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The inherent optimality in steady conduction of heat

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The optimality in heat conduction processes in solid materials is studied analytically. Based on the second law of thermodynamics and the definition of entropy a new thermodynamic property, temheat, is introduced. A balance equation relevant to temheat is derived by modifying the entropy balance. Principle of temheat destruction minimization is applied to several steady heat conduction problems. It is shown that for all steady heat conduction problems, the thermodynamic quantity, the rate of total temheat destruction, is always minimized. The principle of temheat destruction minimization helps to better interpret the natural heat conduction phenomena and to obtain numerical solutions in heat conduction problems.

Key words: Conduction, heat transfer, optimization, temheat; temheat destruction minimization.

INTRODUCTION

Optimization of thermal systems has been a vital focus area since the realization of the importance of sustainability in energy and flow related issues. Scientists make close observations of the natural phenomena and try to imitate highly efficient natural systems and creatures in order to develop and improve the technical and industrial systems. When we look at the natural flow systems, we observe a striking similarity among them. Therefore the governing equations to describe most of the flow processes are the same. For example, Laplace equation is used in the fields of electromagnetism, astronomy, fluid mechanics, heat and mass transfer, elasticity, electrostatics and other areas of physics and engineering to describe the behavior of electric, gravitational, and fluid potentials and steady-state heat conduction (Evans, 2010). Similarly, the Poisson equation is used to describe the problems associated with mechanics and physics such as heat and fluid flow in porous media and theory of gravitation (Slevadurai, 2000). The transient heat equation (parabolic) is not only used in conduction of heat but is also used to describe the processes in diffusion of mass, diffusion of neutrons, diffusion of vorticity, telegraphic transmission, electro-magnetic theory, hydrodynamics, and evolution of probability distributions in random processes.

On the other hand, there is a common understanding that natural phenomena are highly sustainable and optimum within the constraints they occur. One of the important discoveries that explain how nature works is the constructal law introduced by Bejan in 1996. It explains the generation of flow configuration in the nature as a universal phenomenon (Bejan, 1996a, b; Bejan and Lorente, 2010). In all real processes, there is some degree of irreversibility. There is no real process even among the natural ones that can be considered totally reversible or ideal. Then, there may be a lower limit to the irreversibility in natural systems. Investigating this minimum has become the motivation of the current work. Betrola and Cafaro (2008) discussed critically the principle of minimum entropy production which means; "a steady state has the minimum rate of entropy production with respect to other possible states with the same boundary conditions". They stated that this principle was originated in Rayleigh's least dissipation principle (Rayleigh, 1873, 1877), was provided with a proof by Onsager (1931) based on his reciprocity relations and was made famous by Prigogine (Prigogine, 1947, 1962; Glansdorff and Prigogine, 1964), who derived the property of minimum entropy production for discontinuous systems. By providing analysis of two examples (the heat conduction in a fluid at rest and the combined shear flow and heat conduction in an incompressible fluid), Betrola and Cafaro (2008) showed that the principle of minimum entropy production cannot be considered as a general variational principle, but at best an approximation method, which converges to the exact solution as the
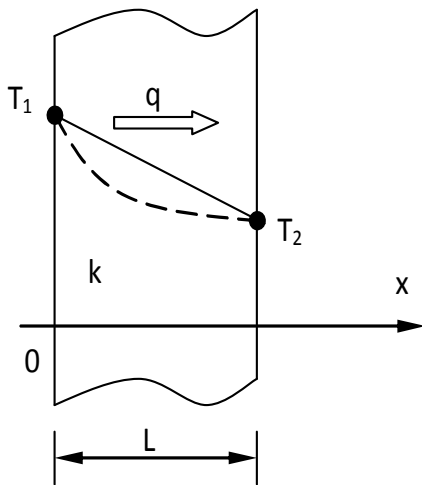
Figure 1. Heat conduction in a solid wall with fixed surface temperatures.

system converges to equilibrium. They were unable to find any special assumption on the temperature dependence on the phenomenological coefficients (such as thermal conductivity and dynamical viscosity) under which a general agreement between standard balance equations and balance equations determined by the minimum entropy production principle can be stated (Bertola and Cafaro, 2008).

Considering the Second Law of Thermodynamics and the definition of entropy as a property, Sahin (2011) introduced a new thermodynamic property and named it *"temheat"* because it carries the unit of 'temperature times heat transfer'. He showed that "the natural heat and fluid flow phenomena occur in such a way that the rate of volumetric total temheat destruction approaches a minimum value at the steady state condition." In the present work, several applications of the temheat destruction minimization principle on the conduction heat transfer problems are discussed.

ENTROPY GENERATION MINIMIZATION IN CONDUCTION OF HEAT

Let us first discuss whether the entropy generation is minimized in steady conduction heat transfer through a slab. Accordingly, consider the slab of thickness L with fixed surface temperatures, T_1 and T_2 respectively, as shown in Figure 1. In the absence of volumetric heat generation and for uniform thermal conductivity, the temperature distribution in the slab is linear and is given by

$$T = T_1 + \frac{T_2 - T_1}{L}x \tag{1}$$

The local entropy generation rate in the slab as a result of heat conduction is (Bejan, 1996):

$$s''' = \frac{k}{T^2}\left(\frac{dT}{dx}\right)^2 = \frac{k}{\left(T_1 + \frac{T_2 - T_1}{L}x\right)^2}\left(\frac{T_2 - T_1}{L}\right)^2 \tag{2}$$

The volumetric rate of total entropy generation is obtained by integrating the local entropy generation rate given in Equation (2) through the volume of the slab, that is,

$$\sigma = \int_0^L s''' A dx = \frac{kA}{L}\frac{(T_1 - T_2)^2}{T_1 T_2} \tag{3}$$

The linear temperature variation given in Equation (1) which yields the steady rate of total entropy generation given in Equation (3) may not be the optimum temperature variation, because the total entropy generation given in Equation (3) may not be a minimum. Now, let us search for the possible optimum temperature variation which minimizes the entropy generation rate. The rate of total entropy generation in the slab is given by

$$\sigma = \int_0^L s''' A dx = \int_0^L \frac{k}{T^2}\left(\frac{dT}{dx}\right)^2 A dx \tag{4}$$

According to the principles of calculus of variations, the optimum temperature variation that minimizes the total entropy generation given in Equation (4) must satisfy the Euler equation

$$\frac{\partial F}{\partial T} - \frac{d}{dx}\left(\frac{\partial F}{\partial T_x}\right) = 0 \tag{5}$$

where $F = \frac{k}{T^2}\left(\frac{dT}{dx}\right)^2$ and T_x denotes $\frac{dT}{dx}$. Carrying out the algebra in Equation (5), the required temperature distribution is found to satisfy the differential equation

$$\frac{d^2T}{dx^2} - \frac{1}{T}\left(\frac{dT}{dx}\right)^2 = 0 \tag{6}$$

which yields a temperature distribution in the transcendental form

$$T(x) = T_1\left(\frac{T_2}{T_1}\right)^{x/L} \tag{7}$$

Indeed the rate of total entropy generation becomes a minimum when the temperature distribution in the slab is given by Equation (7). However, the transcendental temperature distribution given in equation (7) corresponds to the case of heat conduction with internal heat generation (sink), according to equation (6), in the form

$$q(x) = -\frac{k}{T}\left(\frac{dT}{dx}\right)^2 \qquad (8)$$

In this case, the local entropy generation rate becomes

$$s''' = \frac{k}{T^2}\left(\frac{dT}{dx}\right)^2 + \frac{q(x)}{T} \qquad (9)$$

or

$$s''' = \frac{k}{T^2}\left(\frac{dT}{dx}\right)^2 - \frac{k}{T^2}\left(\frac{dT}{dx}\right)^2 = 0 \qquad (10)$$

that means, it is possible to minimize (eliminate) the entropy generation rate in the slab by introducing a heat sink given by Equation (8). The temperature variation in this case is given by Equation (7).

We are faced with a dilemma here. Minimization of entropy generation in steady conduction of heat in a slab required the transcendental temperature variation given in Equation (7). On the other hand we know that the steady temperature variation in a slab is linear (not transcendental) and linear temperature distribution yields a finite entropy generation as given in Equation (3) which is not in agreement with the entropy minimization problem carried out by means of calculus of variations. Then the following question arises: If the rate of the total entropy generation is not minimized in steady heat conduction problems, then is there any other thermo-dynamic quantity that is exactly minimized in such steady natural heat conduction processes? This question was the motivation for the current work. It turns out that there is a thermodynamic quantity, the rate of total temheat destruction, which is always minimized in steady heat conduction problems as discussed in the following.

PRINCIPLE OF TEMHEAT DESTRUCTION MINIMIZATION

Entropy generation during a thermal process is given by the equation

$$\delta S_{gen} = dS - \frac{\delta Q}{T} \qquad (11)$$

Multiplying both sides of Equation (1) by T^2 we obtain

$$\underbrace{T^2 \delta S_{gen}}_{\substack{\text{Temheat} \\ \text{Destruction}}} = \underbrace{T^2 dS}_{\substack{\text{Temheat} \\ \text{Change}}} - \underbrace{T \delta Q}_{\substack{\text{Temheat} \\ \text{Transfer}}} \qquad (12)$$

The first term that appears on the right hand side of equation (12) namely $T^2 dS$ is a thermodynamic property and named "*temheat*" by Sahin (2011). In other words, using the notation M for this new thermodynamic property, the *temheat change* is

$$dM \equiv T^2 dS \qquad (13)$$

The entropy change is defined as (Cengel and Boles, 2007)

$$dS = \left(\frac{\delta Q}{T}\right)_{\text{int rev}} \qquad (14)$$

Therefore the temheat change becomes

$$dM = (T \delta Q)_{\text{int rev}} \qquad (15)$$

where the subscript "int rev" is used to indicate that the temheat change is defined along an internally reversible process, exactly the same way the entropy change is defined.

The term on the left hand side of Equation (12) $T^2 \delta S_{gen}$ is named the *temheat destruction* which is always positive due to the second law of thermodynamics. Thus Equation (12) is referred as the "*temheat balance*". It was shown that the rate of total temheat destruction is always minimized in all kinds of steady flow problems (Sahin, 2011). To illustrate the application of the principle of the temheat destruction minimization in heat conduction problems we consider the rate form of the temheat balance.

Equation (12) can be written in time rate form as

$$\dot{M}_d = \frac{dM}{dt} - T\dot{Q} \qquad (16)$$

where $\dot{M}_d = T^2 \dot{S}_{gen}$ is the rate of temheat destruction. For steady state process, the term $\frac{dM}{dt}$ vanishes and we have

$$\dot{M}_d = -T\dot{Q} \qquad (17)$$

where the term $T\dot{Q}$ is the rate of temheat transfer accompanying heat transfer.

It should be noted that the temheat transfer accompanying heat transfer is also named entransy or heat transport potential capacity in recent literature (Guo et al., 2007; Chen et al., 2009). At this point, it is important to clarify the distinction between entrancy (temheat transfer accompanying heat transfer) and temheat. Entransy and temheat are two different concepts, because

1. Entransy is "a physical quantity" that describes the heat transfer ability (Guo et al., 2007), but temheat is a "thermodynamic property".
2. Entransy is a path dependent quantity, but temheat is independent from the actual path (process).
3. Entransy is defined along an actual (real) process, but temheat is a thermodynamic property similar to entropy and is defined for internally reversible (ideal) process (Equation 15).

Since the rate of temheat destruction $\dot{M}_d$ is always positive for an actual (irreversible) process then the rate of temheat transfer accompanying heat transfer $T\dot{Q}$ is always negative. Accordingly, the rate form of temheat balance for a control volume and steady state process follows

$$\dot{M}_d = \left(T\dot{Q}\right)_{in} - \left(T\dot{Q}\right)_{out} \qquad . \qquad (18)$$

In the following, the principle of temheat destruction minimization is illustrated for heat conduction problems through several examples.

Example 1: Heat conduction through a solid slab

The rate of local temheat destruction in a solid material during steady heat conduction is the product of temperature square with the local entropy generation rate. Making use of the local entropy generation rate for the one-dimensional heat conduction process (Bejan, 1996), the rate of local temheat destruction (per unit volume) becomes

$$\dot{M}_d''' = T^2 \dot{S}_{gen}''' = k\left(\frac{dT}{dx}\right)^2 \qquad (19)$$

where k is the thermal conductivity. The rate of total temheat destruction is obtained by integrating Equation (19) throughout the volume of the solid material as

$$\dot{M}_d = \iiint_V k\left(\frac{dT}{dx}\right)^2 dV \qquad (20)$$

The rate of total temheat destruction given in Equation (20) becomes a minimum for the steady heat conduction process. So the temperature distribution that minimizes the rate of total temheat destruction given in Equation (20), according to the principles of the variational calculus, must satisfy the Euler equation

$$\frac{\partial F}{\partial T} - \frac{d}{dx}\left(\frac{\partial F}{\partial T_x}\right) = 0 \qquad (21)$$

where the function F is

$$F = k\left(\frac{dT}{dx}\right)^2 \qquad (22)$$

and

$$T_x = \frac{dT}{dx}.$$

Substituting function F into Equation (21) and carrying out the algebra, it can be shown that the temperature distribution that yields a minimum rate of total temheat destruction must satisfy the differential equation

$$\frac{d^2T}{dx^2} = 0 \qquad (23)$$

which is the well known steady state heat conduction equation (Laplace equation). Although the cartesian coordinate system has been used in this example to illustrate the principle of minimum rate of total temheat destruction, it can easily be shown that the principle equally applies to the radial heat conduction problems using the cylindrical or spherical coordinate system shown subsequently. The method can also be extended to multi-dimensional heat conduction problems.

Control volume approach

Considering the infinitesimal control volume shown in Figure 2, the temheat balance for steady state heat conduction using Equation (18) is

$$\delta\dot{M}_d = \left(T\dot{Q}\right)_x - \left(T\dot{Q}\right)_{x+dx} \qquad (24)$$

or

$$\delta\dot{M}_d = TAq_x - \left(T+\frac{dT}{dx}dx\right)Aq_{x+dx} \qquad (25)$$
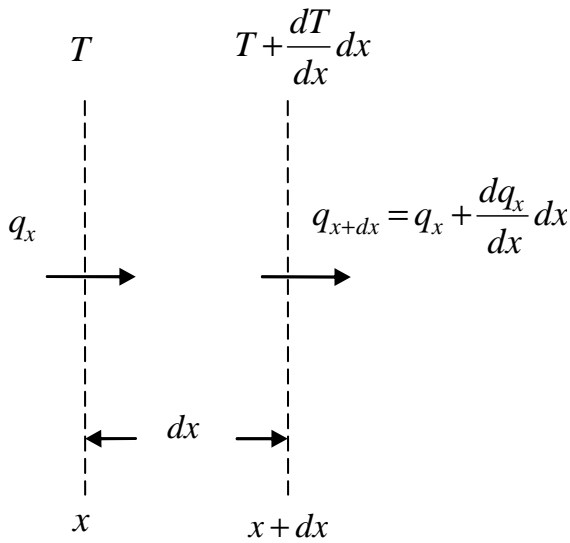
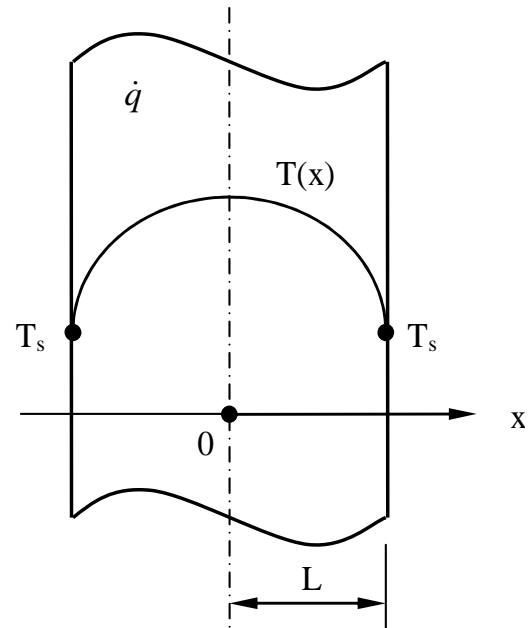**Figure 2.** Control volume in one-dimensional heat conduction.

Figure 3. Temperature distribution in a slab with internal heat generation.

Conservation of energy requires that $Aq_x = Aq_{x+dx}$, therefore

$$\delta \dot{M}_d = Aq_x \left(-\frac{dT}{dx} dx \right) \tag{26}$$

where $q_x = -k \dfrac{dT}{dx}$ and the volume of the infinitesimal control volume is $dV = Adx$. Therefore,

$$\delta \dot{M}_d = k \left(\frac{dT}{dx} \right)^2 dV \tag{27}$$

The total rate of temheat destruction is obtained by integrating Equation (27) over the volume of the solid as

$$\dot{M}_d = \iiint k \left(\frac{dT}{dx} \right)^2 dV \tag{28}$$

which is identical to the rate of total temheat destruction given in Equation (20).

Example 2: Heat conduction with internal energy generation

Consider the problem of steady heat conduction through a slab with uniform internal heat generation $\dot{q}$ as shown in Figure 3. The internal heat generation plays a role of a constraint in the principle of temheat destruction minimization. The local rate of temheat destruction is

$$\dot{M}_d''' = T^2 \dot{S}_{gen}''' = k \left(\frac{dT}{dx} \right)^2 \tag{29}$$

The rate of total temheat destruction is obtained by integrating Equation (29) throughout the volume, that is,

$$\dot{M}_d = \iiint_V \left[k \left(\frac{dT}{dx} \right)^2 \right] dV \tag{30}$$

The total volumetric rate of internal heat generation is constant for steady-state heat conduction. Therefore the rate of total temheat addition that must be used as a constraint for the minimization problem is

$$\iiint_V \dot{q} T dV = cont. \tag{31}$$

In this case, the Euler equation becomes

$$\frac{\partial H}{\partial T} - \frac{d}{dx} \left(\frac{\partial H}{\partial T_x} \right) = 0 \tag{32}$$

Where

$$H = k\left(\frac{dT}{dx}\right)^2 + \lambda \dot{q}T \qquad (33)$$

in which the constant λ is a Lagrange multiplier. It can be shown that Equation (32) yields the Poisson equation

$$\frac{d^2T}{dx^2} - \frac{\lambda}{2k}\dot{q} = 0 \qquad (34)$$

The constant Lagrange multiplier λ in Equation (34) can be determined from the boundary condition, that is, the energy balance on the boundary. For symmetrical heat conduction problem shown in Figure 3, all the heat that is generated within the half of the volume of the slab crosses through one of the surfaces by conduction, that is,

$$\dot{q}L = -k\frac{dT}{dx}\bigg|_{x=L} \qquad (35)$$

Integrating Equation (34) once and applying the boundary condition (Equation 35), the value of the Lagrange multiplier λ is obtained to be -2. Substituting λ = -2 in Equation (34), the differential equation for the heat conduction with uniform internal heat generation is obtained:

$$\frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = 0 \qquad (36)$$

The analysis can easily be extended to steady multi-dimensional heat conduction with internal heat generation.

Control volume approach

Considering the infinitesimal control volume shown in Figure 2, the temheat balance for steady state heat conduction (Equation 18) with the presence of the uniform internal heat generation is

$$\delta \dot{M}_d = TAq_x - \left(T + \frac{dT}{dx}dx\right)Aq_{x+dx} + \dot{q}Adx \quad (37)$$

According to energy balance $Aq_x + A\dot{q}dx = Aq_{x+dx}$, therefore

$$\delta \dot{M}_d = TAq_x - \left(T + \frac{dT}{dx}dx\right)(Aq_x + A\dot{q}dx) + \dot{q}Adx \quad (38)$$

or neglecting (dx)^2 term

$$\delta \dot{M}_d = \left[-\frac{dT}{dx}q_x\right]dV \qquad (39)$$

where $q_x = -k\frac{dT}{dx}$ and

$$\delta \dot{M}_d = \left[k\left(\frac{dT}{dx}\right)^2\right]dV \qquad (40)$$

The total rate of temheat destruction is obtained by integrating Equation (40) over the volume of the solid as

$$\dot{M}_d = \iiint\left[k\left(\frac{dT}{dx}\right)^2\right]dV \qquad (41)$$

which is identical to the rate of total temheat destruction given in Equation (30). Making use of the constraint given in Equation (31), we arrive at the same differential equation for the heat conduction (Equation 36).

Example 3: Heat conduction in radial systems

In this case, consider the radial steady heat conduction in the control volume shown in Figure 4. The rate of local temheat destruction is given by

$$\dot{M}_d''' = T^2\dot{S}_{gen}''' = k\left(\frac{dT}{dr}\right)^2 \qquad (42)$$

Integrating Equation (42) over the volume of the cylindrical solid material of unit length, the rate of total temheat destruction is obtained as

$$\dot{M}_d = \int_0^R 2\pi kr\left(\frac{dT}{dr}\right)^2 dr \qquad (43)$$

The rate of total temheat destruction is minimized during the heat conduction process. Consequently, the Euler equation must be satisfied, that is,

$$\frac{\partial F}{\partial T} - \frac{d}{dr}\left(\frac{\partial F}{\partial T_r}\right) = 0 \qquad (44)$$

where the function F is

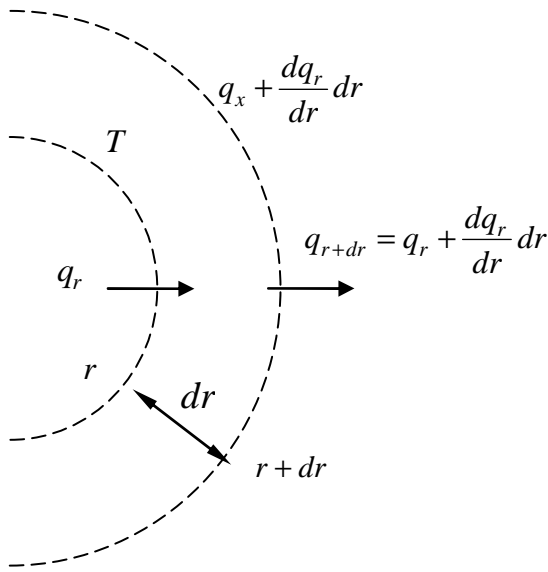$$F = kr\left(\frac{dT}{dr}\right)^2 \qquad (45)$$

Figure 4. Control volume in one-dimensional radial heat conduction.

and $T_r = \dfrac{dT}{dr}$. Substituting Equation (45) into Equation (44) we obtain the differential equation for radial one dimensional heat conduction

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0 \qquad (46)$$

Control volume approach

Considering the control volume shown in Figure 4 and applying the temheat balance

$$\delta \dot{M}_d = \left(T\dot{Q}\right)_r - \left(T\dot{Q}\right)_{r+dr} \qquad (47)$$

or

$$\delta \dot{M}_d = TA_r q_r - \left(T + \frac{dT}{dr}dr\right)A_{r+dr}q_{r+dr} \qquad (48)$$

Conservation of energy requires that $A_r q_r = A_{r+dr}q_{r+dr}$, therefore

$$\delta \dot{M}_d = -A_{r+dr}\left(\frac{dT}{dr}dr\right)q_{r+dr} \qquad (49)$$

or

$$\delta \dot{M}_d = -2\pi(r+dr)\left(\frac{dT}{dr}dr\right)\left(q_r + \frac{dq_r}{dr}dr\right) \qquad (50)$$

where $q_r = -k\dfrac{dT}{dr}$ and the length of the cylindrical material is considered to be unity. Therefore,

$$\delta \dot{M}_d = 2\pi k r\left(\frac{dT}{dr}\right)^2 dr \qquad (51)$$

The total rate of temheat destruction is obtained by integrating Equation (51) over the volume of the solid as

$$\dot{M}_d = \int_0^R 2\pi k r\left(\frac{dT}{dr}\right)^2 dr \qquad (52)$$

which is identical to the rate of total temheat destruction given in Equation (43).

Example 4: Heat conduction in radial systems with internal heat generation

In the case of radial heat conduction with internal heat generation $\dot{q}$, the temheat transfer due to the internal heat generation is considered to be a constraint in the optimization problem. Considering the infinitesimal control volume shown in Figure 4, the equations for the rate of local and total temheat destructions are the same as given in Equations (42) and (43), respectively. On the other hand, the total rate of temheat transfer due to the internal heat generation is

$$\int_0^R 2\pi r\dot{q}Tdr = const. \qquad (53)$$

which is the constraint for the optimization problem. Therefore, the functional (that is, the rate of total temheat destruction with constraint) to be minimized is in the form

$$I = 2\pi\int_0^R\left[kr\left(\frac{dT}{dr}\right)^2 + \lambda r\dot{q}T\right]dr \qquad (54)$$

For this optimization problem the Euler equation is

$$\frac{\partial F}{\partial T} - \frac{d}{dr}\left(\frac{\partial F}{\partial T_r}\right) = 0 \qquad (55)$$

where the function F is

$$F = kr\left(\frac{dT}{dr}\right)^2 + \lambda r\dot{q}T \qquad (56)$$

Substituting Equation (56) into Equation (55), the necessary

condition for the optimization is obtained as

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) - \frac{\lambda}{2k}\dot{q}r = 0 \qquad (57)$$

For a solid cylindrical bar of outer radius R with uniform heat generation all the heat generated within the volume leaves through the outer surface of the bar, that is, on the basis of unit length of the bar

$$\dot{q}\pi R^2 = -k(2\pi R)\frac{dT}{dr}\bigg|_{r=R} \qquad (58)$$

Integrating Equation (57) once and substituting it into Equation (58) the constant Lagrange multiplier is obtained to be - 2. Therefore Equation (57) becomes,

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{\dot{q}r}{k} = 0 \qquad (59)$$

which is the steady heat conduction equation for the radial one-dimensional system with uniform internal heat generation.

Control volume approach

Applying the temheat balance to the control volume shown in Figure 4, in the presence of uniform heat generation we have

$$\delta\dot{M}_d = (T\dot{Q})_r - (T\dot{Q})_{r+dr} + \dot{q}TdV \qquad (60)$$

or

$$\delta\dot{M}_d = TA_rq_r - \left(T + \frac{dT}{dr}dr\right)A_{r+dr}q_{r+dr} + \dot{q}TdV \quad (61)$$

Conservation of energy requires that $A_rq_r + \dot{q}dV = A_{r+dr}q_{r+dr}$, where $dV = 2\pi rdr$. Therefore Equation (61) becomes

$$\delta\dot{M}_d = TA_rq_r - \left(T + \frac{dT}{dr}dr\right)(A_rq_r + \dot{q}dV) + \dot{q}TdV \qquad (62)$$

or neglecting the (dr)^2 term we have

$$\delta\dot{M}_d = -2\pi r\left(\frac{dT}{dr}dr\right)(q_r) \qquad (63)$$

where $q_r = -k\frac{dT}{dr}$ and therefore,

$$\delta\dot{M}_d = 2\pi kr\left(\frac{dT}{dr}\right)^2 dr . \qquad (64)$$

The rate of total temheat destruction is obtained by integrating equation (64) over the volume of the solid as

$$\dot{M}_d = \int_0^R 2\pi kr\left(\frac{dT}{dr}\right)^2 dr \qquad (65)$$

which is identical to the rate of total temheat destruction given in Equation (43). Employing the constraint due to the internal heat generation, Equation (53), we reach to the same result, that is, Equation (59).

Although one-dimensional radial heat conduction system is considered in the present example, it can easily be shown that the analysis can be extended to multi dimensional cylindrical systems. It can also be shown that the analysis is valid for the radial heat conduction using formulation in spherical coordinates.

Example 5: Heat conduction through extended surfaces

In this case, we consider one-dimensional heat conduction through a fin of uniform cross-section as shown in Figure 5. The rate of local temheat destruction is given by

$$\dot{M}_d''' = T^2\dot{S}_{gen}''' = k\left(\frac{dT}{dx}\right)^2 \qquad (66)$$

The rate of total temheat destruction is obtained by integrating Equation (66) throughout the volume, that is,

$$\dot{M}_d = \int\left[k\left(\frac{dT}{dx}\right)^2\right]Adx \qquad (67)$$

where A is the cross sectional area of the extended surface. The rate of total temheat transfer associated with the steady convection heat transfer is

$$\int q_c(T - T_\infty)pdx = const \qquad (68)$$

where p is the perimeter of the outer surface of the extended surface and $q_c = h(T - T_\infty)$. Equation (68) can be used as a constraint in the optimization problem. In this case, the Euler equation becomes,

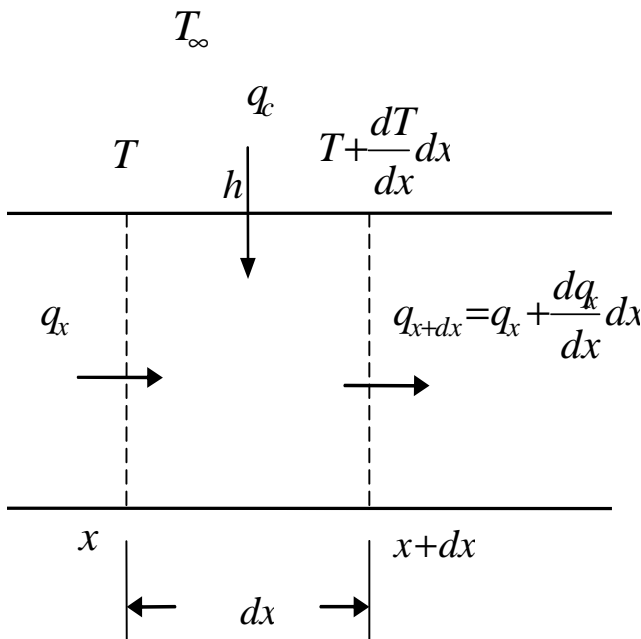$$\frac{\partial H}{\partial T} - \frac{d}{dx}\left(\frac{\partial H}{\partial T_x}\right) = 0 \qquad (69)$$

Figure 5. Conduction heat transfer through an extended surface.

where

$$H = kA\left(\frac{dT}{dx}\right)^2 + \lambda hp(T - T_\infty)^2 \qquad (70)$$

in which the constant λ is a Lagrange multiplier. Substituting Equation (70) into Equation (69), it can be shown that Equation (69) yields the Poisson equation

$$\frac{d^2T}{dx^2} - \frac{\lambda hp}{kA}(T - T_\infty) = 0 \qquad\qquad (71)$$

The constant Lagrange multiplier λ in Equation (71) can be determined from the boundary condition, that is, the energy balance at the base of the extended surface, that is

$$-kA\frac{dT}{dx}\bigg|_{x=0} = \int h(T - T_\infty)p\,dx \qquad (72)$$

Solving Equation (71) and applying the boundary condition (Equation 72) the value of the Lagrange multiplier λ is obtained to be + 1. It should be noted that the value of the Lagrange multiplier is independent from the selected boundary condition at the tip of the extended surface. Substituting λ = + 1 in Equation (71), the differential equation for the steady one-dimensional heat conduction is obtained

$$\frac{d^2T}{dx^2} - \frac{hp}{kA}(T - T_\infty) = 0 \qquad (73)$$

Control volume approach

Considering the infinitesimal control volume shown in Figure 5, the temheat balance for steady state heat conduction (Equation 18) with the presence of the uniform internal heat generation is

$$\delta\dot{M}_d = TAq_x - \left(T + \frac{dT}{dx}dx\right)Aq_{x+dx} + TA_sq_c \qquad (74)$$

where $A_s = pdx$ is the outer surface area of the extended surface. Applying the energy balance we have $Aq_x + A_sq_c = Aq_{x+dx}$, therefore

$$\delta\dot{M}_d = TAq_x - \left(T + \frac{dT}{dx}dx\right)(Aq_x + A_sq_c) + TA_sq_c \qquad (75)$$

or neglecting (dx)² terms we have

$$\delta\dot{M}_d = -A\frac{dT}{dx}q_x dx \qquad (76)$$

Since $q_x = -k\frac{dT}{dx}$, equation (76) becomes

$$\delta\dot{M}_d = \left[kA\left(\frac{dT}{dx}\right)^2\right]dx \qquad . \qquad (77)$$

The total rate of temheat destruction is obtained by integrating Equation (77) over the length of the extended surface as

$$\dot{M}_d = \int kA\left(\frac{dT}{dx}\right)^2 dx \qquad (78)$$

which is identical to the rate of total temheat destruction given in Equation (67). Making use of the constraint given in Equation (68) we arrive at the same differential equation for the steady heat conduction in the extended surface (Equation 73).

The aforementioned examples indicate that the rate of total temheat destruction is minimized during the steady heat conduction processes. The same analogy can be used to extend this principle to other flow problems in various areas of science including fluid mechanics, electromagnetism, astronomy, mass transfer, elasticity, and electrostatics.

Conclusions

The inherent optimality in flow systems and particularly in heat conduction has been investigated. A new thermodynamic property, temheat, has been introduced. The second law of thermodynamics was extended to present the temheat balance that included the temheat destruction. The principle of temheat destruction minimization was discussed. It is shown through several examples that the rate of total temheat destruction is minimized in steady heat conduction problems. The principle of temheat destruction minimization is a useful technique to better describe the heat conduction processes.

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Nomenclature: A, Area (m^2); **H,** heat transfer coefficient (W/m^2K); **K,** thermal conductivity (W/mK); **L,** length (m); **M,** temheat (KJ); **P,** perimeter (m); **q,** heat flux (W/m^2); $\dot{q}$, rate of internal heat; generation (W/m^3); **Q,** heat transfers (J); **R,** radial coordinate (m); **R,** radius (m); **S,** entropy (J/K); **S_{gen},** entropy generation (J/K); **t,** time (s); **T,** temperature (K); **V,** volume (m^3); **x,** axial coordinate (m);

Subscripts: c, convection; **D,** destruction; **gen,** generation; **s,** surface; ∞, ambient.

REFERENCES

Bejan A (1996a). Constructal-theory network of conducting paths for cooling a heat generating volume. Int. J. Heat Mass Transfer., 40: 799–816.

Bejan A (1996b). Entropy Generation Minimization. New York: CRC Press.

Bejan A, Lorente S (2010). The constructal law of design and evolution in nature. Phil. Trans. R. Soc. B. 365: 1335-1347.

Bertola V, Cafaro E (2008). A critical analysis of the minimum entropy production theorem and its application to heat and fluid flow. Int. J. Heat Mass Transfer., 51: 1907–1912.

Cengel YA, Boles MA (2007). Thermodynamics: An Engineering Approach. 6th ed. New York: McGraw Hill.

Chen Q, Wang M, Pan N, Guo Z-Y (2009). Optimization principles for convective heat transfer. Energy. 34: 1199–1206.

Evans L (2010). Partial Differential Equations, 2nd ed. New York: American Mathematical Society.

Glansdorff P, Prigogine I (1964). On a general evolution criterion in macroscopic phys. Phys., 30: 351–374.

Guo Z-Y, Zhu H-Y, Liang X-G (2007). Entransy—A physical quantity describing heat transfer ability. Int. J. Heat Mass Transfer., 50: 2545–2556.

Onsager L (1931). Reciprocal relations in irreversible processes I. Phys. Rev., 37: 405–426.

Prigogine I (1947). Thermodynamic study of irreversible phenomena. Paris: Desoer Lie`ge Dunod.

Prigogine I (1962). Introduction to Non-Equilibrium Thermodynamics. New York: Wiley-Interscience.

Rayleigh JS (1873). Some general theorems relating to vibrations. Proc. Math. Soc. London. 4: 357–368.

Rayleigh JS (1877). Theory of Sound. London: MacMillan. P.1.

Sahin AZ (2011). The Natural Optimality in Heat and Fluid Flow Phenomena. Int. J. Exergy, 9(3): 346-354.

Slevadurai APS (2000). Partial Differential Equations in Mechanics 1. New York: Springer Verlag.