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On solving the nonlinear Schrödinger-Boussinesq equation and the hyperbolic Schrödinger equation by using the $\left(\frac{G'}{G},\frac{1}{G}\right)$-expansion method

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The propagation of the optical solitons is usually governed by the nonlinear Schrödinger equations. In this article, the two variable $\left(\frac{G'}{G},\frac{1}{G}\right)$-expansion method is employed to construct the exact traveling wave solutions with parameters of two nonlinear partial differential equations (PDEs) namely, the (1+1)-dimensional nonlinear Schrödinger-Boussinesq system and the (2+1)-dimensional hyperbolic nonlinear Schrödinger (HNLS) equation which describe the propagation of optical pulses in optic fibers. When the parameters are replaced by special values, the solitary wave solutions of these equations are found from the traveling waves.

Key words: The two variable $\left(\frac{G'}{G},\frac{1}{G}\right)$-expansion method, nonlinear Schrödinger-Boussinesq system, hyperbolic nonlinear Schrödinger (HNLS) equation, exact traveling wave solutions, solitary wave solutions.

INTRODUCTION

In the recent years, investigations of exact solutions to nonlinear partial differential equation (PDEs) play an important role in the study of nonlinear physical phenomena. Many powerful methods have been presented, such as the inverse scattering method (Ablowitz and Clarkson, 1991), the Hirota bilinear transform method (Hirota, 1971), the truncated Painleve expansion method (Weiss et al., 1983; Kudryashov, 1988, 1990, 1991), the Backlund transform method (Miura, 1978; Rogers and Shadwick, 1982), the exp-function method (He and Wu, 2006; Yusufoglu, 2008; Zhang, 2008; Bekir, 2009, 2010), the tanh-function method (Abdou 2010; Fan, 2000; Zhang and Xia, 2008; Yusufoglu and Bekir, 2008), the Jacobi elliptic function expansion method (Chen and Wang, 2005; Liu et al., 2001; Lu, 2005), the $\left(\frac{G'}{G}\right)$-expansion method (Wang et al., 2008; Zhang et al., 2008; Zayed and Gepreel, 2009; Zayed, 2009; Bekir, 2008; Ayhan and Bekir, 2012; Kudryashov, 2010a, b; Aslan, 2010; Zayed, 2010;), the modified $\left(\frac{G'}{G}\right)$-expansion method (Zhang et al., 2011),

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the $\left(\frac{G'}{G}, \frac{1}{G}\right)$-expansion method (Li et al., 2010; Zayed and Abdelaziz, 2012; Zayed et al., 2012; Zayed and Alurrfi, 2014a, b), the Riccati equation method (Ma and Fuchssteiner, 1996), the bilinear method (Ma, 2011, 2013), the transformed rational function method (Ma and Lee, 2009), the multiple exp-function method (Ma and Zhu, 2012) and so on.

The key idea of the one variable $\left(\frac{G'}{G}\right)$-expansion method is that the exact solutions of nonlinear PDEs can be expressed by a polynomial in one variable $\left(\frac{G'}{G}\right)$ in which $G = G(\xi)$ satisfies the second order linear ordinary differential equation (ODE) $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$, where λ, μ are constants and $' = \frac{d}{d\xi}$. The key idea of the two variable $\left(\frac{G'}{G}, \frac{1}{G}\right)$-expansion method is that the exact traveling wave solutions of nonlinear PDEs can be expressed by a polynomial in two variables $\left(\frac{G'}{G}\right)$ and $\left(\frac{1}{G}\right)$ in which $G = G(\xi)$ satisfies the second order linear ODE $G''(\xi) + \lambda G(\xi) = \mu$, where λ and μ are constants. The degree of this polynomial can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in the given nonlinear PDEs. The coefficients of this polynomial can be obtained by solving a set of algebraic equations resulted from the process of using this method. Recently, Li et al. (2010) have applied the $\left(\frac{G'}{G}, \frac{1}{G}\right)$-expansion method and determined the exact solutions of the nonlinear Zakharov equations, while Zayed and Abdelaziz (2012), Zayed et al. (2012), and Zayed and Alurrfi (2014a, b), respectively have used this method to find the exact solutions of the nonlinear combined KdV-mKdV equation, the nonlinear Kadomtsev-Petviashvili equation, the nonlinear PDE for nanobioscinces and two higher order nonlinear evolution equations namely, the nonlinear Klein-Gordon equations and the nonlinear Pochhammer-Chree equations .

The objective of this paper is to apply the two variables $\left(\frac{G'}{G}, \frac{1}{G}\right)$-expansion method obtained in Li et al. (2010), Zayed and Abdelaziz (2012), Zayed et al. (2012), and Zayed and Alurrfi (2014a,b) to find the exact traveling wave solutions of the following two different nonlinear equations which are not yet discussed:

(i) The (1+1)-dimensional Schrödinger-Boussinesq system (SB-system) (Kilicman and Abazari, 2012):

$$\begin{aligned} &iu_t + u_{xx} - auv = 0, \\ &v_{tt} - v_{xx} + v_{xxxx} - b(|u|^2)_{xx} = 0, \end{aligned} \tag{1}$$

Where $t > 0$, $x \in [0, L]$, for some $L > 0$, and a, b are real constants. Here, *u* and *v* are, respectively a complex-valued and a real-valued function.

(ii) The (2+1)-dimensional hyperbolic nonlinear Schrödinger (HNLS) equation (Fen, 2012):

$$iu_y + \frac{1}{2}u_{xx} - \frac{1}{2}u_{tt} + |u|^2 u = 0, \tag{2}$$

Where *u(x,y,t)* is a complex-valued function which represents the slowly varying envelope of propagation, x is the dimensionless variable, y is the propagation coordinate and t is the time.

The SB-system (Equation 1) is considered as a model of interactions between short and intermediate long waves, which is derived in describing the dynamics of Langmuir soliton formation and interaction in a plasma (Makhankov, 1974) and diatomic lattice system (Yajima and Satsuma, 1979). The SB-system has been discussed in Kilicman and Abazari (2012) using the $\left(\frac{G'}{G}\right)$-expansion method and its exact solutions has been found. Equation 2 can be derived from optics (Gorz and Haelterman, 2008) and large-scale Rossby waves (Tan and Wu, 1993). Various types of HNLS equations describing time and space evolutions of slowly varying envelopes have wide applications in various branches of physics (Tang and Shukla 2007; Li, 2007). HNLS equation has been investigated in Fen (2012) using the theory of bifurcations of dynamical system and its exact solutions have been presented.

DESCRIPTION OF THE TWO VARIABLE $\left(\frac{G'}{G}, \frac{1}{G}\right)$-EXPANSION METHOD

Before the main steps of this method are described, the following remarks are needed (Li et al., 2010; Zayed and Abdelaziz, 2012; Zayed et al., 2012; Zayed and Alurrfi, 2014a, b):

Remark 1

If the second order linear ODE is considered:

$$G''(\xi) + \lambda G(\xi) = \mu, \tag{3}$$

and set $\phi = \frac{G'}{G}$, $\psi = \frac{1}{G}$, then we get:

$$\phi' = -\phi^2 + \mu\psi - \lambda, \ \psi' = -\phi\psi. \quad (4)$$

Where λ and μ are constants while $' = \frac{d}{d\xi}$.

Remark 2

If $\lambda < 0$, then the general solution of Equation 3 has the form:

$$G(\xi) = A_1 \sinh(\xi\sqrt{-\lambda}) + A_2 \cosh(\xi\sqrt{-\lambda}) + \tfrac{\mu}{\lambda}, \quad (5)$$

Where A_1 and A_2 are arbitrary constants. Consequently, we have

$$\psi^2 = -\frac{\lambda}{\lambda^2\sigma_1 + \mu^2}(\phi^2 - 2\mu\psi + \lambda), \quad (6)$$

Where $\sigma_1 = A_1^2 - A_2^2$

Remark 3

If $\lambda > 0$, then the general solution of Equation 3 has the form:

$$G(\xi) = A_1 \sin(\xi\sqrt{\lambda}) + A_2 \cos(\xi\sqrt{\lambda}) + \tfrac{\mu}{\lambda}, \quad (7)$$

and hence

$$\psi^2 = \frac{\lambda}{\lambda^2\sigma_2 - \mu^2}(\phi^2 - 2\mu\psi + \lambda), \quad (8)$$

Where $\sigma_2 = A_1^2 + A_2^2$

Remark 4

If $\lambda = 0$, then the general solution of Equation 3 has the form:

$$G(\xi) = \tfrac{\mu}{2}\xi^2 + A_1\xi + A_2, \quad (9)$$

and hence

$$\psi^2 = \frac{1}{A_1^2 - 2\mu A_2}(\phi^2 - 2\mu\psi). \quad (10)$$

Suppose we have the following nonlinear evolution equation.

$$F(u, u_t, u_x, u_{xx}, ...) = 0, \quad (11)$$

Where F is a polynomial in $u(x,t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following, the main steps of the $\left(\frac{G'}{G}, \frac{1}{G}\right)$-expansion method are given (Li et al., 2010; Zayed and Abdelaziz, 2012; Zayed et al., 2012; Zayed and Alurrfi, 2014a, b):

Step 1

The traveling wave transformation

$$u(x,t) = u(\xi), \qquad \xi = x - Ct, \quad (12)$$

Where C is a constant, reduces Equation 11 to an ODE in the form:

$$P(u, u', u'', ...) = 0, \quad (13)$$

Where *P* is a polynomial of $u(\xi)$ and its total derivatives with respect to ξ.

Step 2

Assuming that the solution of Equation 13 can be expressed by a polynomial in the two variables ϕ and ψ as follows:

$$u(\xi) = a_0 + \sum_{i=1}^{N} a_i\phi^i + \sum_{i=1}^{N} b_i\phi^{i-1}\psi, \quad (14)$$

Where a_0, a_i and b_i $(i = 1, 2, ..., N)$ are constants to be determined later satisfying $a_N^2 + b_N^2 \neq 0$.

Step 3

Determine the positive integer N in Equation 14 by using the homogeneous balance between the highest-order derivatives and the nonlinear terms in Equation 13. More precisely we define the degree of $u(\xi)$ as $D[u(\xi)] = N$ which gives rise to the degree of other expressions as follows:

$$D\left[\frac{d^q u}{d\xi^q}\right] = N + q,$$
$$D\left[u^p\left(\frac{d^q u}{d\xi^q}\right)^s\right] = Np + s(q+N). \quad (15)$$

Therefore, we can get the value of N in Equation 14.

Step 4

Substitute Equation 14 into Equation 13 along with Equations 4 and 6, the left- hand side of Equation 13 can be converted into a polynomial in ϕ and ψ, in which the degree of ψ is not longer than 1. Equating each coefficients of this polynomial to 0, yields a system of algebraic equations which can be solved by using the Maple or Mathematica to get the values of $a_i, b_i, C, \mu, A_1, A_2$ and λ where $\lambda < 0$. Similarly, substitute Equation 14 into Equation 13 along with Equations 4 and 8 for $\lambda > 0$ or Equations 4 and 10 for $\lambda = 0$, we obtain the exact solutions of Equation 13 expressed by hyperbolic functions, trigonometric functions and rational functions, respectively.

APPLICATIONS

Here, the method described earlier is applied to find the exact traveling wave solutions of Equations 1 and 2 which are very important in the mathematical physics and have been paid attention by many researchers.

Example 1

The (1+1)-dimensional nonlinear SB-system (Equation 1)

We start with the (1+1)-dimensional nonlinear SB-system (Equation 1). Assume that the solution of Equation 1 can be written as:

$$u(x,t) = U(\xi)e^{i\eta},$$
$$v(x,t) = V(\xi), \quad (16)$$

Where $\xi = kx + \omega t$, $\eta = px + qt$ and k, ω, p, q are constants, $i = \sqrt{-1}$. Substituting Equation 16 into Equation 1, we have the following system of nonlinear ODEs:

$$k^2U'' + i(2kp+\omega)U' - aUV - (p^2+q)U = 0, \quad (17)$$

$$(\omega^2 - k^2)V'' + k^4V^{(4)} - bk^2(U^2)'' = 0. \quad (18)$$

Integrating Equation 18 twice and taking integration constants to be 0, the Equations 17 to 18 reduces to the following system:

$$k^2U'' + i(2kp+\omega)U' - aUV - (p^2+q)U = 0, \quad (19)$$

$$(\omega^2 - k^2)V + k^4V'' - bk^2U^2 = 0. \quad (20)$$

Considering the homogeneous balance between the highest order derivatives and the nonlinear terms in Equations 19 and 20, we obtain $N = M = 2$. Consequently, Equations 19 and 20 have the formal solutions:

$$U(\xi) = \alpha_0 + \alpha_1\phi(\xi) + \alpha_2\phi^2(\xi) + \beta_1\psi(\xi) + \beta_2\phi(\xi)\psi(\xi) \quad (21)$$

$$V(\xi) = c_0 + c_1\phi(\xi) + c_2\phi^2(\xi) + d_1\psi(\xi) + d_2\phi(\xi)\psi(\xi) \quad (22)$$

Where $\alpha_0, \alpha_1, \alpha_2, \beta_1, \beta_2, c_0, c_1, c_2, d_1$ and d_2 are constants to be determined later satisfying $\alpha_2^2 + \beta_2^2 \neq 0$, $c_2^2 + d_2^2 \neq 0$. There are three cases to be discussed as follows:

Case 1: Hyperbolic function solutions ($\lambda < 0$)

If $\lambda < 0$, substituting Equations 21 and 22 into Equations 19 and 20 and using Equations 4 and 6, the left-hand sides are converted into polynomial in ϕ and ψ. Setting each coefficient of this polynomial to 0, yields a system of algebraic equations in $\alpha_0, \alpha_1, \alpha_2, \beta_1, \beta_2, c_0, c_1, c_2, d_1, d_2, \mu, \lambda, \omega, k, p$ and q as follows:

$$-a\alpha_2c_2 + 6k^2\alpha_2 + \frac{a\beta_2d_2\lambda}{\lambda^2\sigma_1 + \mu^2} = 0,$$

$$-2i(2kp+\omega)\alpha_2 + 2k^2\alpha_1 - a\alpha_1c_2 - a\alpha_2c_1 - \frac{\lambda}{\lambda^2\sigma_1+\mu^2}\left(-a\beta_1d_2 - a\beta_2d_1 - 6k^2\beta_2\mu\right) = 0,$$

$$-a\alpha_2d_2 + 6k^2\beta_2 - a\beta_2c_2 = 0,$$

$$-(p^2+q)\alpha_2 - i(2kp+\omega)\alpha_1 - a\alpha_0c_2 - a\alpha_1c_1 - a\alpha_2c_0 + 8k^2\alpha_2\lambda + \frac{\lambda^2a\beta_2d_2}{\lambda^2\sigma_1+\mu^2}$$
$$-\frac{\lambda}{\lambda^2\sigma_1+\mu^2}\left(-a\beta_1d_1 + i(2kp+\omega)\beta_2\mu + k^2\left(2\alpha_2\mu^2 - \beta_1\mu\right)\right) = 0,$$

$$-2i(2kp+\omega)\beta_2 - a\alpha_1d_2 - a\alpha_2d_1 - k^2\left(10\alpha_2\mu - 2\beta_1\right) - a\beta_1c_2 - a\beta_2c_1 - \frac{2\lambda\mu a\beta_2d_2}{\lambda^2\sigma_1+\mu^2} = 0,$$

$$-(p^2+q)\alpha_1-2i\,(2kp+\omega)\alpha_2\lambda-a\alpha_0 c_1-a\alpha_1 c_0+2k^2\alpha_1\lambda$$
$$+\frac{\lambda^2}{\lambda^2\sigma_1+\mu^2}\left(a\beta_1 d_2+a\beta_2 d_1+6k^2\beta_2\mu\right)=0,$$

$$-a\alpha_0 d_2-a\alpha_1 d_1+k^2\left(-3\alpha_1\mu+5\beta_2\lambda\right)-a\beta_1 c_1-a\beta_2 c_0-(p^2+q)\beta_2+i\,(2kp+\omega)\left(-\beta_1+2\alpha_2\mu\right)$$
$$+\frac{2\lambda\mu}{\lambda^2\sigma_1+\mu^2}\left(-a\beta_1 d_2-a\beta_2 d_1-6k^2\beta_2\mu\right)=0,$$

$$k^2\lambda\left(\beta_1-4\alpha_2\mu\right)+i\,(2kp+\omega)\left(-\beta_2\lambda+\alpha_1\mu\right)-a\alpha_0 d_1-(p^2+q)\beta_1-a\beta_1 c_0$$
$$+\frac{2\lambda\mu}{\lambda^2\sigma_1+\mu^2}\left(-a\beta_1 d_1+i\,(2kp+\omega)\beta_2\mu+k^2\left(2\alpha_2\mu^2-\beta_1\mu\right)\right)=0,$$

$$-i\,(2kp+\omega)\lambda\alpha_1+2k^2\alpha_2\lambda^2-a\alpha_0 c_0-\frac{\lambda^2}{\lambda^2\sigma_1+\mu^2}\left(-a\beta_1 d_1+i\,(2kp+\omega)\beta_2\mu+k^2\left(2\alpha_2\mu^2-\beta_1\mu\right)\right)$$
$$-(p^2+q)\alpha_0=0,$$

$$6k^4c_2-bk^2\alpha_2^2+\frac{bk^2\beta_2^2\lambda}{\lambda^2\sigma_1+\mu^2}=0,$$

$$2k^4c_1-2bk^2\alpha_1\alpha_2+\frac{\lambda}{\lambda^2\sigma_1+\mu^2}\left(6k^4d_2\mu+2bk^2\beta_1\beta_2\right)=0,$$

$$-2bk^2\alpha_2\beta_2+6k^4d_2=0,$$

$$\left(\omega^2-k^2\right)c_2+8k^4c_2\lambda-bk^2\left(2\alpha_0\alpha_2+\alpha_1^2\right)-\frac{\lambda}{\lambda^2\sigma_1+\mu^2}\left(k^4\left(2c_2\mu^2-d_1\mu\right)-bk^2\beta_1^2\right)$$
$$+\frac{bk^2\beta_2^2\lambda^2}{\lambda^2\sigma_1+\mu^2}=0,$$

$$-2bk^2\left(\alpha_1\beta_2+\alpha_2\beta_1\right)+k^4\left(2d_1-10c_2\mu\right)-\frac{2bk^2\beta_2^2\lambda\mu}{\lambda^2\sigma_1+\mu^2}=0,$$

$$\left(\omega^2-k^2\right)c_1+2k^4c_1\lambda-2bk^2\alpha_0\alpha_1+\frac{\lambda^2}{\lambda^2\sigma_1+\mu^2}\left(6k^4d_2\mu+2bk^2\beta_1\beta_2\right)=0,$$

$$k^4\left(-3c_1\mu+5d_2\lambda\right)-2bk^2\left(\alpha_0\beta_2+\alpha_1\beta_1\right)+\left(\omega^2-k^2\right)d_2-\frac{2\lambda\mu}{\lambda^2\sigma_1+\mu^2}\left(6k^4d_2\mu+2bk^2\beta_1\beta_2\right)=0,$$

$$k^4\lambda\left(d_1-4c_2\mu\right)-2bk^2\alpha_0\beta_1+\left(\omega^2-k^2\right)d_1+\frac{2\mu\lambda}{\lambda^2\sigma_1+\mu^2}\left(k^4\left(2c_2\mu^2-d_1\mu\right)-bk^2\beta_1^2\right)=0,$$

$$\left(\omega^2-k^2\right)c_0+2k^4\lambda^2c_2-bk^2\alpha_0^2-\frac{\lambda^2}{\lambda^2\sigma_1+\mu^2}\left(k^4\left(2c_2\mu^2-d_1\mu\right)-bk^2\beta_1^2\right)=0.$$

On solving the above algebraic equations using the Maple or Mathematica, we get the following results:

Result 1

Consider

$$\mu=0,\ \lambda=\frac{k^2-\omega^2}{2k^4},\alpha_0=0,\alpha_1=0,\alpha_2=0,\beta_1=0,\beta_2=\pm3\sqrt{\frac{2\sigma_1\left(\omega^2-k^2\right)}{a\,b}},\qquad(23)$$
$$c_0=\frac{3\left(k^2-\omega^2\right)}{a\,k^2},c_1=0,c_2=\frac{6k^2}{a},d_1=0,d_2=0,\ p=-\frac{\omega}{2k},q=\frac{\omega^2-2k^2}{4k^2},$$

Where $\omega>k$.

From Equations 5, 16, 21, 22 and 23, we deduce the traveling wave solutions of SB-system (Equation 1) as follows:

$$u(\xi)=\left[\pm\frac{3\left(\omega^2-k^2\right)}{k^2}\sqrt{\frac{\sigma_1}{a\,b}}\left(\frac{A_1\cosh(\xi\sqrt{\frac{\omega^2-k^2}{2k^4}})+A_2\sinh(\xi\sqrt{\frac{\omega^2-k^2}{2k^4}})}{\left(A_1\sinh(\xi\sqrt{\frac{\omega^2-k^2}{2k^4}})+A_2\cosh(\xi\sqrt{\frac{\omega^2-k^2}{2k^4}})\right)^2}\right)\right]e^{i\eta},\qquad(24)$$

$$v(\xi)=\frac{3\left(k^2-\omega^2\right)}{ak^2}+\frac{3\left(\omega^2-k^2\right)}{ak^2}\left(\frac{A_1\cosh(\xi\sqrt{\frac{\omega^2-k^2}{2k^4}})+A_2\sinh(\xi\sqrt{\frac{\omega^2-k^2}{2k^4}})}{A_1\sinh(\xi\sqrt{\frac{\omega^2-k^2}{2k^4}})+A_2\cosh(\xi\sqrt{\frac{\omega^2-k^2}{2k^4}})}\right)^2,\qquad(25)$$

In particular, by setting $A_1=0$ and $A_2\neq0$ in Equations 24 and 25, we have the kink and bell shaped solitary solutions

$$u(\xi)=\left[\pm\frac{3\left(\omega^2-k^2\right)}{k^2\sqrt{a\,b}}i\,\mathrm{sech}(\xi\sqrt{\frac{\omega^2-k^2}{2k^4}})\tanh(\xi\sqrt{\frac{\omega^2-k^2}{2k^4}})\right]e^{i\eta},\qquad(26)$$

$$v(\xi)=\left[-\frac{3\left(\omega^2-k^2\right)}{ak^2}\mathrm{sech}^2(\xi\sqrt{\frac{\omega^2-k^2}{2k^4}})\right],\qquad(27)$$

while, if $A_1\neq0$ and $A_2=0$, then we have the anti-kink and anti-bell shaped solitary solutions

$$u(\xi)=\left[\pm\frac{3\left(\omega^2-k^2\right)}{k^2\sqrt{a\,b}}\mathrm{csch}(\xi\sqrt{\frac{\omega^2-k^2}{2k^4}})\coth(\xi\sqrt{\frac{\omega^2-k^2}{2k^4}})\right]e^{i\eta},\qquad(28)$$

$$v(\xi)=\left[\frac{3\left(\omega^2-k^2\right)}{ak^2}\mathrm{csch}^2(\xi\sqrt{\frac{\omega^2-k^2}{2k^4}})\right],\qquad(29)$$

where $\xi=kx+\omega t,\ \eta=-\frac{\omega}{2k}x+\frac{\omega^2-2k^2}{4k^2}t.$

Result 2

Consider

$$\mu=0,\ \lambda=\frac{k^2-\omega^2}{k^4},\alpha_0=\pm\frac{2\left(k^2-\omega^2\right)}{k^2\sqrt{ab}},\alpha_1=0,\alpha_2=\pm\frac{3k^2}{\sqrt{ab}},\beta_1=0,$$
$$\beta_2=\pm 3\sqrt{\frac{\sigma_1\left(\omega^2-k^2\right)}{ab}},c_0=\frac{2\left(k^2-\omega^2\right)}{ak^2},c_1=0,c_2=\frac{3k^2}{a},d_1=0, \tag{30}$$
$$d_2=\frac{3\sqrt{\sigma_1\left(\omega^2-k^2\right)}}{a},\ p=-\frac{\omega}{2k},q=\frac{4k^2-5\omega^2}{4k^2},$$

where $\omega>k$.

In this result, we deduce the traveling wave solution of SB-system (Equation 1) as follows:

$$u(\xi)=\left[\pm\frac{2\left(k^2-\omega^2\right)}{k^2\sqrt{ab}}\pm\frac{3\left(\omega^2-k^2\right)}{k^2\sqrt{ab}}\left(\frac{A_1\cosh(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})+A_2\sinh(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})}{A_1\sinh(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})+A_2\cosh(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})}\right)^2\right. \tag{31}$$
$$\left.\pm\frac{3\left(\omega^2-k^2\right)}{k^2}\sqrt{\frac{\sigma_1}{ab}}\left(\frac{A_1\cosh(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})+A_2\sinh(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})}{\left(A_1\sinh(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})+A_2\cosh(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})\right)^2}\right)\right]e^{i\eta},$$

$$v(\xi)=\frac{2\left(k^2-\omega^2\right)}{ak^2}+\frac{3\left(\omega^2-k^2\right)}{ak^2}\left(\frac{A_1\cosh(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})+A_2\sinh(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})}{A_1\sinh(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})+A_2\cosh(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})}\right)^2 \tag{32}$$
$$+\frac{3\left(\omega^2-k^2\right)\sqrt{\sigma_1}}{ak^2}\left(\frac{A_1\cosh(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})+A_2\sinh(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})}{\left(A_1\sinh(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})+A_2\cosh(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})\right)^2}\right),$$

In particular, by setting $A_1\neq 0$ and $A_2=0$ in Equations 31 and 32, we have the anti-kink and anti-bell shaped solitary solutions

$$u(\xi)=\frac{\left(\omega^2-k^2\right)}{k^2\sqrt{ab}}\left[\mp 2\pm 3\coth(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})\right. \tag{33}$$
$$\left.\times\left(\coth(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})+\operatorname{csch}(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})\right)\right]e^{i\eta},$$

$$v(\xi)=\frac{\left(\omega^2-k^2\right)}{ak^2}\left[-2+3\coth(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})\right. \tag{34}$$
$$\left.\times\left(\coth(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})+\operatorname{csch}(\xi\sqrt{\frac{\omega^2-k^2}{k^4}})\right)\right],$$

where $\xi=kx+\omega t$, $\eta=-\frac{\omega}{2k}x+\frac{4k^2-5\omega^2}{4k^2}t$.

Case 2: Trigonometric function solution ($\lambda>0$)

If $\lambda>0$, substituting Equations 21 and 22 into Equations 19 and 20 and using Equations 4 and 8, the left-hand sides are converted into polynomial in ϕ and ψ. Setting each coefficient of this polynomial to 0, yields a system of algebraic equations in $\alpha_0,\alpha_1,\alpha_2,\beta_1,\ \beta_2,c_0,\ \ c_1,c_2,d_1$, d_2, μ, λ, ω, k, p and q as follows:

$$-a\alpha_2c_2+6k^2\alpha_2+\frac{a\beta_2d_2\lambda}{\mu^2-\lambda^2\sigma_2}=0,$$

$$-2i(2kp+\omega)\alpha_2+2k^2\alpha_1-a\alpha_1c_2-a\alpha_2c_1-\frac{\lambda}{\mu^2-\lambda^2\sigma_2}\left(-a\beta_1d_2-a\beta_2d_1-6k^2\beta_2\mu\right)=0,$$

$$-a\alpha_2d_2+6k^2\beta_2-a\beta_2c_2=0,$$

$$-(p^2+q)\alpha_2-i(2kp+\omega)\alpha_1-a\alpha_0c_2-a\alpha_1c_1-a\alpha_2c_0+8k^2\alpha_2\lambda+\frac{\lambda^2a\beta_2d_2}{\mu^2-\lambda^2\sigma_2}$$
$$-\frac{\lambda}{\mu^2-\lambda^2\sigma_2}\left(-a\beta_1d_1+i(2kp+\omega)\beta_2\mu+k^2\left(2\alpha_2\mu^2-\beta_1\mu\right)\right)=0,$$

$$-2i(2kp+\omega)\beta_2-a\alpha_1d_2-a\alpha_2d_1-k^2\left(10\alpha_2\mu-2\beta_1\right)-a\beta_1c_2-a\beta_2c_1-\frac{2\lambda\mu a\beta_2d_2}{\mu^2-\lambda^2\sigma_2}=0,$$

$$-(p^2+q)\alpha_1-2i(2kp+\omega)\alpha_2\lambda-a\alpha_0c_1-a\alpha_1c_0+2k^2\alpha_1\lambda$$
$$+\frac{\lambda^2}{\mu^2-\lambda^2\sigma_2}\left(a\beta_1d_2+a\beta_2d_1+6k^2\beta_2\mu\right)=0,$$

$$-a\alpha_0d_2-a\alpha_1d_1+k^2\left(-3\alpha_1\mu+5\beta_2\lambda\right)-a\beta_1c_1-a\beta_2c_0-(p^2+q)\beta_2+i(2kp+\omega)\left(-\beta_1+2\alpha_2\mu\right)$$
$$+\frac{2\lambda\mu}{\mu^2-\lambda^2\sigma_2}\left(-a\beta_1d_2-a\beta_2d_1-6k^2\beta_2\mu\right)=0,$$

$$k^2\lambda\left(\beta_1-4\alpha_2\mu\right)+i(2kp+\omega)\left(-\beta_2\lambda+\alpha_1\mu\right)-a\alpha_0d_1-(p^2+q)\beta_1-a\beta_1c_0$$
$$+\frac{2\lambda\mu}{\mu^2-\lambda^2\sigma_2}\left(-a\beta_1d_1+i(2kp+\omega)\beta_2\mu+k^2\left(2\alpha_2\mu^2-\beta_1\mu\right)\right)=0,$$

$$-i(2kp+\omega)\lambda\alpha_1+2k^2\alpha_2\lambda^2-a\alpha_0c_0-\frac{\lambda^2}{\mu^2-\lambda^2\sigma_2}\left(-a\beta_1d_1+i(2kp+\omega)\beta_2\mu+k^2\left(2\alpha_2\mu^2-\beta_1\mu\right)\right)$$
$$-(p^2+q)\alpha_0=0,$$

$$6k^4c_2-bk^2\alpha_2^2+\frac{bk^2\beta_2^2\lambda}{\mu^2-\lambda^2\sigma_2}=0,$$

$$2k^4c_1-2bk^2\alpha_1\alpha_2+\frac{\lambda}{\mu^2-\lambda^2\sigma_2}\left(6k^4d_2\mu+2bk^2\beta_1\beta_2\right)=0,$$
$$-2bk^2\alpha_2\beta_2+6k^4d_2=0,$$

$$\left(\omega^2-k^2\right)c_2+8k^4c_2\lambda-bk^2\left(2\alpha_0\alpha_2+\alpha_1^2\right)-\frac{\lambda}{\mu^2-\lambda^2\sigma_2}\left(k^4\left(2c_2\mu^2-d_1\mu\right)-bk^2\beta_1^2\right)+\frac{bk^2\beta_2^2\lambda^2}{\mu^2-\lambda^2\sigma_2}=0,$$

$$-2bk^2\left(\alpha_1\beta_2+\alpha_2\beta_1\right)+k^4\left(2d_1-10c_2\mu\right)-\frac{2bk^2\beta_2^2\lambda\mu}{\mu^2-\lambda^2\sigma_2}=0,$$

$$\left(\omega^2-k^2\right)c_1+2k^4c_1\lambda-2bk^2\alpha_0\alpha_1+\frac{\lambda^2}{\mu^2-\lambda^2\sigma_2}\left(6k^4d_2\mu+2bk^2\beta_1\beta_2\right)=0,$$

$$k^4\left(-3c_1\mu+5d_2\lambda\right)-2bk^2\left(\alpha_0\beta_2+\alpha_1\beta_1\right)+\left(\omega^2-k^2\right)d_2-\frac{2\lambda\mu}{\mu^2-\lambda^2\sigma_2}\left(6k^4d_2\mu+2bk^2\beta_1\beta_2\right)=0,$$

$$k^4\lambda\left(d_1-4c_2\mu\right)-2bk^2\alpha_0\beta_1+\left(\omega^2-k^2\right)d_1+\frac{2\mu\lambda}{\mu^2-\lambda^2\sigma_2}\left(k^4\left(2c_2\mu^2-d_1\mu\right)-bk^2\beta_1^2\right)=0,$$

$$\left(\omega^2-k^2\right)c_0+2k^4\lambda^2c_2-bk^2\alpha_0^2-\frac{\lambda^2}{\mu^2-\lambda^2\sigma_2}\left(k^4\left(2c_2\mu^2-d_1\mu\right)-bk^2\beta_1^2\right)=0.$$

On solving the above algebraic equations using the Maple or Mathematica, we get the following results:

Result 1

Consider

$$\mu=0,\ \lambda=\frac{k^2-\omega^2}{2k^4},\alpha_0=0,\alpha_1=0,\alpha_2=0,\beta_1=0,\beta_2=\pm3\sqrt{\frac{2\sigma_2\left(k^2-\omega^2\right)}{ab}}, \quad (35)$$
$$c_0=\frac{3\left(k^2-\omega^2\right)}{ak^2},c_1=0,c_2=\frac{6k^2}{a},d_1=0,d_2=0,\ p=-\frac{\omega}{2k},q=\frac{\omega^2-2k^2}{4k^2},$$

Where $k>\omega$.

From Equations 7, 16, 21, 22 and 35, we deduce the traveling wave solutions of SB-system (Equation 1) as follows:

$$u(\xi)=\left[\pm\frac{3\left(k^2-\omega^2\right)}{k^2}\sqrt{\frac{\sigma_2}{ab}}\left(\frac{A_1\cos(\xi\sqrt{\frac{k^2-\omega^2}{2k^4}})-A_2\sin(\xi\sqrt{\frac{k^2-\omega^2}{2k^4}})}{\left(A_1\sin(\xi\sqrt{\frac{k^2-\omega^2}{2k^4}})+A_2\cos(\xi\sqrt{\frac{k^2-\omega^2}{2k^4}})\right)^2}\right)\right]e^{i\eta}, \quad (36)$$

$$v(\xi)=\frac{3\left(k^2-\omega^2\right)}{ak^2}\left[1+\left(\frac{A_1\cos(\xi\sqrt{\frac{k^2-\omega^2}{2k^4}})-A_2\sin(\xi\sqrt{\frac{k^2-\omega^2}{2k^4}})}{A_1\sin(\xi\sqrt{\frac{k^2-\omega^2}{2k^4}})+A_2\cos(\xi\sqrt{\frac{k^2-\omega^2}{2k^4}})}\right)^2\right], \quad (37)$$

In particular, by setting $A_1=0$ and $A_2\neq0$ in Equations 36 and 37, we have the periodic solutions

$$u(\xi)=\left[\mp\frac{3\left(k^2-\omega^2\right)}{k^2\sqrt{ab}}\sec(\xi\sqrt{\frac{k^2-\omega^2}{2k^4}})\tan(\xi\sqrt{\frac{k^2-\omega^2}{2k^4}})\right]e^{i\eta}, \quad (38)$$

$$v(\xi)=\frac{3\left(k^2-\omega^2\right)}{ak^2}\sec^2(\xi\sqrt{\frac{k^2-\omega^2}{2k^4}}), \quad (39)$$

while, if $A_1\neq0$ and $A_2=0$, then we have the periodic solutions

$$u(\xi)=\left[\pm\frac{3\left(k^2-\omega^2\right)}{k^2\sqrt{ab}}\csc(\xi\sqrt{\frac{k^2-\omega^2}{2k^4}})\cot(\xi\sqrt{\frac{k^2-\omega^2}{2k^4}})\right]e^{i\eta}, \quad (40)$$

$$v(\xi)=\frac{3\left(k^2-\omega^2\right)}{ak^2}\csc^2(\xi\sqrt{\frac{k^2-\omega^2}{2k^4}}), \quad (41)$$

where $\xi=kx+\omega t$, $\eta=-\frac{\omega}{2k}x+\frac{\omega^2-2k^2}{4k^2}t$.

Result 2

Consider

$$\mu=0,\ \lambda=\frac{k^2-\omega^2}{k^4},\alpha_0=\pm\frac{2\left(k^2-\omega^2\right)}{k^2\sqrt{ab}},\alpha_1=0,\alpha_2=\pm\frac{3k^2}{\sqrt{ab}},\beta_1=0,$$
$$\beta_2=\pm3\sqrt{\frac{\sigma_2\left(k^2-\omega^2\right)}{ab}},c_0=\frac{2\left(k^2-\omega^2\right)}{ak^2},c_1=0,c_2=\frac{3k^2}{a},d_1=0, \quad (42)$$
$$d_2=\frac{3\sqrt{\sigma_2\left(k^2-\omega^2\right)}}{a},\ p=-\frac{\omega}{2k},q=\frac{4k^2-5\omega^2}{4k^2},$$

Where $k>\omega$.

In this result, we deduce the traveling wave solution of SB-system (Equation 1) as follows:

$$u(\xi)=\left[\pm\frac{2\left(k^2-\omega^2\right)}{k^2\sqrt{ab}}\pm\frac{3\left(k^2-\omega^2\right)}{k^2\sqrt{ab}}\left(\frac{A_1\cos(\xi\sqrt{\frac{k^2-\omega^2}{k^4}})-A_2\sin(\xi\sqrt{\frac{k^2-\omega^2}{k^4}})}{A_1\sin(\xi\sqrt{\frac{k^2-\omega^2}{k^4}})+A_2\cos(\xi\sqrt{\frac{k^2-\omega^2}{k^4}})}\right)^2\right. \quad (43)$$
$$\left.\pm\frac{3\left(k^2-\omega^2\right)}{k^2}\sqrt{\frac{\sigma_2}{ab}}\left(\frac{A_1\cos(\xi\sqrt{\frac{k^2-\omega^2}{k^4}})-A_2\sin(\xi\sqrt{\frac{k^2-\omega^2}{k^4}})}{\left(A_1\sin(\xi\sqrt{\frac{k^2-\omega^2}{k^4}})+A_2\cos(\xi\sqrt{\frac{k^2-\omega^2}{k^4}})\right)^2}\right)\right]e^{i\eta},$$

$$v(\xi)=\frac{2\left(k^2-\omega^2\right)}{ak^2}+\frac{3\left(k^2-\omega^2\right)}{ak^2}\left(\frac{A_1\cos(\xi\sqrt{\frac{k^2-\omega^2}{k^4}})-A_2\sin(\xi\sqrt{\frac{k^2-\omega^2}{k^4}})}{A_1\sin(\xi\sqrt{\frac{k^2-\omega^2}{k^4}})+A_2\cos(\xi\sqrt{\frac{k^2-\omega^2}{k^4}})}\right)^2 \quad (44)$$
$$+\frac{3\left(k^2-\omega^2\right)\sqrt{\sigma_2}}{ak^2}\left(\frac{A_1\cos(\xi\sqrt{\frac{k^2-\omega^2}{k^4}})-A_2\sin(\xi\sqrt{\frac{k^2-\omega^2}{k^4}})}{\left(A_1\sin(\xi\sqrt{\frac{k^2-\omega^2}{k^4}})+A_2\cos(\xi\sqrt{\frac{k^2-\omega^2}{k^4}})\right)^2}\right),$$

In particular, by setting $A_1 = 0$ and $A_2 \neq 0$ in Equations 43 and 44, we have the periodic solutions

$$u(\xi) = \pm \frac{\left(k^2 - \omega^2\right)}{k^2\sqrt{ab}}\left[2 + 3\tan(\xi\sqrt{\frac{k^2-\omega^2}{k^4}}) \times \left(\tan(\xi\sqrt{\frac{k^2-\omega^2}{k^4}}) - \sec(\xi\sqrt{\frac{k^2-\omega^2}{k^4}})\right)\right]e^{i\eta}, \tag{45}$$

$$v(\xi) = \frac{\left(k^2 - \omega^2\right)}{ak^2}\left[2 + 3\tan(\xi\sqrt{\frac{k^2-\omega^2}{k^4}})\left(\tan(\xi\sqrt{\frac{k^2-\omega^2}{k^4}}) - \sec(\xi\sqrt{\frac{k^2-\omega^2}{k^4}})\right)\right], \tag{46}$$

while, if $A_1 \neq 0$ and $A_2 = 0$, then we have the periodic solutions

$$u(\xi) = \pm \frac{\left(k^2 - \omega^2\right)}{k^2\sqrt{ab}}\left[2 + 3\cot(\xi\sqrt{\frac{k^2-\omega^2}{k^4}}) \times \left(\cot(\xi\sqrt{\frac{k^2-\omega^2}{k^4}}) + \csc(\xi\sqrt{\frac{k^2-\omega^2}{k^4}})\right)\right]e^{i\eta}, \tag{47}$$

$$v(\xi) = \frac{\left(k^2 - \omega^2\right)}{ak^2}\left[2 + 3\cot(\xi\sqrt{\frac{k^2-\omega^2}{k^4}})\left(\cot(\xi\sqrt{\frac{k^2-\omega^2}{k^4}}) + \csc(\xi\sqrt{\frac{k^2-\omega^2}{k^4}})\right)\right], \tag{48}$$

Where $\xi = kx + \omega t,\ \eta = -\frac{\omega}{2k}x + \frac{4k^2 - 5\omega^2}{4k^2}t.$

Case 3: Rational function solutions ($\lambda = 0$)

If $\lambda = 0$, substituting Equations 21 and 22 into Equations 19 and 20 and using Equations 4 and 10, the left-hand sides are converted into polynomial in ϕ and ψ. Setting each coefficient of this polynomial to 0, yields a system of algebraic equations in $\alpha_0, \alpha_1, \alpha_2, \beta_1, \beta_2, c_0, c_1, c_2, d_1, d_2, \mu, \omega, k$, p and q as follows:

$$-a\alpha_2 c_2 + 6k^2\alpha_2 - \frac{a\beta_2 d_2}{A_1^2 - 2\mu A_2} = 0,$$

$$-2i(2kp+\omega)\alpha_2 + 2k^2\alpha_1 - a\alpha_1 c_2 - a\alpha_2 c_1 - \frac{1}{A_1^2 - 2\mu A_2}\left(a\beta_1 d_2 + a\beta_2 d_1 + 6k^2\beta_2\mu\right) = 0,$$

$$-a\alpha_2 d_2 + 6k^2\beta_2 - a\beta_2 c_2 = 0,$$

$$-(p^2+q)\alpha_2 - i(2kp+\omega)\alpha_1 - a\alpha_0 c_2 - a\alpha_1 c_1 - a\alpha_2 c_0 + \frac{1}{A_1^2 - 2\mu A_2}\left(-a\beta_1 d_1 + i(2kp+\omega)\beta_2\mu + k^2\left(2\alpha_2\mu^2 - \beta_1\mu\right)\right) = 0,$$

$$-2i(2kp+\omega)\beta_2 - a\alpha_1 d_2 - a\alpha_2 d_1 - k^2\left(10\alpha_2\mu - 2\beta_1\right) - a\beta_1 c_2 - a\beta_2 c_1 + \frac{2\mu a\beta_2 d_2}{A_1^2 - 2\mu A_2} = 0,$$

$$-(p^2+q)\alpha_1 - a\alpha_0 c_1 - a\alpha_1 c_0 = 0,$$

$$-a\alpha_0 d_2 - a\alpha_1 d_1 - 3\alpha_1\mu k^2 - a\beta_2 c_1 - a\beta_2 c_0 - (p^2+q)\beta_2 + i(2kp+\omega)\left(-\beta_1 + 2\alpha_2\mu\right) + \frac{2\mu}{A_1^2 - 2\mu A_2}\left(a\beta_1 d_2 + a\beta_2 d_1 + 6k^2\beta_2\mu\right) = 0,$$

$$i(2kp+\omega)\alpha_1\mu - a\alpha_0 d_1 - (p^2+q)\beta_1 - a\beta_1 c_0 - \frac{2\mu}{A_1^2 - 2\mu A_2}\left(-a\beta_1 d_1 + i(2kp+\omega)\beta_2\mu + k^2\left(2\alpha_2\mu^2 - \beta_1\mu\right)\right) = 0,$$

$$-a\alpha_0 c_0 - (p^2+q)\alpha_0 = 0,$$

$$6k^4 c_2 - bk^2\alpha_2^2 - \frac{bk^2\beta_2^2}{A_1^2 - 2\mu A_2} = 0,$$

$$2k^4 c_1 - 2bk^2\alpha_1\alpha_2 - \frac{1}{A_1^2 - 2\mu A_2}\left(6k^4 d_2\mu + 2bk^2\beta_1\beta_2\right) = 0,$$

$$-2bk^2\alpha_2\beta_2 + 6k^4 d_2 = 0,$$

$$\left(\omega^2 - k^2\right)c_2 - bk^2\left(2\alpha_0\alpha_2 + \alpha_1^2\right) + \frac{1}{A_1^2 - 2\mu A_2}\left(k^4\left(2c_2\mu^2 - d_1\mu\right) - bk^2\beta_1^2\right) = 0,$$

$$-2bk^2\left(\alpha_1\beta_2 + \alpha_2\beta_1\right) + k^4\left(2d_1 - 10c_2\mu\right) + \frac{2bk^2\beta_2^2\mu}{A_1^2 - 2\mu A_2} = 0,$$

$$\left(\omega^2 - k^2\right)c_1 - 2bk^2\alpha_0\alpha_1 = 0,$$

$$-3c_1\mu k^4 - 2bk^2\left(\alpha_0\beta_2 + \alpha_1\beta_1\right) + \left(\omega^2 - k^2\right)d_2 + \frac{2\mu}{A_1^2 - 2\mu A_2}\left(6k^4 d_2\mu + 2bk^2\beta_1\beta_2\right) = 0,$$

$$-2bk^2\alpha_0\beta_1 + \left(\omega^2 - k^2\right)d_1 - \frac{2\mu}{A_1^2 - 2\mu A_2}\left(k^4\left(2c_2\mu^2 - d_1\mu\right) - bk^2\beta_1^2\right) = 0,$$

$$\left(\omega^2 - k^2\right)c_0 - bk^2\alpha_0^2 = 0.$$

On solving the above algebraic equations using the Maple or Mathematica, we get the following results:

Result 1

Consider

$$\mu = 0, , \alpha_0 = 0, \alpha_1 = 0, \alpha_2 = \pm\frac{6k^2}{\sqrt{ab}}, \beta_1 = 0, \beta_2 = 0, c_0 = -\frac{(4q+1)}{4a}, \quad c_1 = 0, c_2 = \frac{6k^2}{a}, d_1 = 0, d_2 = 0, p = -\frac{1}{2}, q = q, \omega = k. \tag{49}$$

From Equations 9, 16, 21, 22 and 49, we deduce the traveling wave solutions of SB-system (Equation 1) as follows:

$$u(\xi) = \pm \frac{6k^2}{\sqrt{ab}} \left(\frac{A_1}{A_1\xi + A_2} \right)^2 e^{i\eta}, \qquad (50)$$

$$v(\xi) = -\frac{(4q+1)}{4a} + \frac{6k^2}{a} \left(\frac{A_1}{A_1\xi + A_2} \right)^2, \qquad (51)$$

Where $\xi = kx + \omega t$, $\eta = -\frac{1}{2}x + qt$.

Example 2

The (2+1)-dimensional HNLS equation (**Equation 2*****)***

Here, we study the (2+1)-dimensional HNLS Equation 2. To this end, we assume that the solution of Equation 2 can be written as:

$$u(x,y,t) = W(\xi)e^{i\eta}, \quad \xi = x + ay - ct, \eta = mx + ny + \omega t, \qquad (52)$$

where $W(\xi)$ is a real function of ξ and a, c, m, n, ω are constants to be determined. Substituting Equation 52 into Equation 2, we obtain

$$(c^2-1)W'' - \left[\omega^2 - 2n - (a+\omega c)^2\right]W - 2W^3 = 0, \qquad (53)$$

Where $c^2 \neq 1$.

By balancing between W'' with W^3 in (53) we get $N + 2 = 3N \Rightarrow N = 1$. Consequently, Equation 53 has the formal solution:

$$W(\xi) = \alpha_0 + \alpha_1\phi(\xi) + \beta_1\psi(\xi), \qquad (54)$$

Where α_0, α_1 and β_1 are constants to be determined later satisfying $\alpha_1^2 + \beta_1^2 \neq 0$. There are three cases to be discussed as follows:

Case 1: Hyperbolic function solutions ($\lambda < 0$)

If $\lambda < 0$, substituting Equation 54 into Equation 53 and using Equations 4 and 6, the left-hand side of Equation 53 becomes a polynomial in ϕ and ψ. Setting the coefficients of this polynomial to be 0, yields a system of algebraic equations in $\alpha_0, \alpha_1, \beta_1, a, c, m, n, \omega$, μ and λ as follows:

$$2(c^2-1)\alpha_1 - 2\alpha_1^3 - \frac{6\alpha_1\beta_1^2\lambda}{\lambda^2\sigma_1 + \mu^2} = 0,$$

$$-6\alpha_0\alpha_1^2 + \frac{4\beta_1^3\lambda^2\mu}{(\lambda^2\sigma_1+\mu^2)^2} + \frac{\lambda}{\lambda^2\sigma_1+\mu^2}\left((c^2-1)\beta_1\mu + 6\alpha_0\beta_1^2\right) = 0,$$

$$-6\alpha_1^2\beta_1 + 2(c^2-1)\beta_1 + \frac{2\beta_1^3\lambda}{\lambda^2\sigma_1+\mu^2} = 0,$$

$$-(\omega^2 - 2n - (a+\omega c)^2)\alpha_1 - 6\alpha_0^2\alpha_1 + 2(c^2-1)\alpha_1\lambda + \frac{6\alpha_1\beta_1^2\lambda^2}{\lambda^2\sigma_1+\mu^2} = 0,$$

$$-3(c^2-1)\alpha_1\mu - 12\alpha_0\alpha_1\beta_1 - \frac{12\alpha_1\beta_1^2\lambda\mu}{\lambda^2\sigma_1+\mu^2} = 0,$$

$$-(\omega^2 - 2n - (a+\omega c)^2)\beta_1 - 6\alpha_0^2\beta_1 + (c^2-1)\beta_1\lambda - \frac{8\beta_1^3\lambda^2\mu^2}{(\lambda^2\sigma_1+\mu^2)^2} + \frac{2\beta_1^3\lambda^2}{\lambda^2\sigma_1+\mu^2}$$
$$-\frac{2\lambda\mu}{\lambda^2\sigma_1+\mu^2}\left((c^2-1)\beta_1\mu + 6\alpha_0\beta_1^2\right) = 0,$$

$$-(\omega^2 - 2n - (a+\omega c)^2)\alpha_0 - 2\alpha_0^3 + \frac{4\beta_1^3\lambda^3\mu}{(\lambda^2\sigma_1+\mu^2)^2} + \frac{\lambda^2}{\lambda^2\sigma_1+\mu^2}\left((c^2-1)\beta_1\mu + 6\alpha_0\beta_1^2\right) = 0.$$

On solving the above algebraic equations using the Maple or Mathematica, we get the following results:

Result 1

Consider

$$\mu = 0, \lambda = \lambda, \alpha_0 = 0, \alpha_1 = \pm\sqrt{c^2-1}, \beta_1 = 0, a = a, c = c, \omega = \omega,$$
$$n = (\frac{1}{2}\omega^2 + \lambda)(1-c^2) - a(\frac{1}{2}a + c\omega), \qquad (55)$$

Where $c^2 > 1$.

From Equations 5, 52, 54 and 55, we deduce the traveling wave solution of Equation 2 as follows:

$$u(x,y,t) = \pm\sqrt{-\lambda(c^2-1)}\left(\frac{A_1\cosh(\xi\sqrt{-\lambda}) + A_2\sinh(\xi\sqrt{-\lambda})}{A_1\sinh(\xi\sqrt{-\lambda}) + A_2\cosh(\xi\sqrt{-\lambda})}\right)e^{i\eta}, \qquad (56)$$

In particular, by setting $A_1 = 0$ and $A_2 \neq 0$ in Equation 56, we have the kink shaped solitary solution

$$u(x,y,t) = \pm\sqrt{-\lambda(c^2-1)}\tanh(\xi\sqrt{-\lambda})e^{i\eta}, \qquad (57)$$

While, if $A_1 \neq 0$ and $A_2 = 0$, then we have the anti-kink shaped solitary solution

$$u(x,y,t) = \pm\sqrt{-\lambda(c^2-1)}\coth(\xi\sqrt{-\lambda})e^{i\eta}, \qquad (58)$$

Where

$\xi = x + ay - ct,\ \eta = -(a+\omega c)x + \left((\frac{1}{2}\omega^2 + \lambda)(1-c^2) - a(\frac{1}{2}a + c\omega) \right) y + \omega t.$

Result 2

Consider

$$\mu = \mu, \lambda = \lambda, \alpha_0 = 0, \alpha_1 = \pm\frac{\sqrt{c^2-1}}{2}, \beta_1 = \pm\sqrt{-\frac{(c^2-1)(\lambda^2\sigma_1 + \mu^2)}{4\lambda}}, a = a,$$
$$c = c, \omega = \omega, n = (1-c^2)\left(\frac{1}{2}\omega^2 + \frac{1}{4}\lambda\right) - a(\frac{1}{2}a + c\omega), \quad (59)$$

where $c^2 > 1.$

In this result, we deduce the traveling wave solution of Equation 2 as follows:

$$u(x,y,t) = \left[\pm\frac{\sqrt{-\lambda(c^2-1)}}{2} \left(\frac{A_1\cosh(\xi\sqrt{-\lambda}) + A_2\sinh(\xi\sqrt{-\lambda})}{A_1\sinh(\xi\sqrt{-\lambda}) + A_2\cosh(\xi\sqrt{-\lambda}) + \frac{\mu}{\lambda}} \right) \right. \quad (60)$$
$$\left. \pm\sqrt{-\frac{(c^2-1)(\lambda^2\sigma_1 + \mu^2)}{4\lambda}} \left(\frac{1}{A_1\sinh(\xi\sqrt{-\lambda}) + A_2\cosh(\xi\sqrt{-\lambda}) + \frac{\mu}{\lambda}} \right) \right] e^{i\eta},$$

In particular, by setting $A_1 \neq 0$, $A_2 = 0$ and $\mu = 0$ in Equation 60, we have the anti-kink and anti-bell shaped solitary solution

$$u(x,y,t) = \pm\frac{\sqrt{-\lambda(c^2-1)}}{2}\left(\coth(\xi\sqrt{-\lambda}) + \mathrm{csch}(\xi\sqrt{-\lambda})\right)e^{i\eta}, \quad (61)$$

Where

$\xi = x + ay - ct,\ \eta = -(a+\omega c)x + \left((1-c^2)\left(\frac{1}{2}\omega^2 + \frac{1}{4}\lambda\right) - a(\frac{1}{2}a + c\omega) \right) y + \omega t.$

Case 2: Trigonometric function solution ($\lambda > 0$)

If $\lambda > 0$, substituting Equation 54 into Equation 53 and using Equations 4 and 8, the left-hand side of Equation 53 becomes a polynomial in ϕ and ψ. Setting the coefficients of this polynomial to be 0, yields a system of algebraic equations in $\alpha_0, \alpha_1, \beta_1, a, c, m, n, \omega, \mu$ and λ as follows:

$$2(c^2-1)\alpha_1 - 2\alpha_1^3 + \frac{6\alpha_1\beta_1^2\lambda}{\mu^2 - \lambda^2\sigma_2} = 0,$$

$$-6\alpha_0\alpha_1^2 + \frac{4\beta_1^3\lambda^2\mu}{(\mu^2 - \lambda^2\sigma_2)^2} + \frac{\lambda}{\mu^2 - \lambda^2\sigma_2}\left((c^2-1)\beta_1\mu + 6\alpha_0\beta_1^2\right) = 0,$$

$$-6\alpha_1^2\beta_1 + 2(c^2-1)\beta_1 + \frac{2\beta_1^3\lambda}{\mu^2 - \lambda^2\sigma_2} = 0,$$

$$-(\omega^2 - 2n - (a+\omega c)^2)\alpha_1 - 6\alpha_0^2\alpha_1 + 2(c^2-1)\alpha_1\lambda + \frac{6\alpha_1\beta_1^2\lambda^2}{\mu^2 - \lambda^2\sigma_2} = 0,$$

$$-3(c^2-1)\alpha_1\mu - 12\alpha_0\alpha_1\beta_1 - \frac{12\alpha_1\beta_1^2\lambda\mu}{\mu^2 - \lambda^2\sigma_2} = 0,$$

$$-(\omega^2 - 2n - (a+\omega c)^2)\beta_1 - 6\alpha_0^2\beta_1 + (c^2-1)\beta_1\lambda - \frac{8\beta_1^3\lambda^2\mu^2}{(\mu^2 - \lambda^2\sigma_2)^2} + \frac{2\beta_1^3\lambda^2}{\mu^2 - \lambda^2\sigma_2}$$
$$-\frac{2\lambda\mu}{\mu^2 - \lambda^2\sigma_2}\left((c^2-1)\beta_1\mu + 6\alpha_0\beta_1^2\right) = 0,$$

$$-(\omega^2 - 2n - (a+\omega c)^2)\alpha_0 - 2\alpha_0^3 + \frac{4\beta_1^3\lambda^3\mu}{(\mu^2 - \lambda^2\sigma_2)^2} + \frac{\lambda^2}{\mu^2 - \lambda^2\sigma_2}\left((c^2-1)\beta_1\mu + 6\alpha_0\beta_1^2\right) = 0.$$

On solving the above algebraic equations using the Maple or Mathematica, we get the following results:

Result 1

Consider

$$\mu = 0, \lambda = \lambda, \alpha_0 = 0, \alpha_1 = \pm\sqrt{c^2-1}, \beta_1 = 0, a = a, c = c, \omega = \omega,$$
$$n = (\frac{1}{2}\omega^2 + \lambda)(1-c^2) - a(\frac{1}{2}a + c\omega), \quad (62)$$

Where $c^2 > 1.$

From Equations 7, 52, 54 and 62, we deduce the traveling wave solutions of Equations 2 as follows:

$$u(x,y,t) = \pm\sqrt{\lambda(c^2-1)}\left(\frac{A_1\cos(\xi\sqrt{\lambda}) - A_2\sin(\xi\sqrt{\lambda})}{A_1\sin(\xi\sqrt{\lambda}) + A_2\cos(\xi\sqrt{\lambda})}\right)e^{i\eta}, \quad (63)$$

In particular, by setting $A_1 = 0$ and $A_2 \neq 0$ in Equation 63, we have the periodic solution

$$u(x,y,t) = \mp\sqrt{\lambda(c^2-1)}\tan(\xi\sqrt{\lambda})e^{i\eta}, \quad (64)$$

while, if $A_1 \neq 0$ and $A_2 = 0$, then we have the periodic solution

$$u(x,y,t) = \pm\sqrt{\lambda(c^2-1)}\cot(\xi\sqrt{\lambda})e^{i\eta}, \quad (65)$$

where

$\xi = x + ay - c\,t,\ \eta = -(a+\omega c)x + \left((\frac{1}{2}\omega^2+\lambda)(1-c^2) - a(\frac{1}{2}a + c\omega)\right)y + \omega t.$

Result 2

Consider

$$\mu=\mu,\ \lambda=\lambda,\alpha_0=0,\alpha_1=\pm\frac{\sqrt{c^2-1}}{2},\beta_1=\pm\sqrt{\frac{(c^2-1)(\lambda^2\sigma_2-\mu^2)}{4\lambda}},a=a,\quad (66)$$
$$c=c,\omega=\omega,n=(1-c^2)\left(\frac{1}{2}\omega^2+\frac{1}{4}\lambda\right)-a(\frac{1}{2}a+c\omega),$$

Where $c^2>1.$

In this result, we deduce the traveling wave solution of Equation 2 as follows:

$$u(x,y,t)=\left[\pm\frac{\sqrt{\lambda(c^2-1)}}{2}\left(\frac{A_1\cos(\xi\sqrt{\lambda})-A_2\sin(\xi\sqrt{\lambda})}{A_1\sin(\xi\sqrt{\lambda})+A_2\cos(\xi\sqrt{\lambda})+\frac{\mu}{\lambda}}\right)\right.\quad (67)$$
$$\left.\pm\sqrt{\frac{(c^2-1)(\lambda^2\sigma_2-\mu^2)}{4\lambda}}\left(\frac{1}{A_1\sin(\xi\sqrt{\lambda})+A_2\cos(\xi\sqrt{\lambda})+\frac{\mu}{\lambda}}\right)\right]e^{i\eta},$$

In particular, by setting $A_1=0$, $A_2\neq 0$ and $\mu=0$ in Equation 67, we have the periodic solution

$$u(x,y,t)=\pm\frac{\sqrt{\lambda(c^2-1)}}{2}\left(-\tan(\xi\sqrt{\lambda})+\sec(\xi\sqrt{\lambda})\right)e^{i\eta},\quad (68)$$

while, if $A_1\neq 0$, $A_2=0$ and $\mu=0$, then we have the periodic solution

$$u(x,y,t)=\pm\frac{\sqrt{\lambda(c^2-1)}}{2}\left(\cot(\xi\sqrt{\lambda})+\csc(\xi\sqrt{\lambda})\right)e^{i\eta},\quad (69)$$

Where

$\xi = x + ay - c\,t,\ \eta = -(a+\omega c)x + \left((1-c^2)\left(\frac{1}{2}\omega^2+\frac{1}{4}\lambda\right) - a(\frac{1}{2}a + c\omega)\right)y + \omega t.$

Case 3: Rational function solutions ($\lambda=0$)

If $\lambda=0$, substituting Equation 54 into Equation 53 and using Equations 4 and 10, the left-hand side of Equation 53 becomes a polynomial in ϕ and ψ. Setting the coefficients of this polynomial to be 0, yields a system of algebraic equations in $\alpha_0,\alpha_1,\beta_1,a,c,m,n,\omega$ and μ as follows:

$$2(c^2-1)\alpha_1-2\alpha_1^3-\frac{6\alpha_1\beta_1^2}{A_1^2-2\mu A_2}=0,$$

$$-6\alpha_0\alpha_1^2+\frac{4\beta_1^3\mu}{(A_1^2-2\mu A_2)^2}-\frac{1}{A_1^2-2\mu A_2}\left((c^2-1)\beta_1\mu+6\alpha_0\beta_1^2\right)=0,$$

$$-6\alpha_1^2\beta_1+2(c^2-1)\beta_1-\frac{2\beta_1^3}{A_1^2-2\mu A_2}=0,$$

$$-(\omega^2-2n-(a+\omega c)^2)\alpha_1-6\alpha_0^2\alpha_1=0,$$

$$-3(c^2-1)\alpha_1\mu-12\alpha_0\alpha_1\beta_1+\frac{12\alpha_1\beta_1^2\mu}{A_1^2-2\mu A_2}=0,$$

$$-(\omega^2-2n-(a+\omega c)^2)\beta_1-6\alpha_0^2\beta_1-\frac{8\beta_1^3\mu^2}{(A_1^2-2\mu A_2)^2}$$
$$+\frac{2\mu}{A_1^2-2\mu A_2}\left((c^2-1)\beta_1\mu+6\alpha_0\beta_1^2\right)=0,$$

$$-(\omega^2-2n-(a+\omega c)^2)\alpha_0-2\alpha_0^3=0.$$

On solving the above algebraic equations using the Maple or Mathematica, we get the following results:

Result 1

Consider

$$\mu=0,\alpha_0=0,\alpha_1=0,\beta_1=\pm A_1\sqrt{c^2-1},a=a,c=c,\omega=\omega,\quad (70)$$
$$n=\frac{1}{2}\omega^2(1-c^2)-a(\frac{1}{2}a+c\omega),$$

where $c^2>1.$

From Equations 9, 52, 54 and 70, we deduce the traveling wave solutions of Equation 2 as follows:

$$u(x,y,t)=\pm\sqrt{c^2-1}\left(\frac{A_1}{A_1\xi+A_2}\right)e^{i\eta},\quad (71)$$

Where

$\xi = x + ay - c\,t,\ \eta = -(a+\omega c)x + \left(\frac{1}{2}\omega^2(1-c^2) - a(\frac{1}{2}a + c\omega)\right)y + \omega t.$

Result 2

Consider

$$\mu=\mu,\alpha_0=0,\alpha_1=\pm\frac{\sqrt{c^2-1}}{2},\beta_1=\pm\frac{\sqrt{(c^2-1)(A_1^2-2\mu A_2)}}{2},a=a, \quad (72)$$
$$c=c,\omega=\omega,n=\frac{1}{2}\omega^2(1-c^2)-a(\frac{1}{2}a+c\omega),$$

Where $c^2>1, A_1^2>2\mu A_2$.

In this result, we deduce the traveling wave solution of Equation 2 as follows:

$$u(x,y,t)=\left[\frac{\pm\sqrt{c^2-1}}{2}\left(\frac{\mu\xi+A_1}{\frac{\mu}{2}\xi^2+A_1\xi+A_2}\right)\pm\frac{\sqrt{(c^2-1)(A_1^2-2\mu A_2)}}{2}\times\left(\frac{1}{\frac{\mu}{2}\xi^2+A_1\xi+A_2}\right)\right]e^{i\eta}, \quad (73)$$

where

$$\xi=x+ay-ct,\ \eta=-(a+\omega c)x+\left(\frac{1}{2}\omega^2(1-c^2)-a(\frac{1}{2}a+c\omega)\right)y+\omega t.$$

PHYSICAL EXPLANATIONS OF SOME OBTAINED SOLUTIONS

The obtained solutions for the two equations (1) and (2) include the kink, anti-kink soliton solutions, bell and anti-bell soliton solutions as well as periodic and rational solutions. The graphical representations of some of these solutions are plotted by taking suitable values of involved unknown parameters to visualize the mechanism of the original equations (Figures 1 to 6).

CONCLUSIONS

The two variable $\left(\frac{G'}{G},\frac{1}{G}\right)$-expansion method is used in this article to obtain new exact solutions of two nonlinear PDEs namely, the (1+1)-dimensional nonlinear Schrödinger-Boussinesq system and the (2+1)-dimensional HNLS equation. These exact solutions are presented in terms of the hyperbolic, trigonometric and rational functions. As the two parameters A_1 and A_2 takes special values, we obtain the solitary wave solutions. From Equations 3 and 14, we can deduce that the two variable $\left(\frac{G'}{G},\frac{1}{G}\right)$-expansion method reduces to the $\left(\frac{G'}{G}\right)$-expansion method. So the two variable $\left(\frac{G'}{G},\frac{1}{G}\right)$-expansion method is an extension of the $\left(\frac{G'}{G}\right)$-expansion method. The used method in this paper is more effective and more general than the $\left(\frac{G'}{G}\right)$-expansion method because it gives exact solutions in more general forms. In summary, the advantage of the two variable $\left(\frac{G'}{G},\frac{1}{G}\right)$-expansion method over the $\left(\frac{G'}{G}\right)$-expansion method is that the solutions obtained by using

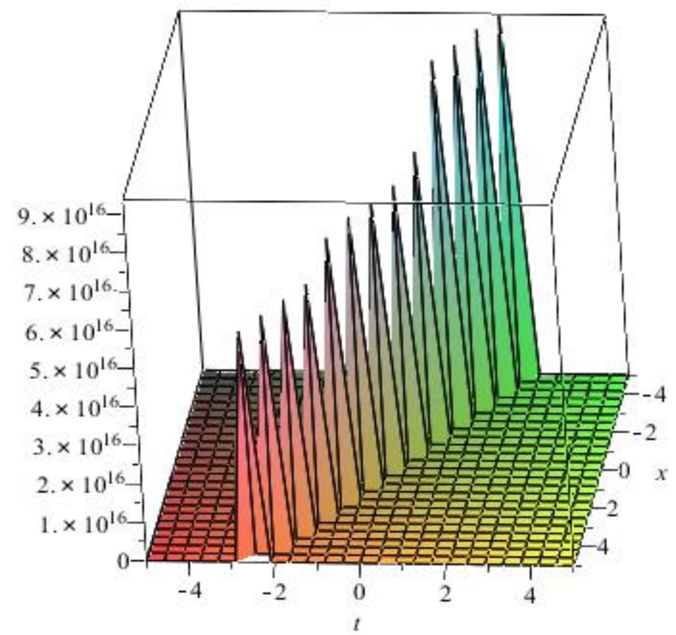

Figure 1. The plot of $U(x,t)$ of Equation 33 when $k=1,\omega=2,a=1,b=1$.

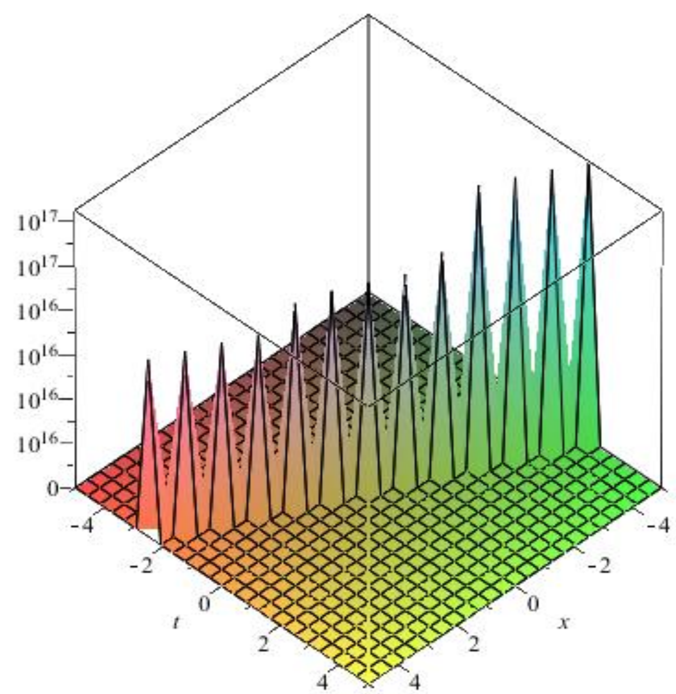

Figure 2. The plot of $v(x,t)$ of Equation 34 when $k=2,\omega=3,a=3$.

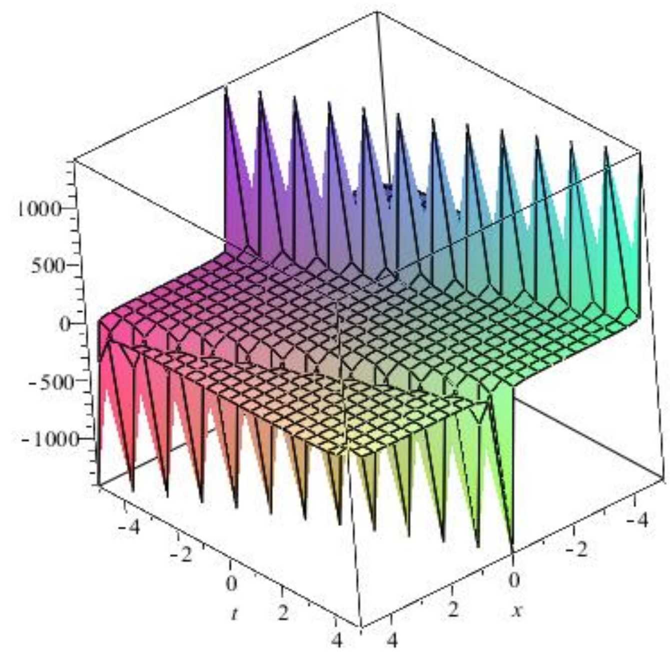

Figure 3. The plot of $U(x,t)$ of Equation 38 when $k=2, \omega=1, a=1, b=1.$

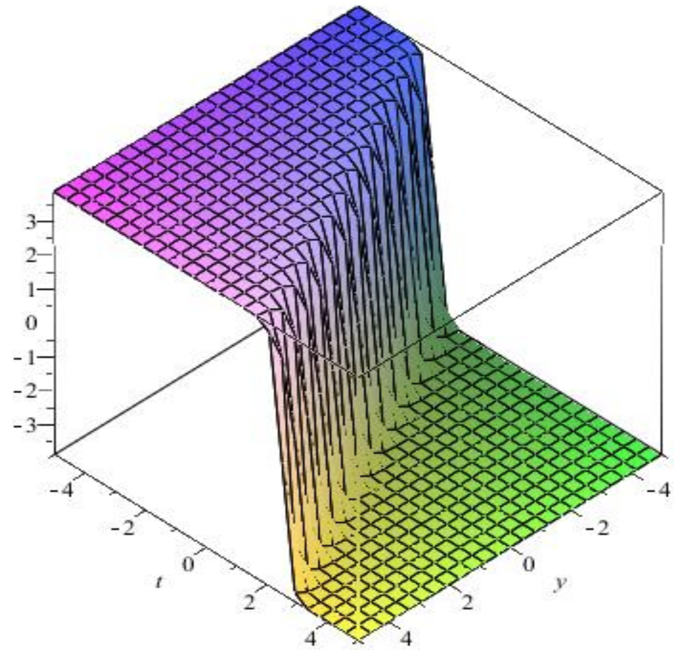

Figure 5. The plot of $W(0,y,t)$ of Equation 57 when $\lambda=-1, c=4, a=2.$

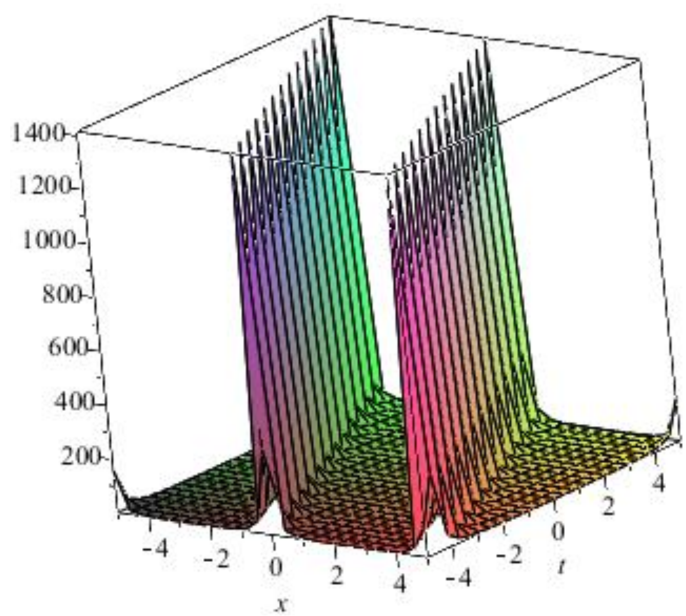

Figure 4. The plot of $v(x,t)$ of Equation 39 when $k=2, \omega=1, a=1.$

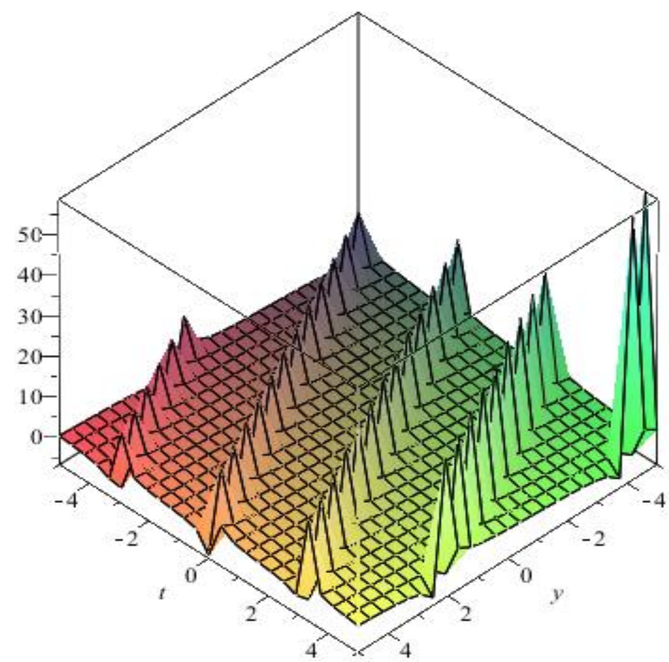

Figure 6. The plot of $W(0,y,t)$ of Equation 68 when $\lambda=1, c=2, a=1.$

the first method recover the solutions obtained by using the second one. On comparing our results obtained in this article with the well-know results obtained in Kilicman and Abazari (2012) and Fen (2012), we conclude that our results are new and not published elsewhere.

Finally, all solutions obtained in this article have been checked with the Maple by putting them back into the original equations.

Conflict of Interest

The authors have not declared any conflict of interest.

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