International Journal of the Physical Sciences Vol. 5(12), pp. 1818-1823, 4 October, 2010
Available online at http://www.academicjournals.org/IJPS
ISSN 1992 - 1950 ©2010 Academic Journals

Full Length Research Paper

Hypersurfaces with harmonic curvature in a space of constant Hessian sectional curvature and Betti numbers of Hessian manifolds

Münevver Yildirim Yilmaz

Department of Mathematics, Faculty of Science and Arts, Firat University, 23119 Elaziğ, Turkey.
E-mail: myildirim@firat.edu.tr.

Accepted 27 September, 2010

In this paper, the notion of harmonic curvature for Hessian manifolds of constant Hessian sectional curvature is introduced and two theorems are proved on this subject. Betti numbers of Hessian manifolds that have special features are also studied.

Key words: Hypersurface, harmonic curvature, Hessian manifold, Betti numbers.

INTRODUCTION

According to O. Veblen, a famous American mathematician, " Geometry consists of a sequence of statements arranged in a certain logical order but void of all physical meaning. In order to apply them to nature we identify the undefined terms (points, lines, etc.) as names of recognizable objects.The unproved propositions (axioms) are then given a meaning, and we can ask whether they are true statements or not. If they are true, we expect that the theorems which are their logical consequences are also true and that the abstract geometry will take its place as a useful branch of physics ", Ritter (1997).

In fact in recent years we have witnessed a remarkable development in the interaction between geometry and physics. For instance the curvature of a manifold plays an important role in general relativity especially Ricci curvature is the keyterm in the Einstein field equations. There are many studies on this subject. For instance in El Naschie (2006), El Naschie studies the particle content of the standard model of high energy elementary particles. Also in El Naschie (2005), connections between Gödel's classical solutions of Einstein field equation and E-infinity were mentioned. On the other hand, the curvature of spacetime has been another research area for mathematicians and physicians for along time. Einstein's famous formula, which describes the relation between the curvature of spacetime and mass energy density, gives a great contribution science.

Moreover, another research area for scientists is hyperbolic geometry and its relevance with the universe. In the past two decades, the mathematicians discuss on this subject with different aspects. (Stakhov and Rozin, 2007; Abdel-All and Abd-Ellah, 2003; El Naschie, 2002, 2004; Yildirim Yilmaz and Bektaş, 2008).

Scientists working in the theoretical physics and applied mathematics have many objectives in common. One of the major issues of this field is Betti numbers. It is well-known a Betti number is the maximum number of cuts that can be made without dividing a surface into two seperate pieces. The Betti number of an object simply describes its features such as the number of holes and cavities that it possesses. Betti numbers have a wide scale of applications ranging from graph theory to electromagnetic fields. In this study we focus on the Betti numbers of a homogeneous Hessian manifold and also comment on the Betti numbers of Hessian manifolds that have special feature.

The present work consists of three parts. In the first part our purpose is to recall the basic concepts of Hessian manifolds and constructions of Hessian manifolds of constant Hessian sectional curvature which corresponds Euclidean space, sphere or hyperbolic space according to the sign of the curvature.

Let M^{n+1} be a flat affine manifold with flat affine connection D. Among Riemannian metrics on M^{n+1} there exists an important class of Riemannian metrics compatible with the flat affine connection D. A

Riemannian metric g on M^{n+1} is said to be Hessian metric if g is locally expressed by $g=D^2u$, where u is a local smooth function.We call such a pair (D,g) a Hessian structure on M^{n+1} and a triple (M^{n+1},D,g) a Hessian manifold, (Shima, 1980, 1986, 1995, 1997). Geometry of Hessian manifold is deeply related to Kählerian geometry and affine differential geometry Shima (1995).

It is well known that a compact convex hypersurface with constant mean curvature in a Euclidean space is a sphere. On the other hand, Simons (1968) has recently done an important suggestive contribution to the study of minimal submanifolds in a Riemannian manifold, in which he has given a formula for the Laplacian of the square of the norm of the second fundamental form of the submanifold. Under the stimulus of the Simons' study Do Cormo, Chern, Kobayashi (1970) and (Nomizu and Smyth, 1969), using the similar formula to that of Simons, have obtained some theorems on a compact minimal submanifold or a complete hypersurface with constant mean curvature in a Riemannian manifold of constant curvature. (Nakagawa and Yokote, 1972) generalize this result by applying Simons' formula to a compact hypersurface with constant scalar curvature in a Riemannian manifold of constant curvature. Then Omachi,(1986) has obtained some results in the case of hypersurfaces in a space of non-negative constant curvature making use of harmonic curvature.The following section is on hypersurfaces with harmonic curvature under the stimulus of above studies. We also prove two theorems on Hessian manifolds.

The last part of the study is on Betti numbers of Hessian manifolds that have special features.

Preliminaries

Let M^{n+1} be a Hessian manifold with Hessian structure (D,g).We express various geometric concepts for the Hessian structure *(D,g)*in terms of affine coordinate system $\{x^1,...,x^{n+1}\}$ with respect to D , i.e $Ddx^A=0$. Here *A, B, C*,... run from 1 to n+1.

The Hessian metric ;

$$g_{AB}=\frac{\partial^2 u}{\partial x^A \partial x^B}.$$

Let γ be a tensor field of type (1, 2) defined by

$$\gamma(X,Y)=\nabla_X Y-D_X Y,$$

where ∇ is the Riemannian connection for *g*. Then we have

$$\gamma^A_{BC}=\Gamma^A_{BC}=\frac{1}{2}g^{AD}\frac{\partial g_{DB}}{\partial x^C},$$

$$\gamma_{ABC}=\frac{1}{2}\frac{\partial g_{AB}}{\partial x^C}=\frac{1}{2}\frac{\partial^3 u}{\partial x^A\partial x^B\partial x^C},$$

$$\gamma_{ABC}=\gamma_{BAC}=\gamma_{CBA},$$

where Γ^A_{BC} are the Christoffel 's symbols of ∇. Define a tensor field *S* of type (1, 3) by

$$S=D\gamma$$

and call it the Hessian curvature tensor for (D, g). Then, we have

$$S^A_{BCE}=\frac{\partial\gamma^A_{BE}}{\partial x^C},$$

$$S^A_{BCE}=\frac{1}{2}\frac{\partial^4 u}{\partial x^A\partial x^B\partial x^C\partial x^E}-\frac{1}{2}g^{DF}\frac{\partial^3 u}{\partial x^A\partial x^C\partial x^D}-\frac{\partial^3 u}{\partial x^B\partial x^E\partial x^F},$$

$$S_{ABCE}=S_{AECB}=S_{CBAE}=S_{BAEC}=S_{CEAB}.$$

The Riemannian curvature tensor for ∇ ;

$$R^A_{BCE}=\gamma^A_{DC}\gamma^D_{BE}-\gamma^A_{DE}\gamma^D_{BC},$$

$$R_{ABCE}=\frac{1}{2}(S_{BACE}-S_{ABCE}). \quad (1)$$

Shima (1995).

Definition 1: For a non-zero contravariant symmetric tensor ξ_x of degree 2 at *x* we set

$$h(\xi_x)=\frac{\langle\varsigma(\xi_x),\xi_x\rangle}{\langle\xi_x,\xi_x\rangle}$$

and call it the Hessian sectional curvature in the direction ξ_x ,Shima(1995).

Theorem 1. Let *(M, D, g)* be a Hessian manifold of dimension ≥ 2. If the Hessian sectional curvature $h(\xi_x)$depends only *x* then *(M, D, g)* is of constant Hessian sectional curvature. *(M, D, g)* is of constant Hessian sectional curvature *c* if and only if

$$S_{ABCE}=\frac{c}{2}(g_{AB}g_{CE}+g_{AE}g_{CB}). \quad (2)$$

Shima(1995).

Corollary 1: If a Hessian manifold *(M, D, g)* is a space of constant Hessian sectional curvature *c*, then the

Riemannian manifold *(M, g)* is a space of constant sectional curvature $-\frac{c}{4}$, Shima (1995).

Constructions of Hessian manifolds of constant Hessian sectional curvature

In this section we shall construct, for each constant c, a Hessian manifold with constant Hessian sectional curvature c. We now recall the following result due to Shima and Yagi (1995) . Let *(M^{n+1}, D, g)* be a simply connected Hessian manifold. If g is complete, then *(M^{n+1},D,g)* is isomorphic to $(\Omega, \tilde{D}, \tilde{D}^2\varphi)$ where Ω is a convex domain in IR^{n+1}, $\tilde{D}$ is the canonical flat connection on IR^{n+1} and φ is a smooth convex function on Ω.

A. The case c = 0.

It is obvious that the Euclidean space (IR^{n+1}, $\tilde{D}$, g=(1/2) $\tilde{D}^2$ (∑ $(x^A)^2$}) is a simply connected Hessian manifold of constant Hessian sectional curvature 0.

B. The case c > 0.

Theorem 2. Let be a domain in R^{n+1} given by

$$x^{n+1} > \frac{c}{2}\sum_{A=1}^{n}(x^A)^2,$$

where c is a positive constant, and let be a smooth function on defined by

$$\varphi = -\frac{1}{c}\log\{x^{n+1} - \frac{c}{2}\sum_{A=1}^{n}(x^A)^2\}.$$

Then $(\Omega, \tilde{D}, g = \tilde{D}^2\varphi)$ is a simply connected Hessian manifold of positive constant Hessian sectional curvature c. As Riemannian manifold (Ω,g) is isometric to the hyperbolic space (H($-\frac{c}{4}$),g) of constant sectional curvature -c/4;

$$H = \{(\xi^1, \ldots \xi^n, \xi^{n+1}) \in IR^{n+1} \,\big|\, \xi^{n+1} > 0\},$$

$$g = \frac{1}{(\xi^{n+1})^2}\left\{\sum_{A=1}^{n}(d\xi^A)^2 + \frac{4}{c}(d\xi^{n+1})^2\right\}.$$

C. The case c<0.

Theorem 3. Let φ be a smooth function on IR^{n+1} defined by

$$\varphi = -\frac{1}{c}\log(\sum_{A=1}^{n+1} e^{-cx^A} + 1),$$

where c is a negative constant. Then (IR^{n+1}, $\tilde{D}$, g= $\tilde{D}^2\varphi$) is a simply connected Hessian manifold of negative constant Hessian sectional curvature c. The Riemannian manifold (IR^{n+1},g) is isometric a domain of the sphere

$\sum_{i=1}^{n+2} \xi^2{}_A = -\frac{4}{c}$ defined by $\xi_A > 0$ for all A.Shima (1995).

For the proof of the theorems we refer to Shima(1995).

Hypersurfaces with harmonic curvature in a space of constant Hessian sectional curvature

Definition 2. A Riemannian curvature tensor *R* is said to be harmonic if it satisfies

$$\nabla_i R_{jk} - \nabla_j R_{ik} = 0,$$

where R_{ij} means the component of the Ricci tensor, i.e. $R_{jk} = R_{jik}{}^i$. If the Ricci tensor is parallel, the curvature is harmonic. However, the converse is generally not true, Omachi (1986).

We consider a hypersurface M^n with harmonic curvature immerse in an (n+1)- dimensional Hessian manifold M^{n+1}(c) of constant Hessian sectional curvature c by an isometric immersion ϕ:M^n →M^{n+1}(c) and denote the induced metric tensor,the induced metric connection, the curvature tensor of M^n and the second fundamental form by g, ∇, R and h respectively. We assume that the mean curvature $trh = h^k{}_k$ is constant. Under these conditions the following formulas hold

$$R_{ijkl} = -\frac{c}{4}(g_{ik}g_{jl} - g_{il}g_{jk}) + h_{ik}h_{jl} - h_{il}h_{jk} \text{ (Gauss)} \quad (3)$$

$\nabla_i h_{jk} - \nabla_j h_{ik} = 0$ (Codazzi), (4)

$\nabla_i h^k{}_k = 0$ (Mean curvature constant) , (5)

$\nabla_i R_{jk} - \nabla_j R_{ik} = 0$ (harmonic curvature) , (6)

where the indices i,j,k,..., run from 1 to n.
The equation (6) implies that the scalar curvature is constant that is;

$\nabla_i R_k{}^k = 0$. (7)

On the other hand, we get from equation (3)

$$R_{jk} = (1-n)\frac{c}{4} g_{jk} + h_l{}^l h_{jk} - h_j{}^l h_{lk} . \quad (8)$$

For simplification, we shall write $h_{ij}{}^2, h_{ij}{}^3$,... instead of $h_1{}^k h_{kj}, h_2{}^k h_{kj}$,... And using equation (5)

$$\nabla_i R_{jk} = h_l{}^l \nabla_i h_{jk} - \nabla_i (h_{jk})^2 . \quad (9)$$

Hence we know from equations (4) and (9) that

$$\nabla_i (h_{jk})^2 - \nabla_j (h_{ik})^2 = 0 . \quad (10)$$

is equivalent to equation (6). It is easy to see

$$\nabla_i h^2{}_k{}^k = 0 \quad (11)$$

from equations (9), (5) and (7)
First, we shall give two equations about

$$\left\|\nabla h^2\right\|^2 = \nabla_i (h^l{}_j h_{lk}).\nabla^i (h^{jm} h_{mk}) ,$$

where $\nabla^i = g^{ik}\nabla_k$

Lemma 1.

$$\left\|\nabla h^2\right\|^2 = \frac{1}{2}\nabla_i\nabla^i (trh^4) n\frac{c}{4} trh^4 - trh\,trh^5 - \frac{c}{4}(trh^2)^2 + (trh^3)^2 \quad (12)$$

Proof:

$$\left\|\nabla h^2\right\|^2 = \nabla_i (h^2{}_{jk}).\nabla^i (h^{2jk})$$
$$= \nabla_i (h^2{}_{jk}\nabla^i h^{2jk}) - (h^2{}_{jk}\nabla_i\nabla^i h^{2jk}) \quad (13)$$
$$= \frac{1}{2}\nabla^i\nabla_i (trh^4) - h^2{}_{jk}\nabla_i\nabla^i h^{2jk}$$

holds. Using equations (3),(5),(10),(11) and the Ricci identity we get

$$\nabla_i\nabla^i h^{2jk} = \nabla_i\nabla^j h^{2ik}$$
(14)
$$= \nabla^j\nabla_i h^{2ik} + R_i{}^{jil} h^2{}_{lk} + R_i{}^{jkl} h_l{}^{2i}$$
$$= \left\{(1-n)\frac{c}{4} g^{jl} + h_i{}^i h^{jl} - h_i{}^l h^{ji} h_l{}^2{}_k\right\}$$
$$+ \left\{-\frac{c}{4}(\delta_i{}^k g^{jl} - \delta^l{}_i g^{jk} + h_i{}^k h^{jl} - h_i{}^l h^{jk}\right\} h_i{}^{2l} .$$

From equations (13) and (14) the equation (12) follows.

Lemma 2.

$$\left\|\nabla h^2\right\|^2 = \frac{1}{3}\nabla_i\nabla^i (trh^4) + \frac{4}{3}[trh^4(trh^2 + n\frac{c}{4}) + trh(-\frac{c}{4}trh^3 - trh^5)] . \quad (15)$$

Proof. We remark that

$$\nabla_i h_{jk}{}^2 = 2h_j{}^m \nabla_i h_{mk} \quad (16)$$

holds. In fact

$$\nabla_i h_{jk}{}^2 = \left(\nabla_i h_j{}^m\right) h_{mk} + h_j{}^m \nabla_i h_{mk} \quad (17)$$

implies together with equations (4) and (10) that the second term of the right side of equation (7) is symmetric with respect to i,j and k, from which equation (16) follows. Hence

$$\|\nabla h\|^2 = \left(\nabla_i h^2{}_{jk}\right)\left(\nabla^i h^{2jk}\right)$$
$$= 4h^{jl}{}_m (\nabla_i h_{lk})\left(\nabla^i h^{mk}\right) . \quad (18)$$

On the other hand, we have

$$\left\|\nabla h^2\right\|^2 = 2h^l{}_j (\nabla_i h_{lk})(\nabla^i h^{2jk}) \quad (19)$$
$$= 2\nabla^i (h^{3kl}\nabla_i h_{lk}) - 2h^{2jk}(\nabla^i h_{jl})(\nabla^i h_{lk}) - 2h^{3kl}\nabla_i\nabla^i h_{lk}$$

by equation (16). The first and second terms of the right side of equation (19) are reduced to

$\frac{1}{2}\nabla_i\nabla^i h^k{}_k$ and $-\frac{1}{2}\left\|\nabla h^2\right\|^2$

by equation (14), respectively. Using equations (3),(4),(5), (11) and the Ricci identity, we get

$$\nabla^i \nabla_i h_{lk} = \nabla^i \nabla_l h_{ik} \quad (20)$$
$$= \nabla_l \nabla^i h_{ik} + R^{im}{}_{li} h_{mk} + R^{im}{}_{lk} h_{im}$$
$$= \left\{ (1-n)\frac{c}{4}\delta^m{}_l + h^i{}_i h^m{}_l - h^{im} h_{li} \right\} h_{mk}$$
$$+ \left\{ -\frac{c}{4}\left(\delta_k{}^i \delta_l{}^m - g^{im} g_{lk} \right) + h^i{}_k h^m{}_l - h^{im} h_{lk} \right\} h_{im}.$$

Therefore the third term of equation (19) can be reduced to

$$2\left(n\frac{c}{4} trh^4 - trhtrh^5 - \frac{c}{4} trhtrh^3 + trh^2 trh^4 \right).$$

Finally, equation (19) becomes the equation (15).

We eliminate the term of $trhtrh^5$ from equations (12) and (15) and we have

$$\left\| \nabla h^2 \right\|^2 = \nabla_i \nabla^i (trh^4) + 4\left[\left(trh^3 \right)^2 - trh^2 trh^4 - \frac{c}{4}(trh^2)^2 + \frac{c}{4} trhtrh^3 \right] \quad (21)$$

Taking the suitable orthonormal frame, we diagonize h and denote its diagonal components by $\alpha_1, ..., \alpha_n$. Then, the equation (21) can be written as

$$\left\| \nabla h^2 \right\|^2 = \Delta\left(trh^4 \right) - 2\sum_{i \neq j} \alpha_i \alpha_j (\alpha_i \alpha_j - \frac{c}{4})(\alpha_i - \alpha_j)^2 \quad , \quad (22)$$

where Δ means $\nabla^i \nabla_i$.

In the case of c=0 it is obvious that the Euclidean space $(IR^{n+1}, \tilde{D}, g = \left(\frac{1}{2}\right)\tilde{D}^2 \{\sum (x_i)^2\})$ is simply connected Hessian manifold of constant Hessian sectional curvature 0. If trh^4 is constant on M^n, then all the non-zero eigenvalues of h have a constant unique value on M^n by (5) and (22). Therefore we can apply K. Nomizu and B.Smyth's (1969) arguement if M^n is complete. Thus the following theorem is proved.

Theorem 4. Let M^n be a connected hypersurface with harmonic curvature isometrically immersed in (n+1)-dimensional Hessian manifold $M^{n+1}(c)$ by an isometric immersion ϕ with constant mean curvature. We denote the second fundamental form by h

(i) If M^n is complete and trh^4 is constant on $\phi(M^n)$ then $\phi(M^n)$ is of the form $S^p \times E^{n-p}$, 0≤p≤n.

(ii) If M^n is compact, then $\phi(M^n)$ is S^n.

In order to consider the case of c<0 according to Theorem 2.3 and following equation

$$\left\| \nabla h^2 \right\|^2 = \frac{1}{2}\left(trh^2 \right) - \frac{1}{2}\sum_{i \neq j} \left(\alpha_i - \alpha_j \right)^2 \left(\alpha_i \alpha_j - \frac{c}{4} \right) \quad (23)$$

This equation follows from equations (3),(4),(5) and the Ricci identity, and in our situation, the first term of the right side of equation (23) vanished by equation (11). So, we have

$$\left\| \nabla h^2 \right\| = -\frac{1}{2}\sum_{i \neq j} \left(\alpha_i - \alpha_j \right)^2 \left(\alpha_i \alpha_j - \frac{c}{4} \right). \quad (24)$$

And from equations (22) and (24), we have

$$\left\| \nabla h^2 \right\|^2 - c\left\| \nabla h \right\|^2 = \Delta\left(trh^4 \right) - 2\sum_{i \neq j} (\alpha_i \alpha_j - \frac{c}{4})(\alpha_i - \alpha_j)^2 \ ,$$

and following theorem is proved as in the case of c=0.

Theorem 5. Let M^n be a connected hypersurface with harmonic curvature isometrically immersed in (n+1)-dimensional Hessian manifold $M^{n+1}(c)$ by an isometric immersion ϕ with constant mean curvature. If M^n is compact, then $\phi(M^n)$ is the form $S^p(r) \times S^{n-p}(s)$, 0≤p≤n where r= α^2-(c/4), s= β^2-(c/4) and α and β satisfy $\alpha\beta - \frac{c}{4}$ =0 and p α +(n-p) β =*trh.*

Hessian manifolds and Betti numbers

Definition 3. Let M be a Hessian manifold. A diffeomorphism of M onto itself is called an automorphism o M if it preserves both the flat affine structure and the Hessian metric. The set of all automorphism of M denoted by Aut(M), forms a Lie group. A Hessian manifold M is said to be homogeneous if the group Aut(M) acts transitively on M, Shima (1980).

Theorem 6. Let M be a connected homogeneous Hessian manifold. Then we have:

(1) The domain of definition E_X for the exponential mapping $\exp_X$ at $x \in M$ given by the flat affine structure is a convex domain. Moreover E_X is the universal covering manifold of M with affine projection $\exp_X$: $E_X \rightarrow$M.

(2) The universal covering manifold E_X of M has a decomposition $E_X = E^0{}_X + E_X{}^+$ where $E^0{}_X$ is a uniquely determined vector subspace of the tangent space $T_X M$ of M at x and full straight line. Thus E_X admits a unique fibering with the following properities:

(i) The base space is $E_X{}^+$.

(ii) The projection p: $E_X \to E^+{}_X$ is given by the canonical projection from $E_X = E^0{}_X + E_X{}^+$ onto E_{x}⁺.

(iii) The fiber $E^0{}_X + v$ through $v \in E_X$ is a characterized as the set of all points which can be joined with *v* by full straight lines contained in E_X. Moreover each fiber is an affine subspace of $T_X M$ and is an Euclidean space with respect to the induced metric.

(iv) Every automorphism of E_X is fiber preserving.

(v) The group of automorphisms of E_X which preserve every fiber, acts transitively on the fibers, Shima (1980).

Corollary 2. A compact connected homogeneous Hessian manifold is a Euclidean torus, Shima (1980). Acoording to the Definition 3 and Corollary 2 the following theorem holds.

Theorem 7. Let M be a compact connected homogeneous Hessian manifold. Then the Betti number of M equals to 2.

Theorem 8. Let φ be a smooth function on IR^3 defined by

$$\varphi = -\left(\frac{1}{c}\right)\log\left\{x^3 - \frac{c}{2}\sum_{A=1}^{2}\left(x^A\right)^2\right\}.$$

Let *(M,D,* $g = D^2\varphi$) be a three dimensional Hessian manifold with constant Hessian sectional curvature c. If g is complete and c<0, the Betti number of M equals to 0.

Proof. We now recall the following result due to Shima and Yagi (1997) Let (M^{n+1}, D, g) be a simply connected Hessian manifold. If g is complete, then (M^{n+1},D,g) is isomorphic to ($\Omega, \tilde{D}, \tilde{D}^2\varphi$) where Ω is a convex domain in IR^{n+1}, $\tilde{D}$ is the canonical flat connection on IR^{n+1} and φ is a smooth convex function on Ω. And according to Theorem 3 if c<0, the Riemannian manifold (R^3,g) is isometric a domain of the sphere $\sum_{i=1}^{4}\xi^2{}_A = -\frac{4}{c}$ defined by ξ_A >0 for all *A*. This completes the proof.

Theorem 9. Let *M* be a connected hypersurface with harmonic curvature isometrically immersed in 3-dimensional Hessian manifold $M^3(c)$ by an isometric immersion ϕ with constant mean curvature.Then the Betti number of M equals to 0 if M is compact.

Proof. Using Theorem 4, the proof can be made immediately.

Conclusion

The aim of the present work is to find the relation between a Hessian manifold and a hyperbolic space, hypersurfaces with harmonic curvature in a space of constant Hessian sectional curvature on one hand and Betti numbers of a special type Hessian manifolds on the other. We discuss the constructions of a Hessian manifold of constant Hessian sectional curvature, and consider the hypersurfaces in terms of harmonic curvature. Both subjects also have physical applications. Defining Hessian manifolds of fuzzy type and Betti numbers of it is still a great puzzle for the author.

REFERENCES

Abdel-All NH, Abd-Ellah H (2003). Deformation of a kinematic surface in a hyperbolic space. Chaos, Solitons and Fractals, 15: 631-638.

Chern SS, Do Cormo M, Kobayashi S (1970). Minimal submanifolds of a sphere with second fundamental form of constant length. Funct. Analysis and Related fields, Springer-Verlag.59-75.

El Naschie MS (2002). Wild topology, hyperbolic geometry and fusion algebra of high energy particle physics. Chaos, Solitons and Fractals, 13: 1935-1945.

El Naschie MS (2004). A review of the E - infinity theory and the mass spectrum of high energy particle physics. Chaos, Solitons and Fractals, 19: 209-236.

El Naschie MS (2005). Gödel universe, dualities and high energy particles in E-infinity. Chaos, Solitons & Fractals. 25(3): 759-764.

El Naschie MS (2006). Is Einstein's general field equation more fundamental than quantum field theory and particle physics? Chaos,Solitons and Fractals, 30(3): 525-531.

Nakagawa H, Yokote I (1972). On hypersurfaces with constant scalar curvature in a Riemannian manifold of constant curvature. Kodai Math. Sem. Rep., 24: 471-481.

Nomizu K, Smyth B (1969). A formula of Simons' type and hypersurfaces with constant mean curvature, J. Diff. Geo., 3: 367-377.

Omachi E (1986). Hypersurfaces with harmonic curvature in a space of constant curvature. Kodai Math. J., 9: 170-174.

Ritter J (2007). Abstract Geometry, a Useful Branch of Physics.The Princeton School in the 1920s, Universite′ de Paris 8/IMJ, pp. 1-22.

Shima H (1980). Homogeneous Hessian manifolds, Ann. Inst. Fourier (Grenoble), 30(3): 91-128.

Shima H (1986). Vanishing theorems for compact Hessian manifolds. Ann. Inst. Fourier (Grenoble), 36-3: 183-205.

Shima H (1995). Hessian manifolds of constant Hessian sectional curvature. J. Math. Soc. Jpn, 47(4): 735-753.

Shima H, Yagi K (1997). Geometry of Hessian manifolds, Diff. Geo. Its Appl., 7: 277-290.

Simons J (1968). Minimal variety in Riemannian manifolds. Ann. Math., 88: 62-105.

Stakhov A, Rozin B (2007). The "golden"hyperbolic models of Universe. Chaos, Solitons and Fractals, 34: 159-171.

Yildirim Yilmaz M, Bektaş M (2008). A survey on curvatures of Hessian manifolds, Chaos, Solitons and Fractals, 38: 620-630.