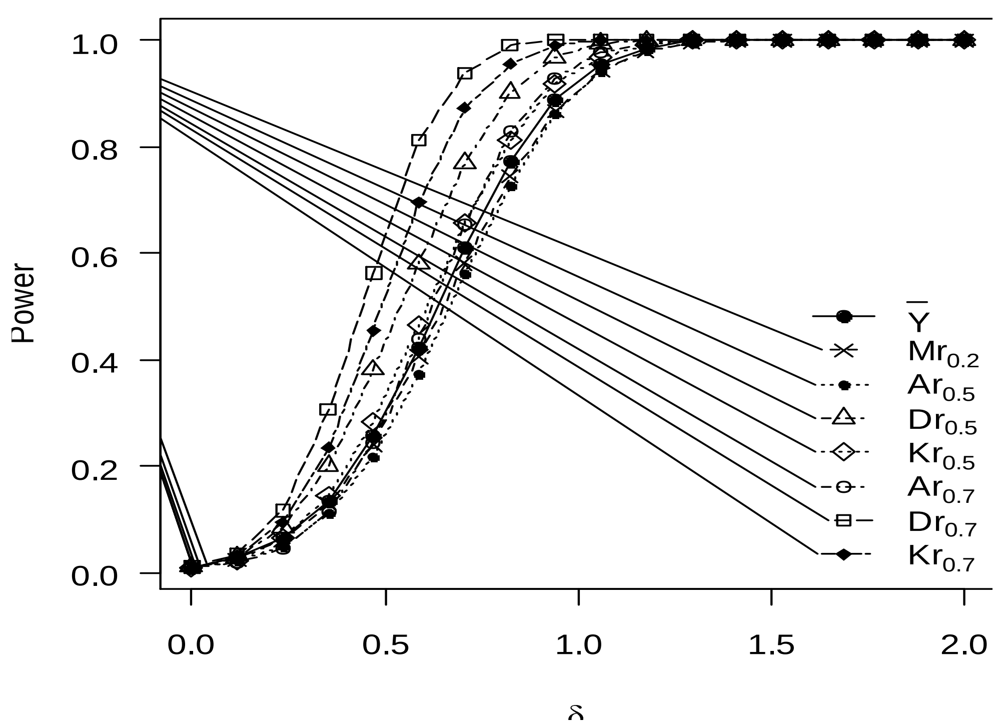

Figure 7. Power curves of $\overline{Y}, M_r, A_r, D_r$ and K_r charts for $\rho_{xz} = 0.6, \rho_{yx} = 0.2$ and $\rho_{yz} = 0.5 \text{ and } 0.7$.

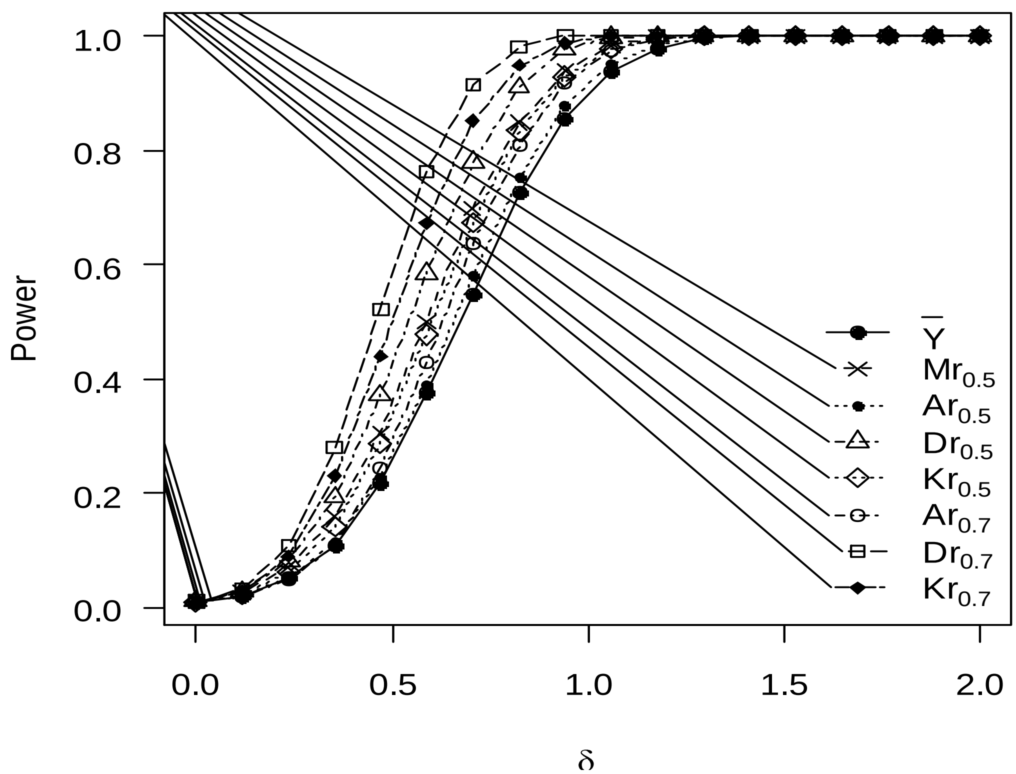

Figure 8. Power curves of $\bar{Y}, M_r, A_r, D_r$ and K_r charts for $\rho_{xz} = 0.6, \rho_{yx} = 0.5$ and $\rho_{yz} = 0.5$ and 0.7

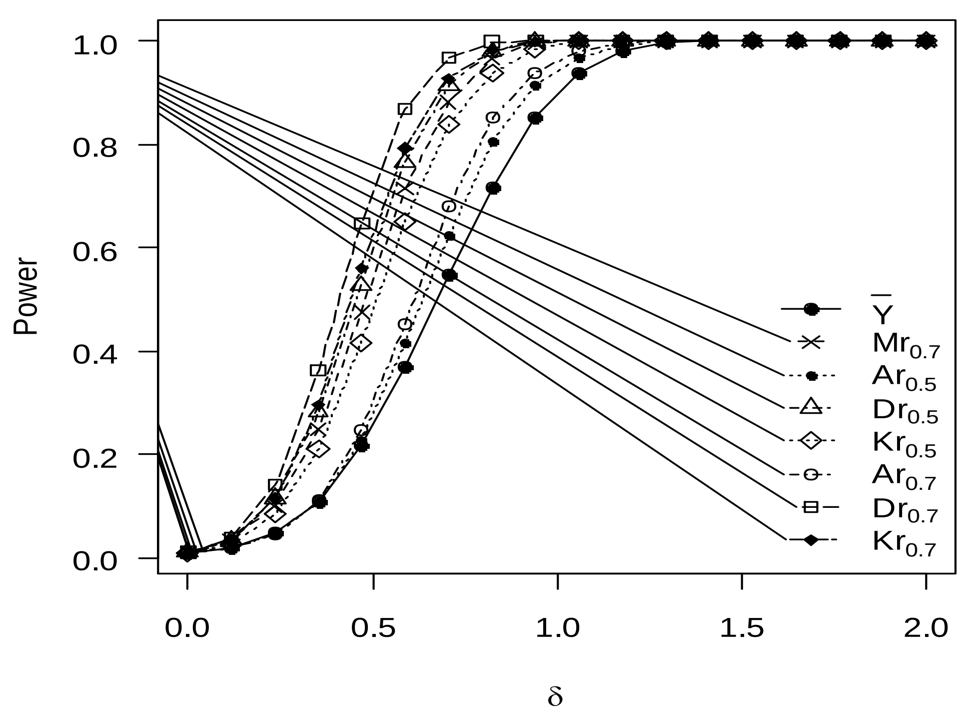

Figure 9. Power curves of $\bar{Y}, M_r, A_r, D_r$ and K_r charts for $\rho_{xz} = 0.6, \rho_{yx} = 0.7$ and $\rho_{yz} = 0.5$ and 0.7 .

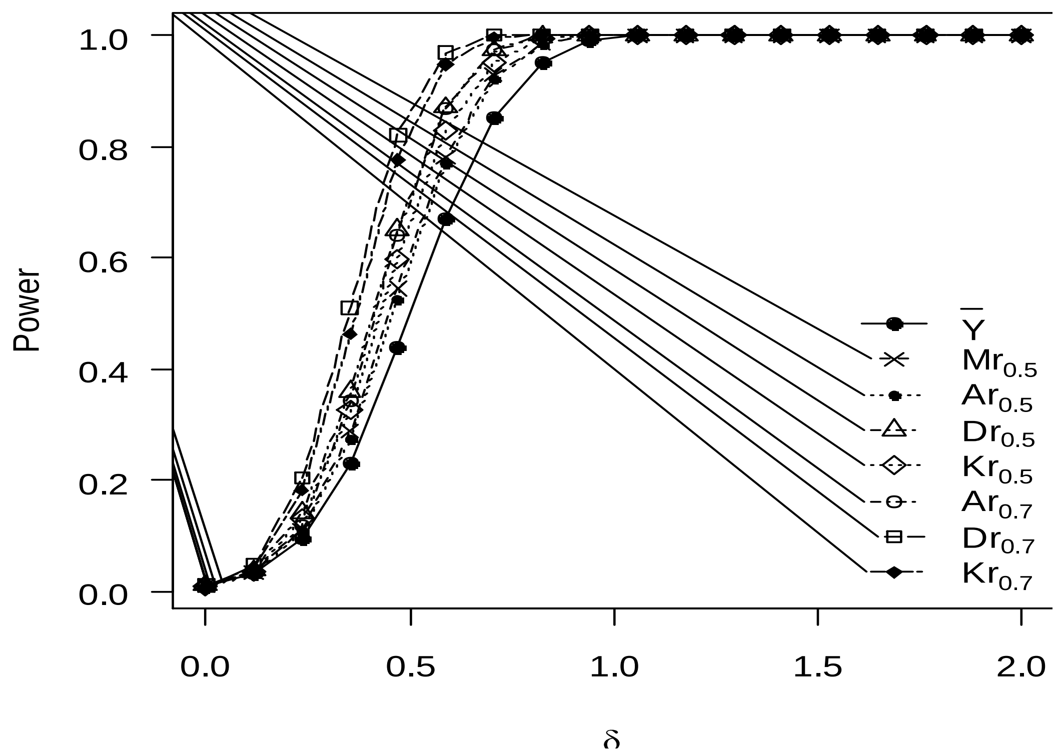

Figure 10. Power curves of $\bar{Y}, M_r, A_r, D_r$ and K_r charts for $\rho_{xz} = 0.4, \rho_{yx} = 0.5$ and $\rho_{yz} = 0.5$ and 0.7 when $n = 25$.

rest of the 10% come from $N_3\left(\begin{pmatrix}1.0\\0.0\\0.0\end{pmatrix},\begin{bmatrix}1.0 & 0.5 & 0.6\\0.5 & 1.0 & 0.1\\0.6 & 0.1 & 1.0\end{bmatrix}\right)$.

Example 2: First 20 samples are from $N_3\left(\begin{pmatrix}0.0\\0.0\\0.0\end{pmatrix},\begin{bmatrix}1.0 & 0.5 & 0.6\\0.5 & 1.0 & 0.1\\0.6 & 0.1 & 1.0\end{bmatrix}\right)$ and the last 10 samples are from $N_3\left(\begin{pmatrix}1.0\\0.0\\0.0\end{pmatrix},\begin{bmatrix}1.0 & 0.5 & 0.6\\0.5 & 1.0 & 0.1\\0.6 & 0.1 & 1.0\end{bmatrix}\right)$. We have simulated the datasets for the above mentioned two mechanisms of examples 1 and 2 so that we can have datasets with the known characteristics. The resulting datasets are used to calculate the required quantities for the proposed chart. The sample statistic A_r as defined in (1) and R_y (the sample range) are calculated for both the examples and are provided in the following Table 5.

The control limits for the proposed A_r chart are calculated using $\alpha = 0.01$ for the two examples under discussion. The two sets of limits are given as:

(For example 1)

$$LCL = \bar{A}_r + G_{0.005}\hat{\sigma}_y/\sqrt{n} = -0.535$$
$$CL = \bar{A}_r = 0.121$$
$$UCL = \bar{A}_r + G_{0.995}\hat{\sigma}_y/\sqrt{n} = 0.781$$
where $\hat{\sigma}_y = \bar{R}_y/d_2 = 2.874/3.078 = 0.934$, $G_{0.005} = -2.220$, $G_{0.995} = 2.236$, $\rho_{xz} = 0.1$, $\rho_{yx} = 0.5$, $\rho_{yz} = 0.6$, $n = 10$, $\alpha = 0.01$

(For example 2)

$$LCL = \bar{A}_r + G_{0.005}\hat{\sigma}_y/\sqrt{n} = -0.728$$
$$CL = \bar{A}_r = -0.061$$
$$UCL = \bar{A}_r + G_{0.995}\hat{\sigma}_y/\sqrt{n} = 0.611$$
where $\hat{\sigma}_y = \bar{R}_y/d_2 = 2.925/3.078 = 0.9502$, $G_{0.005} = -2.220$, $G_{0.995} = 2.236$, $\rho_{xz} = 0.1$, $\rho_{yx} = 0.5$, $\rho_{yz} = 0.6$, $n = 10$, $\alpha = 0.01$

Now values of the statistic A_r given in Table 5 are plotted against their respective control limits given above. The resulting control chart displays of the A_r chart are presented in Figures 11 and 12 for Examples 1 and 2 respectively.

According to our decision rule for the proposed A_r chart, we received out-of-control signals at time points

Table 5. Sample statistics A_r and R_y fro the simulated data sets.

Sample	Example 1		Example 2	
Number	A_r	R_y	A_r	R_y
1	-0.1953	3.4072	-0.2522	2.3536
2	0.9956	3.4576	-0.5479	3.7193
3	0.3632	3.3511	-0.0228	3.1412
4	0.3848	2.4604	0.3154	5.0663
5	0.0504	3.2620	-0.3051	2.4668
6	-0.2662	1.8949	-0.0692	2.6307
7	-0.3812	1.7086	0.2455	3.8516
8	1.4919	2.2800	0.1176	1.4283
9	-0.2341	2.2169	0.0573	2.9638
10	0.1919	3.9100	-0.2906	2.8699
11	-0.2110	2.3610	-0.0198	3.2805
12	0.0577	2.9506	-0.0261	2.6952
13	-0.1008	2.3926	0.0244	2.1876
14	-0.4243	2.4553	-0.4366	2.8410
15	-0.0554	2.8706	-0.1345	2.8980
16	0.1396	2.7692	0.1654	1.9662
17	-0.1152	2.3398	-0.0683	1.9612
18	0.6356	2.9082	0.2092	3.6413
19	0.1922	3.4963	0.1461	3.5290
20	-0.0131	2.1525	-0.3310	3.0069
21	0.1314	2.2191	1.0648	2.4157
22	0.1363	2.5550	0.7475	3.1311
23	1.0418	3.4011	0.7258	2.7343
24	-0.0354	2.9528	0.6691	2.3613
25	-0.2625	2.3061	0.7959	3.3186
26	-0.1491	2.9588	0.9752	5.4543
27	0.1727	2.3281	0.9220	2.6533
28	0.2465	4.1356	1.0267	2.7691
29	-0.5724	3.9148	1.0901	3.8828
30	0.4033	4.8015	0.9544	2.9581

#2, 8, 23 and 29 as can be seen in Figure 11. Similarly the out-of-control signals are received at time point's #21 to 30 as can be seen in Figure 12 (which is infact a permanent shift in the location parameter μ_y).

Conclusions

In this study we proposed three Shewhart type control charts, namely the A_r chart, the D_r chart and the K_r chart. These charts exploit the information of two auxiliary characteristics for the sake of an improved monitoring of the location parameter of the quality characteristic of interest. The design structure of the proposed charts has been developed in the form of the three sigma and the probability limits based on trivariate normality assumption. Comparison of the proposals has been made with the existing counterparts including the usual $\bar{Y}$ chart and the proposals of Zhang (1984); Wade and Woodall (1993); Riaz (2008a). The comparisons revealed that the D_r chart is best among all the charts investigated in this study. All the new proposals outperformed the said existing counterparts in terms of discriminatory power. This superiority of the proposed chart demands the auxiliary characteristics to be highly/moderately correlated with the main quality characteristic of interest but not having high correlations with each other.

The proposals of this article are of Shewhart type, focusing on an improved monitoring of the location parameter. However EWMA and CUSUM type structures can also be devised by using these location estimators,

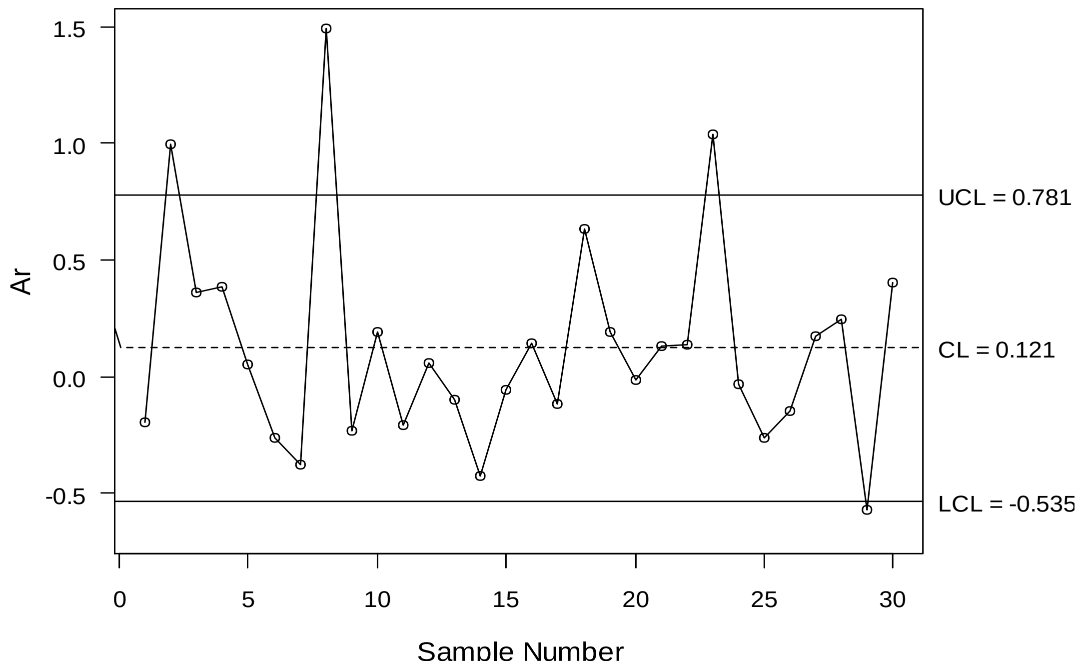

Figure 11. A_r chart for Example 1.

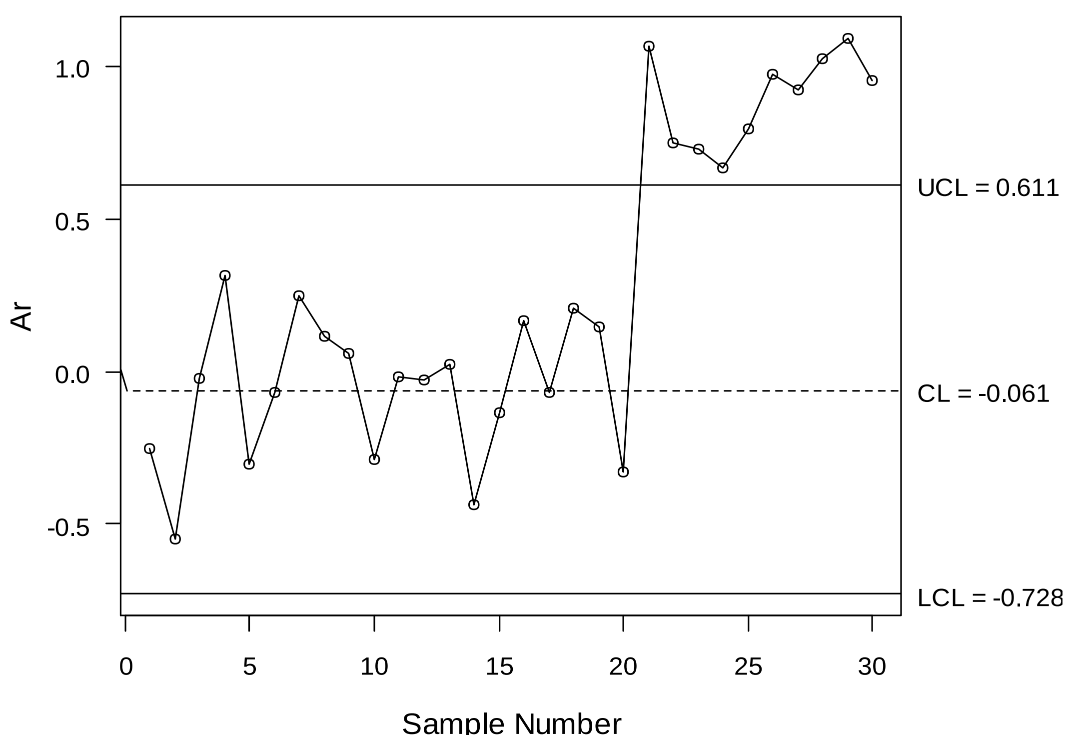

Figure 12. A_r chart for Example 2.

for efficient detection of small deviations from the parameter value or alternatively some extra sensitizing rules can also be carefully planned to be used with its control structure.

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APPENDIX

Here we will present asymptotic results of g_2 and g_3 for different choices of T

1. When $T = A_r$

From Kadilar and Cingi (2005), we have:

$$g_2 \approx 0$$
$$g_3 = \sqrt{1 - \rho_{yx}^2 - \rho_{yz}^2 + 2\rho_{yx}\rho_{yz}\rho_{xz}}$$

2. When $T = D_r$

From Abu-Dayyeh et al. (2003), we have:

$$g_2 = \frac{1}{2\sqrt{n}}\left[\frac{\sigma_x}{\mu_x} b_{yx.z} + \frac{\sigma_z}{\mu_z} b_{yz.x} - \frac{\sigma_y}{\mu_y} b_{y.xz}\right]$$
$$g_3 = \sqrt{(1 - \rho_{y.xz}^2) - \frac{1}{4n}\left(\frac{\sigma_x}{\mu_x} b_{yx.z} + \frac{\sigma_z}{\mu_z} b_{yz.x} - \frac{\sigma_y}{\mu_y} b_{y.xz}\right)^2}$$

3. When $T = K_r$

From Kadilar and Cingi (2005), we have:

$$g_2 = \frac{\sqrt{n}}{\sigma_y} E\left\{\left(-\alpha_1 \frac{\mu_y}{\mu_x} - b_1\right)(\bar{x} - \mu_x) + \left(-\alpha_2 \frac{\mu_y}{\mu_z} - b_2\right)(\bar{z} - \mu_z) + (\bar{y} - \mu_y)\right\}$$

$$g_3 = \sqrt{\begin{array}{l} 1 + (\rho_1^* + \rho_{yx})^2 + (\rho_2^* + \rho_{yz})^2 + 2(\rho_1^* + \rho_{yx})(\rho_2^* + \rho_{yz})\rho_{xz} - 2(\rho_1^* + \rho_{yx})\rho_{yx} - 2(\rho_2^* + \rho_{yz})\rho_{yz} - \\ \left(\frac{n}{\sigma_y}\right)^2 \left(E\left\{\left(-\alpha_1 \frac{\mu_y}{\mu_x} - b_1\right)(\bar{x} - \mu_x) + \left(-\alpha_2 \frac{\mu_y}{\mu_z} - b_2\right)(\bar{z} - \mu_z) + (\bar{y} - \mu_y)\right\}\right)^2 \end{array}}$$