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On conformally Osserman Lorentzian manifolds satisfying a certain condition on the Ricci tensor

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Let \((M^n, g)\), \(n \geq 4\), be an \(n\)-dimensional homogeneous Lorentzian manifold of which the Jacobi operator associated to the Weyl conformal curvature tensor has constant eigenvalues on the bundle of unit timelike (spacelike) tangent vectors (known as conformally Osserman Lorentzian manifolds). Then \(M^n\) is a conformally Osserman Lorentzian manifold if and only if \(M^n\) is a conformally flat manifold, (Blazic, 2005). In this paper, by utilizing this equivalence and the similar arguments in Erdogan and Ikawa (1995) and Sekigawa and Takagi (1971), we classify locally conformally flat homogeneous Lorentzian manifolds and, equivalently, as well as conformally Osserman Lorentzian manifolds which satisfy a condition on the Ricci tensor.

**Key words:** Conformally manifold, Weyl conformal tensor, Conformally Osserman manifold, Conformal Jacobi operator.

INTRODUCTION

Let \((M^n, g)\) be an \(n\)-dimensional semi-Riemannian manifold \(R(X,Y)Z=[\nabla_X, \nabla_Y]Z-\nabla_{[X,Y]}Z\) its curvature operator and \(R(X,Y,Z,W)=g(R(X,Y)Z,W)\) its curvature tensor.

Let \(\{e_i\}\) be a local orthonormal frame for the tangent bundle, \(g(e_i, e_j) = \delta_{i j}, \quad \epsilon_i = \pm 1\). The Ricci curvature and the scalar curvature are defined by \(\text{Ric}(X,Y) = \sum_{j} e_j g(R(e_j,e_i,e_j), e_i)\) and \(\tau = \sum_{i} e_i \text{Ric}(e_i, e_i)\).

The Weyl conformal curvature tensor is defined by

\[
W(X,Y) = R(X,Y) + \frac{\tau}{(n-1)(n-2)} \left\{ g(Y,\ldots)X - g(X,\ldots)Y \right\} + \frac{1}{n-2} \left\{ g(QY,\ldots)X - g(QX,\ldots)Y \right\}, \quad n > 2,
\]

where \(Q\) is the symmetric endomorphism which corresponds to the Ricci tensor \(\text{Ric}\) that is \(\text{Ric}(X,Y) = g(QX,Y)\).

The Jacobi operator and the Weyl (conformal) Jacobi operator are defined by \(J_{R_{\epsilon}}(X) = Y \rightarrow R_{\epsilon}(Y, X)X\) and \(J_{W_{\epsilon}}(X) = Y \rightarrow W_{\epsilon}(Y, X)X\). The study of the Jacobi operator is a central topic in Semi-Riemannian geometry. The geodesic deviation is measured by this part of the curvature tensor and because of its fundamental role in the Jacobi equation, many geometric properties can be derived from the knowledge of the Jacobi operators.
The Weyl curvature tensor denotes \( J_{W_0} = J_{W_0} \). The Weyl curvature tensor \( W \) as a conformal invariant is important in the understanding of Conformal Semi-Riemannian Geometry. We say \((M^n, g)\) is **conformally Osserman** if the eigenvalues of the symmetric Weyl Jacobi operator \( J_{W_0}(X) \) are independent on the choice of the unit timelike (spacelike) direction \( X \).

It is well known that an \( n \)-dimensional semi-Riemannian manifold, \( n \geq 4 \), is conformally flat if and only if its Weyl operator \( W \) vanishes. It was proved that if \( M^n \) is conformally Osserman Lorentzian manifold then it is conformally flat (Blazic, 2005). In this case, \( W = 0 \) implies that the curvature operator \( R \) of \( M^n \) satisfies

\[
R(X, Y) = \frac{1}{n-2} \left\{ g(QX, Y) - g(QY, X) \right\} - \frac{r}{(n-1)(n-2)} \left\{ g(Y, X) - g(X, Y) \right\}
\]  

(2)

for any tangent vector fields \( X \) and \( Y \). If \( X \wedge Y \) denotes the endomorphism defined by \((X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y\), then we have

\[
R(X, Y) = \frac{1}{n-2} \left\{ (X \wedge Y) + X \wedge QY \right\} - \frac{TrQ}{(n-1)(n-2)} X \wedge Y.
\]

(3)

In a recent paper, Honda and Tsukada (2007) classified three-dimensional conformally flat homogeneous Lorentzian manifolds depending on the form of the Ricci operators. In their proof, they use the same method as in Hahn (1984) which is relating the principal curvatures of an isoparametric hypersurface with the eigenvalues of the shape operator \( A \). In the present work, assuming that \( Q \) is diagonalizable, we aim to prove a local classification theorem for \( n \)-dimensional conformally flat Lorentzian manifolds and equivalently also for the conformally Osserman Lorentzian manifolds which satisfy the condition;

\[
R(X, Y)Q = 0
\]

(4)

for any tangent vectors \( X \) and \( Y \).

**THEOREM**

We prove the following:

1) An Osserman Lorentzian manifold of constant curvature.
2) A locally product manifold of an Osserman Lorentzian manifold of constant curvature \( k \) and a Riemannian Osserman manifold of constant curvature \(-k\), or a Riemannian Osserman manifold of constant curvature \( k \) and a Lorentzian Osserman manifold of constant curvature \(-k\); that is, \( M^n(k) \times M^{m-n}(-k) \) or \( M^n(k) \times M^{m-n}(-k) \), where \( k \neq 0 \).
3) A locally product space of a Lorentzian Osserman manifold of constant curvature \( k \) and a 1-dimensional Riemannian Osserman manifold, or a Riemannian Osserman manifold of constant curvature \( k \) and a 1-dimensional Lorentzian Osserman manifold; that is, \( M^{n-1}(k) \times M^1 \) or \( M^{n-1}(k) \times M^1 \), where \( k \neq 0 \).

Now (Equation 3) and the condition (Equation 4) imply

\[
\frac{1}{n-2} \left( g(QY, Z)X - g(QX, Z)Y \right) + g(QY, Z)X - g(QX, Z)Y
\]

\[
- \frac{TrQ}{(n-1)(n-2)} \left( g(Y, Z)X - g(X, Z)Y \right)
\]

\[
- \frac{1}{n-2} \left( g(QY, Z)QX - g(QX, Z)QY \right)
\]

\[
+ g(Y, Z)QX - g(X, Z)QY
\]

\[
+ \frac{TrQ}{(n-1)(n-2)} \left( g(Y, Z)QX - g(X, Z)QY \right)
\]

\[
= 0
\]

(5)

Since \( g \) is the Lorentzian metric and \( Q \) is a symmetric endomorphism of the tangent space \( T_p M^n \) at point \( p \), with respect to suitably chosen frames for \( T_p M^n \), \( Q_p \) has one of the following four forms (O'Neill, 1983):

\[
Q_p = \begin{pmatrix}
\lambda_i & 0 \\
0 & \lambda_n
\end{pmatrix}
\]

(i)

1988; Erdog and Ikawa, 1995; Garcia-Rio et al., 1997; Garcia et al., 2002). Since for each vector \( X \), the Jacobi operator is a self-adjoint operator, the study of its eigenvalues is of great interest. In the Lorentzian case especially, they play a fundamental role in the construction of mathematical models in general relativity. On the other hand, the eigenvalues of the Jacobi operator depend both on a point \( p \in M^n \) and a direction \( X \in T_p M^n \). The conformally Jacobi operator is conformally invariant, namely, if \( h = e^\varphi g \) is a conformally equivalent to the metric \( g \), then

\[
J_{W_0} = J_{W_0}.
\]
Let \( (M^n, g) \) be a Semi-Riemannian manifold of dimension \( n \geq 3 \) and let \( g \) be a non-degenerate symmetric metric on \( M^n \) of signature \( (p, q) \). For \( \epsilon = \pm \) and \( p \in M^n \), let \( S^\epsilon_p := \{ X \in T_p M^n : g(X, X) = \epsilon \} \) be the set of all unit spacelike \((\epsilon = +)\) and timelike \((\epsilon = -)\) tangent vectors at \( p \in M^n \). If \( X \in S^\epsilon_p \), \( T_X(S^\epsilon_p) = \{ Y \in T_X M^n : g(X, Y) = 0 \} \). If \( X \) is not a null vector, and \( X \in T_X M^n, p \in M^n \), then one can split a tangent space \( T_X M^n = X \oplus T_X^\perp \). The endomorphism \( Q_x \) of \( T_x(S^\epsilon_p) \) is diagonalizable if \( X^\perp \) has the definite induced metric, in other case it can be also undiagonalized. Therefore, we only deal with the Lorentzian case, namely \( Q \) is always diagonalizable. In this case, \( Q \) has at most two real eigenvalues (Honda, 2003). It is known that any local Lorentzian Osserman manifold has constant sectional curvature (Blazic, 1997). One says that \( M^n \) is a conformal space form if \( W = \lambda R \). We easily see that this implies \( \lambda = 0 \) so \( M^n \) is conformally flat, where \( R \) is defined by \( R(X, Y)Z = g(Y, Z)X - g(X, Z)Y \). Then, since \( J_{W(X)} X = 0 \) and \( J_{W(X)}(X) \) is self-adjoint, \( J_{W^r} \) preserves the subspace \( X^\perp \). So we may define the reduced conformal Jacobi operator by letting \( \lambda \), and only if \( J_{W^r} \) has constant eigenvalues on \( S^\epsilon_p \).

Now we express a theorem and its proof will follow from the purely algebraic lemma which has been given in Blazic, Bokan and Gilkey (1997):

**Theorem 2**

Let \( V \) be a Lorentzian vector space of the signature \((1, n-1)\) and let \( R \) be an algebraic curvature tensor and \( W \) the corresponding Weyl operator. The following conditions are equivalent:

1) \( R \) is a conformally Osserman curvature tensor,
2) \( W = \lambda R, \lambda = const. \),
3) \( W = 0 \).

If a connected manifold \( M^r \) is a locally spacelike Osserman or a locally timelike Osserman manifold, then \( M^r \) has constant curvature \( c \) for some \( c \) independent of the point of the manifold. So if \( n \geq 4 \), then Theorem 2 implies that the following conditions are equivalent:

(1) \( M^r \) is a conformally Osserman manifold,
(2) \( M^r \) is a conformally flat manifold.

**Corollary 2**

A Lorentzian manifold \( M^r \) is timelike conformally Osserman if and only if it is spacelike conformally Osserman.

**PROOF OF THEOREM**

First we assume that \( \dim M^r > 3 \). The proof of the theorem will be devided into four parts, according to the four possible forms of \( Q \).

**Case (i)**

Suppose that \( Q \) is of the Form (i). We know that \( Q \) has at most two real eigenvalues (Honda, 2003). Now by

\[
Q_p = \begin{pmatrix}
\lambda & \mu & 0 & \ldots & 0 \\
-\mu & \lambda & 0 & \ldots & 0 \\
0 & 0 & \lambda_1 & 0 & \ldots \\
0 & 0 & 0 & \ddots & \ddots \\
0 & 0 & 0 & 0 & \lambda_k
\end{pmatrix}
\]
putting \( X = e_i, \ Y = Z = e_j \) in Equation (5), at each point, it follows that

\[
(\lambda_j - \lambda_i) (\lambda^*_j + \lambda^*_i - \frac{\text{Tr}Q}{n-1}) = 0
\]  

(6)

for any \( j, \ 2 \leq j \leq n \). If for all \( 2 \leq j \leq n \), \( \lambda_j = \lambda_i \), then \( Q \) is reduced to \( Q = \lambda I \), where \( I \) is the identity transformation. Hence \( M^*_1 \) is Einstein, and from Equation (3), it follows that \( M^*_1 \) is a space of constant curvature. If \( \lambda_1 = \lambda_2 = \ldots = \lambda_{n-m} = \lambda \) and \( \lambda_{n-m+1} = \ldots = \lambda_n = \mu \), then Equation (6) implies

\[
\lambda + \mu - \frac{\text{Tr}Q}{n-1} = 0
\]

which gives \((n-m-1)\mu = (1-m)\lambda\).  

(7)

Now if \( \lambda = 0 \) then Equation (7) implies that \( \mu = 0 \) or \( Q_p \) has the following form:

\[
Q_p = \begin{pmatrix} 
0 & \mu \\
\mu & \mu \\
& \ddots \\
& & \mu 
\end{pmatrix}
\]

(8)

If \( \lambda \neq 0 \) and \( m = 1 \) then Equation (7) implies that \( \mu = 0 \) and so \( Q \) is reduced to

\[
Q_p = \begin{pmatrix} 
\lambda \\
\lambda \\
& \ddots \\
& & \lambda \\
0
\end{pmatrix}
\]

(9)

Otherwise, if \( m \neq 1 \) and \( \lambda \mu < 0 \) then Equation (7) implies that

\[
Q_p = \begin{pmatrix} 
\lambda \\
& \lambda \\
& & \lambda \\
& & & \lambda \\
0 & & & & \mu \\
& & & & \mu \\
& & & & \mu \\
& & & & \mu 
\end{pmatrix}
\]

(10)

where \( \lambda \) and \( \mu \) have the algebraic multiplicities \( n - m \) and \( m \), respectively.  

First let us consider the case (Equation 10). We assume that \( \lambda > 0 \) and set \( U = \{ x \in M^*_1 : O \} \) has the form (Equation 10), then, by the continuity argument for the characteristic polynomial of \( Q \), \( U \) is an open subset of \( M^*_1 \). By \( U_0 \), we denote a connected component of \( U \). Then \( m \) is constant on \( U_0 \) and \( \lambda(x) \) and \( \mu(x) \) are differentiable functions on \( U_0 \). Let us define two distributions on \( U_0 \) by \( T_1(x) = \{ X \in S^*_1 : QX = \lambda(x)X \} \) and \( T_2(x) = \{ X \in S^*_1 : QX = \mu(x)X \} \). Then \( T_1(x) \) and \( T_2(x) \) are differentiable and the restrictions of the metric on \( T_1M^*_1 \) to \( T_1(x) \) and \( T_2(x) \) are nondegenerate. Since \( M^*_1 \) is conformally flat and \( X, Y \in T_1(x) \) satisfies \( QX = \lambda(x)X \) and \( QY = \lambda(x)Y \), it follows that \( R(X, Y) = kX \wedge Y \), \( X, Y \in T_1(x) \), \( k = \frac{\lambda - \mu}{n-2} \). Similarly, for \( X, Y \in T_2(x) \), we have \( R(X, Y) = -kX \wedge Y \). By the second Bianchi identity, we can see that \( k \) is a constant. Therefore, \( M^*_1 \) is locally a product space of an \( m \)-dimensional Osserman Lorentzian space of constant curvature \( k \) and a \((n-m)\)-dimensional Osserman Riemannian space of constant curvature \(-k\), or an \( m \)-dimensional Osserman Riemannian space of constant curvature \( k \) and an \((n-m)\)-dimensional Osserman Lorentzian space of constant curvature \(-k\); that is, \( M^*_1 \times M^{n-m}(-k) \) or \( M^*_1 \times M^{n-m}(-k) \), where \( 1 \leq m < n-1 \) and \( k = \frac{\lambda - \mu}{n-2} \).

Next assume that the rank of \( Q \) is \( n-1 \) at some point \( p \). Let \( U = \{ x \in M^*_1 : Q \} \) has the rank \( n-1 \}. Then \( U \) is open and non-zero eigenvalue of \( Q \), say \( \lambda' \), is a differentiable function on \( U \). Two distributions \( T_0 \) and \( T_1 \) on \( U \) are defined by \( T_0(x) = \{ X \in S^*_1 : QX = 0 \}, T_1(x) = \{ X \in S^*_1 : QX = \lambda'(x)X \} \). Then, it follows that they are differentiable, \( T_0 \) is involutive and geodesics whose tangents belong to \( T_0 \) are infinitely extendible. Moreover, \( T_0 \) and \( T_1 \) are parallel distributions, (Sekigawa and Takagi, 1971). The restriction of the metric on \( T_1M^*_1 \) to \( T_0(x) \) and \( T_1(x) \) are non-degenerate. Hence \( T_1 \) (resp. \( T_0 \)) has maximal integral submanifold \( M^{n-1} \) (resp. \( M^n \)) of \( M^*_1 \). Since \( M^*_1 \) is conformally flat and \( X \in T_1(x) \) satisfies \( QX = \lambda(x)X \), \( M^{n-1} \) has constant curvature \( k = \frac{\lambda - \mu}{n-2} \). By virtue of the second Bianchi identity, therefore, \( M^*_1 \) is locally a product space of \((n-1)\)-dimensional Osserman Lorentzian space \( M^{n-1}_1 \) of constant curvature \( k \) and a \( 1 \)-dimensional
Osserman Riemannian space $M^1$, or (n-1)-dimensional Osserman Riemannian space $M^{n-1}$ of constant curvature and a 1-dimensional Osserman Lorentzian space $M^1$.

Case (ii)

Suppose that $Q_e$ is of the Form (ii). Then it follows that

$Qe_1 = \lambda e_1 - e_1, \quad Qe_2 = \mu e_1 + \lambda e_2, \quad Qe_j = \lambda j e_j (j = 3, \ldots, n).$

Putting $X = e_1, Y = Z = e_j$ in Equation (5), we have

$\frac{(\lambda_j^2 - \lambda^2) e_1 + 2 \lambda \mu e_2}{n-2} = 0.$

(11)

Equation (11) implies that $\frac{\mu(2\lambda - 2\lambda^2)}{n-1} = 0$ or since $\mu \neq 0$, $\lambda = \frac{TrQ}{2(n-1)}$. Next, putting $X = e_1, Y = e_j, Z = e_2$, in Equation (5) and taking an inner product with $e_j$, we get

$2\lambda \mu + \frac{(\lambda_j - \lambda) TrQ}{n-1} = 0$. Substituting $\lambda = \frac{TrQ}{2(n-1)}$ into this equation, we have $\lambda_j = 0$. Similarly, putting $X = e_1, Y = e_j, Z = e_1$ in Equation (5), we obtain

$(\lambda_j^2 - \lambda^2) - \frac{2\lambda(\lambda - \lambda_j)}{n-2} = 0.$

(12)

If $\lambda = 0$ then Equation (12) reduces to $\lambda_j^2 + \mu^2 = 0$. This contradicts the assumption that $\mu \neq 0$. Therefore, $\lambda \neq 0$ and $\lambda_j = 0$. So from Equation (12), it follows that $n = 2$ which contradicts the assumption on the dimension. Thus this case can not occur.

Case (iii)

Suppose that $Q_e$ is of the Form (iii). Then it follows that

$Qu_1 = \lambda u_1 + u_1, \quad Qu_2 = \mu u_1, \quad Qu_j = \lambda_j u_j (j = 3, \ldots, n)$. Putting $X = u_1, Y = u_2, Z = u_j$ in Equation (1.5), we obtain

$2\lambda u_1 - \frac{(TrQ)u_2}{n-1} = 0$, so that

$\lambda = \frac{2TrQ}{n-1}.$

(13)

Putting $X = u_1, Y = u_j, Z = u_j$ in Equation (5), we have

$\frac{(\lambda_j^2 - \lambda^2) u_1 + (TrQ)(\lambda_j - \lambda) u_2}{n-2} = 0$. By taking the inner product with $u_2$, it follows that $\lambda_j = \lambda$ or $\lambda_j = -2\lambda_j$ by virtue of Equation (13). Hence Equation (13) reduces to

$\lambda = \frac{2\lambda_j}{n-1}$ which implies $\lambda = 0$ because $\dim M^1 \geq 4$.

Therefore, it follows that $\lambda = 0$ and that $R(X, Y) = 0$ for any tangent vectors $X$ and $Y$ in $S^1$, by virtue of Equation (3). Therefore, $M^1$ is flat, namely it has constant curvature zero.

Case (iv)

By using the semi-orthogonal basis to evaluate $Q$, and following similar argument in Case (iii) we have $R(X, Y) = 0$ for any tangent Vectors $X$ and $Y$ in $S^1$, by virtue of Equation (2). Thus, $M^1$ has constant curvature zero.

In the case $\dim M = 3$, if $Q$ is of the Form (i), then the result is accomplished by similar arguments in Sekigawa and Takagi (1971) and also for the other cases (Honda, 2003).

REFERENCES