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Full Length Research Paper

Reliable approach of iterative method for nonlinear fractional differential equations

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In this paper, a new approach of the iterative method is developed to handle nonlinear differential equations of fractional order. For this reason, an efficient modification of iterative method, namely iterative-Laurent method, is introduced based on iterative method and Laurent series expansion. The proposed approach is capable of reducing the size of calculations and easily overcome the difficulty arising in calculating complicated integrals. Furthermore, the new approach is compared with the variational iteration method and Adomian decomposition method in various nonlinear fractional differential equations and the obtained results reveal that the proposed method is more accurate and efficient.

Key words: Fractional differential equations, Iterative method, laurent series expansion, caputo's fractional differential operator.

INTRODUCTION

Fractional order ordinary differential equations, as generalizations of classical integer order ordinary differential equations, are increasingly used to model problems in fluid flow, mechanics, viscoelasticity, biology, physics and engineering and other applications. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. Half-order derivatives and integrals proved to be more useful for the formulation of certain electrochemical problems than the classical models (Podlubny, 1999). Fractional differentiation and integration operators are also used for extensions of the diffusion and wave equations (Schneider and Wyess, 1989). Most nonlinear fractional differential equations do not have exact analytic solutions, therefore approximation and numerical techniques must be used. The variational iteration method (VIM) and the Adomian decomposition method (ADM) and their modifications (Jafari et al., 2011; Ghorbani, 2008; Jafari and Gejji, 2006; Lensic, 2005; Momani and Odibat, 2007; Odibat and Momani, 2006) are relatively new approaches to provide an analytical approximation to linear and nonlinear differential equations of fractional order.

The ADM and VIM are limited, that the former has complicated algorithms in calculating Adomian polynomials for nonlinear problems and the latter has an inherent inaccuracy in identifying the Lagrange multiplier for fractional operators, which is necessary for constructing variational iteration formula. Gejji and Jafari (2006) employed the basic ideas of decomposition method to propose a general method for nonlinear functional equations, namely the iterative method (IM). The IM was successfully applied to solve several types of nonlinear problems such as nonlinear fractional differential equations (Bhalekar and Gejji, 2008; Gejji and Jafari, 2006). The aim of present paper is to introduce reliable approach of the IM to handle nonlinear fractional differential equations. The main advantage of this approach is the capability to reduce the computational work and to overcome the difficulty that arising in calculating complicated integrals. Moreover, this method is examined by comparing the results with the VIM and ADM. Numerical

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Abbreviations: VIM, Variational iteration method; **ADM,** Adomian decomposition method; **IM,** iterative method; **ILM,** Iterative- Laurent method.

results show the efficiency of the proposed method of this paper.

Basic definitions

We give some basic definitions and properties of the fractional calculus theory (Ghorbani, 2008) which are used further in this paper.

Definition 1

A real function $f(x), x > 0$, is said to be in the space $C_\mu, \mu \in R$ if there exists a real number $p > \mu$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0,\infty]$. Clearly $C_\mu \subset C_\beta$ if $\beta \le \mu$.

Definition 2

A function $f(x), x > 0$, is said to be in the space $C_\mu^m, m \in N \cup \{0\}$, if $f^{(m)} \in C_\mu$.

Definition 3

The Riemann-Liouville fractional integral operator of order $\alpha \ge 0$, of a function, $f \in C_\mu, \mu \ge -1$, is defined as

$$D^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \quad (1)$$

$$D^0 f(x) = f(x). \quad (2)$$

Here, we have (Miller and Ross, 1993).

$$D^{-\alpha} D^{-\beta} f(x) = D^{-\alpha+\beta} f(x), \quad (3)$$

$$D^{-\alpha} D^{-\beta} f(x) = D^{-\beta} D^{-\alpha} f(x), \quad (4)$$

$$D^{-\alpha} x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)} x^{\gamma+\alpha}, \quad (5)$$

where $\alpha, \beta \ge 0, x > 0$ and $\gamma > -1$.

Definition 4

The fractional derivative of $f(x)$ in the Riemann-Liouville sense is defined as

$$D^\alpha f(x) = \frac{d^m}{dx^m} \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt, \quad (6)$$

where $m \in N$ and satisfies the relation $m-1 < \alpha < m$, and $f \in C_{-1}^m$.

Properties of the operators can be found in (Podlubny, 1999), we mention only the following:

$$D^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} x^{\gamma-\alpha}, \quad (7)$$

where $\alpha \ge 0, x > 0$ and $\gamma > -1$.

The Riemann–Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D_*^α proposed by Caputo in his work on the theory of viscoelasticity (Caputo, 1967).

Definition 5

The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$D_*^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt, \quad (8)$$

for $m-1 < \alpha \le m, m \in N, x > 0$ and $f \in C_{-1}^m$.

Also, we need here two of its basic properties.

$$D_*^\alpha D^{-\alpha} f(x) = f(x), \quad (9)$$

$$D^{-\alpha} D_*^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \; x > 0. \quad (10)$$

The ADM, VIM and IM

In recent years, ADM has been applied to a wide class of stochastic and deterministic problems in many areas of mathematics and physics. This computational method yield analytical solutions and has certain advantage over standard numerical methods. Recently (Jafari and Gejji, 2006; Lensic, 2005), the solution of fractional ordinary differential equations has been obtained through the Adomian decomposition method. To illustrate the decomposition procedure of the ADM, we consider the following nonlinear fractional differential equation (more general form can be considered without loss of generality):

$$D_*^\alpha y = f(x,y),\ y^{(k)} = c_k,\ k = 0,1,\cdots,m-1, \quad (11)$$

where the fractional differential operator D_*^α is defined as in Equation (8), $m-1 < \alpha \le m, m \in N, f$ is a nonlinear functional of y, and y is an unknown function to be determined later. Applying the operator $D^{-\alpha}$, the inverse of the operator D_*^α, to both sides of (11) yields

$$y(x) = \sum_{k=0}^{m-1} y^{(k)}(0^+)\frac{x^k}{k!} + D^{-\alpha}\left(f(x,y)\right). \quad (12)$$

The ADM suggests the solution be decomposed into the infinite series of components

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \quad (13)$$

And the nonlinear function in (12) is decomposed into a series of the so-called Adomian polynomials and is as follows:

$$f(x,y) = \sum_{n=0}^{\infty} A_n(y_0,\cdots,y_n), \quad (14)$$

which the terms can be calculated recursively form

$$A_n = \frac{1}{n!}\frac{\partial^n}{\partial\lambda^n}\left(f\left(x,\sum_{k=0}^{n} y_k\lambda^k\right)\right)_{\lambda=0}. \quad (15)$$

Substituting (13) and (14) into both sides of (12) gives

$$\sum_{n=0}^{\infty} y_n(x) = \sum_{k=0}^{m-1} y^{(k)}(0^+)\frac{x^k}{k!} + D^{-\alpha}\left(\sum_{n=0}^{\infty} A_n\right). \quad (16)$$

From this equation, the iterates are determined by the following recursive way:

$$y_0(x) = \sum_{k=0}^{m-1}\frac{c_k}{k!}x^k,$$
$$y_{n+1}(x) = D^{-\alpha}(A_n),\ n = 0,1,2,\cdots \quad (17)$$

Finally, we approximate the solution by the truncated series

$$\phi_N(x) = \sum_{n=0}^{N-1} y_n(x), \text{ and } \lim_{N\to\infty}\phi_N(x) = y(x). \quad (18)$$

The VIM was first proposed by the Chinese mathematician, He. It has been shown that this procedure is a powerful tool for solving various kinds of problems. To illustrate its basic idea of the method, we consider (11) as

$$D_*^\alpha - f(x,y) = 0,\ m-1 < \alpha \le m. \quad (19)$$

The basic character of the method is to construct a correction functional for (19). He (1998) constructed the following correction functional for (19):

$$y_{n+1}(x) = y_n(x) + D^{-\alpha}\{\lambda(D_*^\alpha y_n(x) - f(x,\tilde{y}_n(x)))\}, \quad (20)$$

Where λ is a Lagrange multiplier which can be identified optimally via variational theory (He, 1997), y_n is the nth approximate solution, and $\tilde{y}_n$ denotes a restricted variation, that is. $\delta\tilde{y}_n = 0$. To approximately identify the Lagrange multiplier when there does not exist a derivative with an integer order, there is no way to directly obtain the stationary conditions from a functional with fractional integrate, so, in order to approximately identify the multiplier, one has to find a minimal integer $m = ceil(\alpha) > \alpha,$ or a maximal integer $m-1 = floor(\alpha) < \alpha$. Therefore, the correction functional (20) can be approximately expressed as follows:

$$y_{n+1}(x) = y_n(x) + \int_0^x \left\{\lambda_1\left(\frac{d^m y_n(t)}{dt^m} - f(t,\tilde{y}_n(t))\right)\right\}dt, \quad (21)$$

and

$$y_{n+1}(x) = y_n(x) + \int_0^x \left\{\lambda_2\left(\frac{d^{m-1} y_n(t)}{dt^{m-1}} - f(t,\tilde{y}_n(t))\right)\right\}dt. \quad (22)$$

The multiplier, therefore, can be determined by the VIM, substituting the identified Lagrange multipliers, respectively, into (20) resulting in the following iteration procedures:

$$y_{n+1}(x) = y_n(x) + D^{-\alpha}\{\lambda_1(D_*^\alpha y_n(x) - f(x,y_n(x)))\}, \quad (23)$$

$$y_{n+1}(x) = y_n(x) + D^{-\alpha}\{\lambda_2(D_*^\alpha y_n(x) - f(x,y_n(x)))\}, \quad (24)$$

or

$$y_{n+1}(x) = y_n(x) + D^{-\alpha}\{(a\lambda_1 + b\lambda_2)(D_*^\alpha y_n(x) - f(x,y_n(x)))\}, \quad (25)$$

where a and b are weighted factors with $a+b=1$. It should be emphasized that the aforementioned process was only suggested for $1 < \alpha \le 2$ (He, 1998). For instance, Momani and Odibat (2007) and Odibat and Momani (2007) have constructed the correction functional for (19) as follows:

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda\left(\frac{d^m}{dt^m} y_n(t) - f(x, \tilde{y}_n(t))\right)dt, \quad (26)$$

Therefore, the multiplier can be obtained by the VIM, substituting the identified Lagrange multiplier into (26) results in the following iteration procedure:

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(D_*^\alpha y_n(t) - f(x, y_n(t)))dt. \quad (27)$$

Now we can start with the given initial approximation and by the above iteration formulas we can obtain the approximate solutions. Consequently, the exact solution may be obtained by using $y(x) = \lim_{n\to\infty} y_n(x)$, (Ghorbani, 2008). The IM has been successfully applied to solve several types of nonlinear problems (Bhalekar and Gejji, 2008; Gejji and Jafari, 2006). To illustrate its basic idea of the IM, we consider (11). Applying the operator $D^{-\alpha}$, the inverse of the operator D_*^α, to both sides of (11) yields

$$y(x) = \sum_{k=0}^{m-1} y^{(k)}(0^+)\frac{x^k}{k!} + D^{-\alpha}(f(x,y)). \quad (28)$$

The iterative method suggests the solution be decomposed into the infinite series of components

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \quad (29)$$

And the nonlinear function in (12) is decomposed as

$$f(x, \textstyle\sum_{n=0}^{\infty} y_n) = f(x, y_0) + \sum_{n=1}^{\infty}\left\{f\left(x, \sum_{j=0}^{n} y_j\right) - f\left(x, \sum_{j=0}^{n-1} y_j\right)\right\} \quad (30)$$

Now substituting (29) and (30) into both sides of (28) gives

$$\sum_{n=0}^{\infty} y_n(x) = \sum_{k=0}^{m-1} y^{(k)}(0^+)\frac{x^k}{k!} + D^{-\alpha}(f(x,y_0)) + D^{-\alpha}\left(\sum_{n=1}^{\infty}\left\{f\left(x, \sum_{j=0}^{n} y_j\right) - f\left(x, \sum_{j=0}^{n-1} y_j\right)\right\}\right), \quad (31)$$

From this equation, the iterates are determined by the following recursive way:

$$y_0(x) = La_M\left(\sum_{k=0}^{m-1} y^{(k)}(0^+)\frac{x^k}{k!}\right),$$
$$y_1(x) = La_M\left(D^{-\alpha}(f(x, y_0))\right),$$
$$y_2(x) = La_M\left(D^{-\alpha}(f(x, y_0 + y_1) - f(x, y_0))\right), \quad (32.a)$$

And in general

$$y_{n+1}(x) = La_M\left(D^{-\alpha}(f(x, y_0 + \cdots + y_n) - f(x, y_0 + \cdots + y_{n-1}))\right), \; n \geq 1 \quad (32.b)$$

Finally, we approximate the solution by the truncated series

$$\phi_N(x) = \sum_{n=0}^{N-1} y_n(x), \text{ and } \lim_{N\to\infty} \phi_N(x) = y(x). \quad (33)$$

From here, we can clearly conclude that the main demerit of the ADM is to calculate Adomian polynomials for a nonlinear operator where the procedure is very complex and the main demerit of the VIM is to identify the Lagrange multiplier for a fractional operator which is merely approximate. The main demerit of the IM is to calculate the components y_n, in (32), and it may also requires a large amount of computational work in determining these components. In order to overcome these disadvantages, subsequently, we propose a new approach of the IM for solving nonlinear fractional differential equations where there is no complicated process. The procedure solution becomes easier and more straightforward.

ANALYSIS OF THE NEW METHOD

Although the IM has many advantages such as simple solver and it does not require using the Adomian polynomials, it may be difficult to calculate the components y_n and it may also require a large amount of computational work in determining these components. Here, a new approach of the IM using the Laurent series expansion, namely Iterative- Laurent method (ILM) is proposed to overcome these difficulties. Main idea of the method proposed is to be used the Laurent series expansion in (32). Here, for a given function f, we denote its M^{th}-order Laurent series expansion at zero by $La_M(f)$. Let us replace the right hand side functions of (32) with their M^{th}-order Laurent series expansion, then we have

$$y_0(x) = La_M\left(\sum_{k=0}^{m-1} y^{(k)}(0^+)\frac{x^k}{k!}\right),$$
$$y_1(x) = La_M\left(D^{-\alpha}(f(x, y_0))\right),$$
$$y_2(x) = La_M\left(D^{-\alpha}(f(x, y_0 + y_1) - f(x, y_0))\right), \quad (34.a)$$

And in general

$$y_{n+1}(x) = La_M\left(D^{-\alpha}(f(x, y_0 + \cdots + y_n) - f(x, y_0 + \cdots + y_{n-1}))\right), \; n \geq 1 \quad (34.b)$$

Finally, we approximate the solution by the truncated series

$$\phi_N(x) = \sum_{n=0}^{N-1} y_n(x), \text{ and } \lim_{N\to\infty} \phi_N(x) = y(x).$$

Generally speaking, the presented method will show the simplicity and accuracy in solving various fractional problems, as well as the rapid convergence of approximations to accurate solutions. Furthermore, the solution procedure is much more fascinating and straightforward.

CONVERGENCE ANALYSIS

Consider the general functional equation

$$u = f + N(y), \tag{35}$$

where N is a nonlinear operator from a banach is space $B \to B$ and f is a known function. Suppose that y having the series form $y = \sum_{n=0}^{+\infty} y_n$ and $Tl_M oN$, be a contraction, that is,

$$\left\|Tl_M\big(N(x)\big) - Tl_M\big(N(y)\big)\right\| \le L\|x - y\|, \qquad 0 < L < 1,$$

Then

$$\|u_{m+1}\| = \left\|Tl_M\big(N(u_0 + \cdots + u_m)\big) - Tl_M\big(N(u_0 + \cdots + u_{m-1})\big)\right\| \le L\|u_m\| \le L^{m+1}\|u_0\|, \quad m \ge 1$$

So the series $\sum_{i=0}^{+\infty} u_i$ which is obtained by the ILM converges to the unique solution of Equation (35) absolutely and uniformly, in view of the Banach fixed point theorem (Jerri, 1999).

APPLICATION OF ITERATIVE-LAURENT METHOD (ILM)

Here, for the sake of comparison, we have selected some examples where the exact solution already exists, which will ultimately show the simplicity, effectiveness and exactness of the proposed method. Maple 12 is used for computations here.

Example 1

Consider the following nonlinear fractional differential equation (Ghorbani, 2008).

$$D_*^{2.4} y(x) - y^3(x) = \frac{10}{\Gamma(3/5)} x^{3/5} - x^6 - 3x^7 - 3x^8 - x^9, \tag{36}$$

With the initial conditions

$$y(0) = y'(0) = 0, \; y''(0) = 2, \tag{37}$$

And the exact solution $y(x) = x^2 + x^3$.

Solution by the IM

Applying the operator $D^{-2.4}$, the inverse of the operator $D_*^{2.4}$, to both sides of (36) yields

$$y(x) = x^2 + \frac{1}{\Gamma(2.4)} \int_0^x (x-t)^{1.4} \left(\frac{10}{\Gamma\left(\frac{8}{5}\right)} t^{\frac{3}{5}} - t^6 - 3t^7 - 3t^8 - t^9 \right) dt,$$
$$+ \frac{1}{\Gamma(2.4)} \int_0^x (x-t)^{1.4} y^3(t)\, dt. \tag{38}$$

Starting with the initial approximation.

$$y_0(x) = x^2 + \frac{1}{\Gamma(2.4)} \int_0^x (x-t)^{1.4} \left(\frac{10}{\Gamma\left(\frac{8}{5}\right)} t^{\frac{3}{5}} - t^6 - 3t^7 - 3t^8 - t^9 \right)$$
$$= x^2 + 1.000000000x^3 - 0.007514915720x^{8.4} - 0.01678864150x^{9.4}$$
$$+ 0.01291433961x^{10.4} - 0.003398510425x^{11.4}.$$

We get

$$y_1(x) = \frac{1}{\Gamma(2.4)} \int_0^x (x-t)^{1.4} y_0^3(t)\, dt = 0.007514915720x^{8.4} + 0.0167886415x^{9.4}$$
$$+0.01291433961x^{10.4} + 0.003398510425x^{11.4} - 0.00003944215654x^{14.8}$$
$$-0.0001416327156x^{15.8} - 0.0002060553054x^{16.8} - 0.0001518956893x^{17.8}$$
$$-0.00005675978963x^{18.8} - 0.000008599968130x^{19.8} + 1.205736548 \times 10^{-7} x^{21.2}$$
$$+5.880305728 \times 10^{-7} x^{22.2} + 0.000001243356536x^{23.2} + 0.000001477425017x^{24.2}$$
$$+0.000001065293658x^{25.2} + 4.659968248 \times 10^{-7} x^{26.2} + 1.144725357 \times 10^{-7} x^{27.2}$$
$$-1.572724516 \times 10^{-10} x^{27.6} + 1.217792933 \times 10^{-8} x^{28.2} - 9.656074567 \times 10^{-10} x^{28.6}$$
$$-2.664849348 \times 10^{-9} x^{29.6} - 4.336432100 \times 10^{-9} x^{30.6} - 4.583249440 \times 10^{-9} x^{31}$$
$$-3.261441653 \times 10^{-9} x^{32.6} - 1.561994621 \times 10^{-9} x^{33.6} - 4.853348693 \times 10^{-10} x^{34.6}$$
$$-8.874816952 \times 10^{-11} x^{35.6} - 7.274440129 \times 10^{-12} x^{35.6} \tag{39}$$

Here, calculating y_n, for ≥ 2, require a large amount of computational work.

Solution by ILM

For this example we choose $M = 20$. According to (34) and (38), we have

$$y_0(x) = x^2 + \frac{1}{\Gamma(2.4)} \int_0^x (x-t)^{1.4} La_{20} \left(\frac{10}{\Gamma\left(\frac{8}{5}\right)} t^{\frac{3}{5}} - t^6 - 3t^7 - 3t^8 - t^9 \right) dt,$$
$$= x^2 + 1.000000000x^3 - 0.007514915720x^{8.4} - 0.01678864150x^{9.4}$$
$$+0.01291433961x^{10.4} - 0.003398510425x^{11.4},$$
$$y_1(x) = \frac{1}{\Gamma(2.4)} \int_0^x (x-t)^{1.4} La_{20}(y_0^3(t))\, dt = 0.007514915720x^{8.4} + 0.0167886415x^{9.4}$$
$$+0.01291433961x^{10.4} + 0.003398510425x^{11.4} - 0.00003944215654x^{14.8}$$
$$-0.0001416327156x^{15.8} - 0.0002060553054x^{16.8} - 0.0001518956893x^{17.8}$$
$$-0.00005675978963x^{18.8} - 0.000008599968130x^{19.8} + 1.205736548 \times 10^{-7} x^{21.2}$$
$$+5.880305728 \times 10^{-7} x^{22.2},$$
$$y_2(x) = \frac{1}{\Gamma(2.4)} \int_0^x (x-t)^{1.4} La_{20} \left((y_0(t) + y_1(t))^3 - y_0^3(t) \right) dt,$$
$$= 0.00003944215654x^{14.8} + 0.0001416327156x^{15.8} + 0.0002060553054x^{16.8}$$
$$+0.0001518956893x^{17.8} + 0.00005675978963x^{18.8} + 0.000008599968130x^{19.8}$$
$$-2.047838839 \times 10^{-7} x^{21.2} - 0.000001007942823x^{22.2},$$
$$y_3(x) = 8.421022916 \times 10^{-8} x^{21.2} + 4.199122500 \times 10^{-7} x^{22.2}, \tag{40}$$

The 3^{th}-approximate solution of the ILM is,

$$y(x) \approx \sum_{n=0}^{3} y_n(x) = x^2 + x^3 + 1.6 \times 10^{-16} x^{21.2} - 2.10^{-16} x^{22.2}. \tag{41}$$

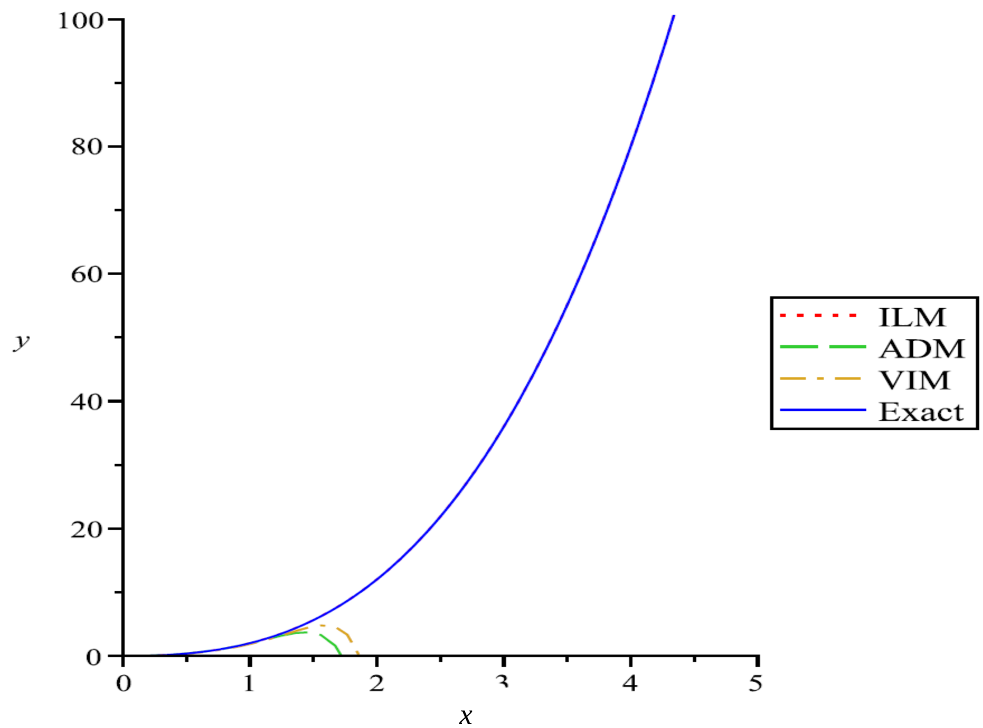

Figure 1. Approximate solutions for example 1 using ILM: Dat, ADM: Dash, VIM: dash dat and exact solution is $x^2 + x^3$: solid line.

Solution by the VIM: According to relations (26) and (27), we have the following variational iteration formula for solving (36):

$$y_{n+1}(x) = y_n(x) - \frac{1}{2}\int_0^x (t-x)^2 \left(D_*^{2.4} y_n(t) - y_n{}^3(t) - \frac{10}{\Gamma\left(\frac{8}{5}\right)} t^{\frac{8}{5}} + t^6 + 3t^7 + 3t^8 + t^9 \right) dt. \quad (42)$$

We start with the initial approximation $y_0(x) = x^2$. Therefore, we obtain

$$y_0(x) = x^2,$$

$$y_1(x) = y_0(x) - \frac{1}{2}\int_0^x (t-x)^2 \left(D_*^{2.4} y_0(t) - y_0{}^3(t) - \frac{10}{\Gamma\left(\frac{8}{5}\right)} t^{\frac{8}{5}} + t^6 + 3t^7 + 3t^8 + t^9 \right) dt,$$

$$= x^2 + 0.4483874014x^{3.6} - 0.004166666666x^{10} - 0.003030303030x^{11}$$
$$-0.0007575757575x^{12},$$

$$y_2(x) = y_1(x) - \frac{1}{2}\int_0^x (t-x)^2 \left(D_*^{2.4} y_1(t) - y_1{}^3(t) - \frac{10}{\Gamma\left(\frac{8}{5}\right)} t^{\frac{8}{5}} + t^6 + 3t^7 + 3t^8 + t^9 \right) dt,$$

$$= x^2 + 0.4483874014x^{3.6} - 0.1841728255x^{4.2} - \cdots - 7.92918003010^{-15}x^{39},$$

$$y_3(x) = y_2(x) - \frac{1}{2}\int_0^x (t-x)^2 \left(D_*^{2.4} y_2(t) - y_2{}^3(t) - \frac{10}{\Gamma\left(\frac{8}{5}\right)} t^{\frac{8}{5}} + t^6 + 3t^7 + 3t^8 + t^9 \right) dt,$$

$$= x^2 + 0.4483874014x^{3.6} - 0.1841728255x^{4.2} - \cdots - 2.958520758 \times 10^{-49}x^{120}. \quad (43)$$

Solution by the ADM

In accordance with the ADM, we have

$$y_0(x) = x^2 + D^{-2.4}\left(\frac{10}{\Gamma(3/5)}x^{3/5} - x^6 - 3x^7 - 3x^8 - x^9\right),$$

$$= x^2 + 1.000000000x^3 - 0.007514915720x^{8.4} - 0.01678864150x^{9.4}$$
$$+0.01291433961x^{10.4} - 0.003398510425x^{11.4},$$

$$y_1(x) = \frac{1}{\Gamma(2.4)}\int_0^x (x-t)^{1.4} A_0(t)\,dt,$$

$$= 0.0003485035552x^{11} + 0.0006098812216x^{12} + \cdots - 1.068446179 \times 10^{-14}x^{39.2},$$

$$y_2(x) = \frac{1}{\Gamma(2.4)}\int_0^x (x-t)^{1.4} A_1(t)\,dt,$$

$$= 3.102171845 \times 10^{-8}x^{21} + 3.564966712 \times 10^{-8}x^{22} + \cdots 6.172042871 \times 10^{-27}x^{67},$$

$$y_{3(x)} = \frac{1}{\Gamma(2.4)}\int_0^x (x-t)^{1.4} A_2(t)\,dt,$$

$$= 5.381692194 \times 10^{-13}x^{29} + 2.031278137 \times 10^{-12}x^{30} + \cdots - 3.870040959 \times 10^{-39}x^{94.8}. \quad (44)$$

where $A_n(x)$ are A domian polynomials for nonlinear operator $f(x,y) = y^3(x)$. The 3^{th}-approximate solution of the ADM is stated as:

$$y(x) \approx \sum_{n=0}^{3} y_n(x) = x^2 + x^3 - 0.007514915720x^{8.4} + \cdots - 3.870040959 \times 10^{-39}x^{94.8}. \quad (45)$$

In Figure 1, the approximation solutions $y(x) \approx \sum_{n=0}^{3} y_n(x)$ of the

ILM and ADM and the approximation solution $y_3(x)$ of the VIM have been plotted. We observe that the obtained solution using the ILM coincide with the exact solution. It is noteworthy that the obtained results confirm the proposed ILM is easier, more effective and much more accurate than the IM, VIM and ADM.

Example 2

Consider the nonlinear differential equation of the fractional order (Ghorbani, 2008):

$$D_*^{1.3}y(x) = \frac{20}{7\Gamma(7/10)}x^{7/10} + x^4 - y^2(x), \qquad (46)$$

With the initial conditions

$$y(0) = 0, \quad y'(0) = 0. \qquad (47)$$

And the exact solution, $y(x) = x^2$.

Solution via the IM

Applying the operator $D^{-1.3}$, the inverse of the operator $D_*^{1.3}$, to both sides of (46) yields

$$y(x) = \frac{1}{\Gamma(1.3)}\int_0^x (x-t)^{.3}\left(\frac{20}{7\Gamma(7/10)}t^{7/10} + t^4\right)dt - \frac{1}{\Gamma(1.3)}\int_0^x (x-t)^{.3}y^2(t)dt, \quad (48)$$

It is easy to verify that to compute y_n, for $n > 4$ require a large amount of computational work.

Solution via ILM: With selecting $M = 20$ and According to (34), we have

$$y_0(x) = \frac{1}{\Gamma(1.3)}\int_0^x (x-t)^{.3}La_{20}\left(\frac{20}{7\Gamma(7/10)}t^{7/10} + t^4\right)dt,$$
$$= 0.9999999997x^2 + 0.1189218102x^{5.3},$$
$$y_1(x) = -\frac{1}{\Gamma(1.3)}\int_0^x (x-t)^{.3}La_{20}(y_0{}^2(t))dt,$$
$$-0.1189218101x^{5.3} - 0.01484584632x^{8.6} - 0.0005748986958x^{11.9},$$
$$y_2(x) = -\frac{1}{\Gamma(1.3)}\int_0^x (x-t)^{.3}La_{20}(y_1{}^2(t) + 2y_0(t)y_1(t))dt,$$
$$= 0.01484584631x^{8.6} + 0.001781887543x^{11.9} + 0.00003387626469x^{15.2}$$
$$-0.000005018006622x^{18.5}, \qquad (49)$$

The 5^{th}-approximate solution of the ILM is stated below.

$$y(x) \approx \sum_{n=0}^{5} y_n(x) = 0.9999999997x^2 + 1\times10^{-10}x^{5.3} - 1\times10^{-11}x^{8.6} + \dots + 7\times10^{-15}x^{18.5}. \quad (50)$$

Solution via the VIM: According to relations (26) and (27), we have the following variational iteration formula for solving (46):

$$y_{n+1}(x) = y_n(x) + \int_0^x (t-x)\left(D_*^{1.3}y_n(t) - \frac{20}{7\Gamma(7/10)}t^{7/10} - t^4 + y_n{}^2(t)\right)dt. \quad (51)$$

We start with the initial approximation $y_0(x) = 0$ and the approximation solution $y_5(x)$ of (51) is stated below,

$$y(x) \approx y_5(x) = 2.397706766x^{2.7} - 1.973145130x^{3.4} + 0.7160309441x^{4.1} + \cdots$$
$$-1.495161834\times10^{-65}x^{126}. \qquad (52)$$

Solution via the ADM: According to the ADM, we have

$$y_0(x) = \frac{1}{\Gamma(1.3)}\int_0^x (x-t)^{.3}\left(\frac{20}{7\Gamma(7/10)}t^{7/10} + t^4\right)dt,$$
$$= 0.9999999997x^2 + 0.1189218102x^{5.3},$$
$$y_1(x) = -\frac{1}{\Gamma(1.3)}\int_0^x (x-t)^{.3}A_0(t)dt,$$
$$= -0.8571096218x^{5.3} - 0.2038580556x^{8.6} - 0.01212158450x^{11.9},$$

The 5^{th}-approximate solution of the ADM is stated below.

$$y(x) \approx \sum_{n=0}^{5} y_n(x) = 0.9999999997x^2 - 0.7381878116x^{5.3} + \dots - 0.00005495411418x^{38.3}. \quad (53)$$

In Figure 2, the approximation solutions $y(x) \approx \sum_{n=0}^{5} y_n(x)$ of the ILM and ADM and the approximation solution $y_5(x)$ of the VIM have been plotted. We observe that the obtained solution using the ILM is much more accurate and efficient than the approximate solutions obtained using the VIM and ADM.

Example 3

Consider the following nonlinear fractional differential equation with variable coefficient (Ghorbani, 2008).

$$D_*^{0.25}y(x) + xy^2(x) = \frac{32}{21\Gamma(3/4)}x^{7/4} + x^5. \qquad (54)$$

With initial condition

$$y(0) = 0. \qquad (55)$$

And the exact solution.

$$y(x) = x^2. \qquad (56)$$

Applying the operator $D^{-0.25}$, the inverse of the operator $D_*^{0.25}$, to both sides of (54) yields

$$y(x) = \frac{1}{\Gamma(0.25)}\int_0^x (x-t)^{-0.75}\left(\frac{32}{21\Gamma\left(\frac{3}{4}\right)}t^{\frac{7}{4}} + t^5\right)dt - \frac{1}{\Gamma(0.25)}\int_0^x (x-t)^{-0.75}(ty^2(t))dt. \quad (57)$$

Solution by the IM

Here, calculating y_n for $n \geq 3$, require a large amount of computational work.

Solution by the ILM: As processed before, we obtain

$$y(x) \approx \sum_{n=0}^{3} y_n(x) = x^2 + 1\times10^{-10}x^{5.25} - 0.7520906892x^{8.5} + \dots - 0.7962569272x^{20.25}. \quad (58)$$

where $M = 20$ is considered.

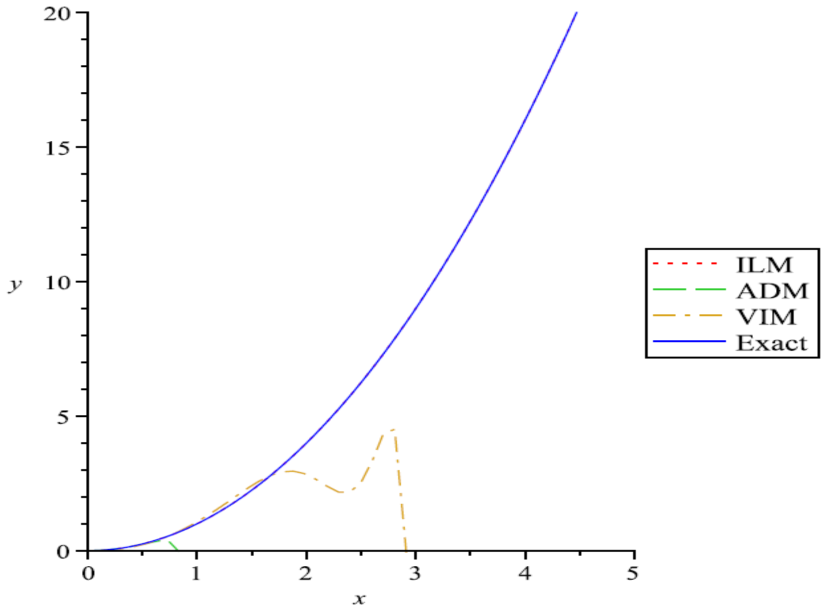

Figure 2. Approximate solutions for example 2 using ILM: Dat, ADM: Dash, VIM: Dash Dat and exact solution is x^2: solid line.

Solution by the VIM

As before processing, we obtain

$$y_{n+1}(x) = y_n(x) - \int_0^x (D_*^{0.25} y_n(t) + t y_n^{\,2}(t) - \frac{32}{21\Gamma\left(\frac{3}{4}\right)} t^{\frac{7}{4}} - t^5) dt \,, \quad (59)$$

And

$$y(x) \approx y_3(x) = 1.356548887x^{2.75} - 0.5158304769x^{3.5} + 0.05679944478x^{4.25} + \cdots - 1.312253298 \times 10^{-7} x^{30}. \quad (60)$$

where $y_0(x) = 0$ is supposed.

Solution by the ADM

As processed before, we obtain

$$y_0(x) = D^{-0.25}\left(\frac{32}{21\Gamma(3/4)} x^{7/4} + x^5\right),$$

$$y_{n+1} = D^{-0.25}(A_n(x)),\ n \geq 0. \quad (61)$$

And

$$y(x) \approx \sum_{n=0}^{3} y_n(x) = x^2 - 1.53519260x^{15} - 1.752433093x^{18.25} - 0.7204334669x^{21.5} - 0.1045974660x^{24.75}. \quad (62)$$

In Figure 3, the approximation solutions $y(x) \approx \sum_{n=0}^{3} y_n(x)$ of (58) of the ILM, the approximation solution $y_3(x)$ of (60) of the VIM and the approximation solution $y(x) \approx \sum_{n=0}^{3} y_n(x)$ of (62) of the ADM have been plotted.

Example 4

Consider the following nonlinear differential equation of the fractional order (Ghorbani, 2008).

$$D_*^{\alpha} y(x) + e^{y(x)} = 0, \qquad 0 < \alpha \leq 1, \quad (63)$$

With initial condition

$$y(0) = 0. \quad (64)$$

And the exact solution (when $\alpha = 1$).

$$y(x) = -\ln(1+x). \quad (65)$$

Applying the operator $D^{-\alpha}$, the inverse of the operator D_*^{α}, to both sides of (63) yields

$$y(x) = -\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} e^{y(t)} dt \,. \quad (66)$$

Solution via the IM

We have:

$$y_0(x) = 0,$$
$$y_1(x) = -\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} e^{y_0(t)} dt = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} dt = -\frac{x^{\alpha}}{\Gamma(1+\alpha)},$$
$$y_2(x) = -\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \left(e^{y_0(t)+y_1(t)} - e^{y_0(t)}\right) dt$$
$$= -\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \left(e^{-\frac{t^{\alpha}}{\Gamma(1+\alpha)}} - 1\right) dt, \quad (67)$$

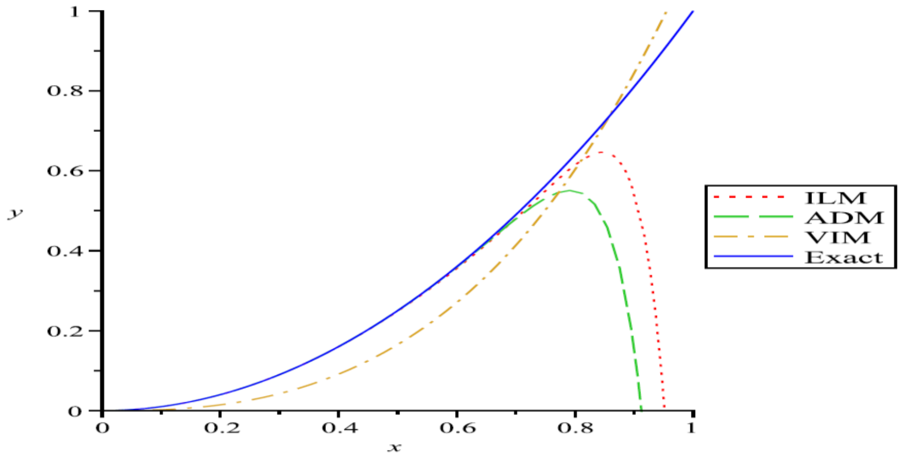

Figure 3. Approximate solutions for example 3 using ILM: Dat, ADM: Dash, VIM: Dash Dat and exact solution is x^2: solid line.

Here, calculating y_n, for $n \geq 2$ is difficult, when $0 < \alpha < 1$.

Solution via the ILM

With selecting $M = 20$ and According to (34), we have

$$y_0(x) = 0,$$
$$y_1(x) = -\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} La_{20}\left(e^{y_0(t)}\right)dt = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}dt,$$
$$y_2(x) = -\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} La_{20}\left(e^{y_0(t)+y_1(t)} - e^{y_0(t)}\right)dt,$$
$$y_3(x) = -\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} La_{20}\left(e^{y_0(t)+y_1(t)+y_2(t)} - e^{y_0(t)+y_1(t)}\right)dt, \tag{68}$$

Here, y_n, $n \geq 0$, can be easily calculate, when $0 < \alpha < 1$.

Solution via the VIM

Since the integration of the nonlinear term in (66) is not easily evaluated, thus here we replace the nonlinear term with a series of finite components. Under this assumption, therefore, we consider the following fractional iteration scheme: According to (26) and (27), and the above assumption, we have the following variational iteration formula for solving (63):

$$y_{n+1}(x) = y_n(x) - \int_0^x \left(D_*^\alpha y_n(t) + 1 + y_n(t) + \frac{1}{2}y_n^2(t) + \frac{1}{6}y_n^3(t)\right)dt, \tag{69}$$

where $y_0(x) = 0$ is supposed.

Solution via the ADM

According to ADM, we have the following recursive relation:

$$y_0(x) = 0,$$
$$y_{n+1}(x) = -D^{-\alpha}(A_n(x)),\ n \geq 0, \tag{70}$$

where

$$A_0(x) = e^{y_0(x)},$$
$$A_1(x) = y_1(x)e^{y_0(x)},$$
$$A_2(x) = \left(\frac{y_1^2(x)}{2} + y_2(x)\right)e^{y_0(x)}, \tag{71}$$

Figure 4 shows the approximate solutions obtained for (63) using the ILM, VIM and ADM when $\alpha = 1$ versus the exact solution, $y(x) = -\ln(1+x)$. The value of $\alpha = 1$ is the only case for which we know the exact solution and our approximate solutions using the fractional iteration method are in good agreement with the exact values. It is to be noted that only three terms of ILM and ADM and third-term of VIM were used in evaluating the approximate solutions for Figures 4 and 5. From the numerical results in Figure 4, it is easy to conclude that our approximate solution using the ILM is more accurate than the approximate solutions obtained using the VIM and ADM. Figure 5 shows the approximate solutions obtained for (63) using the ILM, VIM and ADM when $\alpha = 0.75$ versus the exact solution of (63) when $\alpha = 1$. The results demonstrate that the ILM is more effective and accurate than the IM, VIM and ADM in solving these nonlinear fractional problems. One of the biggest advantages the ILM has over the ADM, VIM and IM is that it overcomes the difficulty that arises in calculating the Adomian polynomials, in identifying the Lagrange multiplier, and in the difficulty arising in calculating complicated integrals, respectively.

DISCUSSION AND CONCLUSIONS

In this work, we carefully proposed an efficient modification of the iterative method to handle nonlinear fractional differential equations. Efficient approximate solutions have been derived for fractional differential equations and the results have been shown remarkable performance.

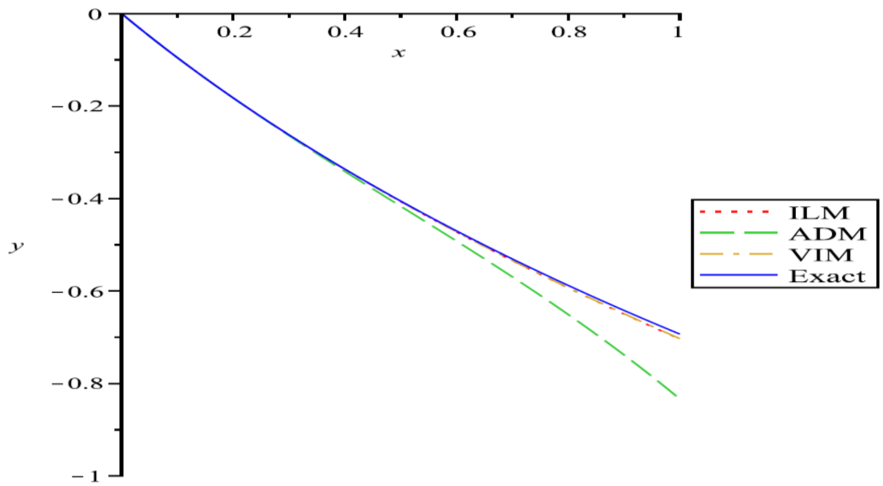

Figure 4. Approximate solutions for Example 4 when $\alpha = 1$ using ILM: Dat, ADM: Dash, VIM: Dash Dat and exact solution: solid line.

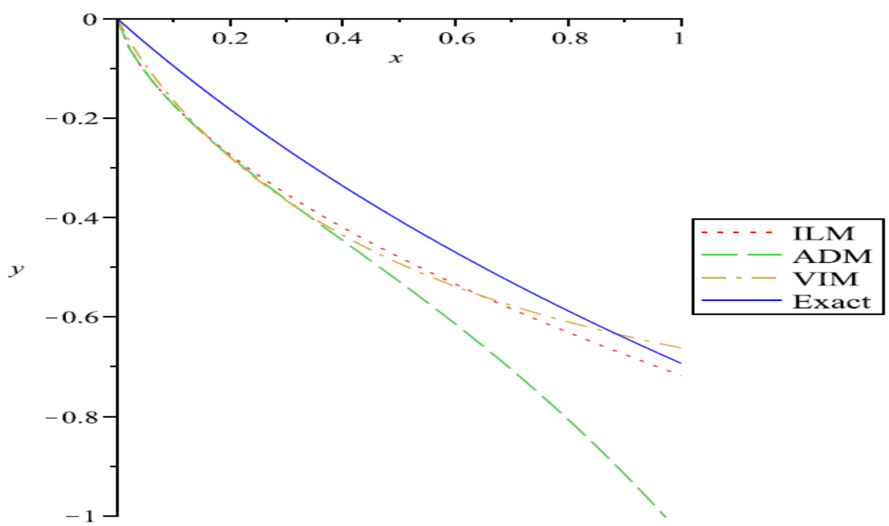

Figure 5. Approximate solutions for example 4 when $\alpha = 0.75$ using ILM: Dat, ADM: Dash, VIM: Dash Dat and exact solution: Solid line.

are two important points to make here. First, the proposed approach can provide the suitable approximate solution by using only a few numbers of iterations, as shown in Examples 1 to 4. Also it may be conclude that this approach require less computational work when compared with the standard iterative method as shown in Examples 1 to 3. Secondly, the new approach overcomes the difficulty arising in calculating complicated integrals, as shown in Example 4. Unlike ADM, the LIM method is free from the need to use Adomian polynomials. This method has no need for the Lagrange multiplier, correction functional, stationary conditions, the variational theory, etc., which eliminates the complications that exist in VIM.

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