International Journal of the Physical Sciences Vol. 6(20), pp. 4706-4711, 23 September, 2011
Available online at http://www.academicjournals.org/IJPS
ISSN 1992 - 1950 ©2011 Academic Journals

Full Length Research Paper

Natural lift curves and the geodesic sprays for the spherical indicatrices of the involutes of a timelike curve in Minkowski 3-space

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Mathematics Subject Classification (2000): 53B30, 53C50.

Accepted 16 August, 2011

The aim of this paper is to determine criteria of being integral curve for the geodesic spray of the natural lift curves of the spherical indicatrices of the involutes of a given timelike curve in Minkowski 3-space.

Key words: Minkowski space, involute-evolute curve couple, geodesic spray, natural lift curve.

INTRODUCTION

C. Huygens, who is also known for his works in optics, discovered involutes while trying to build a more accurate clock. The original curve is called an evolute. A curve can have any number of involutes, thus a curve is an evolute of each of its involutes and an involute of its evolute. The normal to a curve is tangent to its evolute and the tangent to a curve is normal to its involutes. In addition to this, involute-evolute curve couple is a well known concept in the classical differential geometry, (Millman and Parker, 1977; Hacısalihoğlu, 2000; Çalışkan and Bilici, 2002).

In differential geometry, especially the theory of space curves, the Darboux vector is the areal velocity vector of the Frenet frame of a space curve. It is named after Gaston Darboux who discovered it. In terms of the Frenet-Serret apparatus, the Darboux vector w can be expressed as $w=\tau t+\kappa b$, details is given in Lambert et al. (2010). Çalışkan and Bilici (2006) have obtained the relationships between the Frenet frames of the timelike curve α and the spacelike involute curve α^* depending on hyperbolic timelike angle $(\theta>0)$ between the unit tangent vector *t* and the normalisation of the Darboux vector $c = w/\|w\|$ in the Minkowski 3-space IR_1^3. In addition to this, the concepts of the natural lift and the geodesic sprays have been given by Thorpe (1979). On the other hand, Çalışkan et al. (1984) have studied the natural lift curves and the geodesic sprays in the Euclidean 3-space IR^3. Recently, Bilici et al. (2003) have proposed the natural lift curves and the geodesic sprays for the spherical indicatrices of the the involute-evolute curve couple in IR^3. Also, Yılmaz (2010) introduced Bishop spherical images of a spacelike curve in terms of Bishop trihedra in Minkowski 3-space.

In the present paper, the natural lift curves and geodesic sprays for the spherical indicatrices of the involutes of a given timelike curve have been investigated in Minkowski 3-space and some new results were obtained.

MATERIALS AND METHODS

Let *M* be a hypersurface in IR_1^3 equipped with a metric *g*, where the metric *g* means a symmetric non-degenerate $(0,2)$ tensör field on *M* with constant signature. For a hypersurface *M*, let *TM* be the set $\bigcup\{T_p(M): p\in M\}$ of all tangent vectors to *M*. A technicality: For each $p\in M$ replace $0\in T_p(M)$ by 0_p (other-wise the zero tangent vector is in every tangent space). Then each $v\in TM$ is in a unique $T_p(M)$, and the projection $\pi: TM\to M$ sends *v* to *p*. Thus $\pi^{-1}(p)=T_p(M)$. There is a natural way to make *TM* a manifold, called the tangent bundle of *M*.

A vector field $X \in \chi(M)$ is exactly a smooth section of *TM*, that is, a smooth function $X : M \to TM$ such that $\pi \circ X = \text{identity}$.

Let *M* be a hypersurface in IR_1^3. A curve $\alpha : I \to M$ is an integral curve of $X \in \chi(M)$ provided $\dot{\alpha} = X_\alpha$; that is,

$$\frac{d}{dt}(\alpha(t)) = X(\alpha(t)) \text{ for all } t \in I \text{ (O'neill, 1983).} \quad (1)$$

For any parametrized curve $\alpha : I \to M$, the parametrized curve $\overline{\alpha} : I \to TM$ given by

$$\overline{\alpha}(t) = (\alpha(t), \dot{\alpha}(t)) = \dot{\alpha}(t)\big|_{\alpha(t)} \quad (2)$$

is called the *natural lift* of α on *TM*.

Thus, we can write

$$\frac{d\overline{\alpha}}{dt} = \frac{d}{dt}(\dot{\alpha}(t))\big|_{\alpha(t)} = D_{\dot{\alpha}(t)}\dot{\alpha}(t), \quad (3)$$

where *D* is the standard connection on IR_1^3.

For $v \in TM$, the smooth vector field $X \in \chi(TM)$ defined by

$$X(v) = \varepsilon\, g(v, S(v))\xi\big|_{\alpha(t)},\ \varepsilon = g(\xi, \xi) \quad (4)$$

is called the *geodesic spray* on the manifold *TM*, where ξ is the unit normal vector field of *M* and *S* is the shape operator of *M*.

Let α be a unit speed timelike curve with curvature κ and torsion τ. Let Frenet frames of α be $\{t, n, b\}$. In this trihedron, *t* is timelike vector, *n* and *b* are spacelike vectors. For these vectors, we can write

$$t \times n = -b\ ,\ n \times b = t\ ,\ b \times t = -n,$$

where $\times$ is the Lorentzian cross product in space IR_1^3 (O'neill, 1983). In this situation, the Frenet formulas are given by

$$\dot{t} = \kappa n, \quad \dot{n} = \kappa t - \tau b, \quad \dot{b} = \tau n \text{ (Woestijne, 1990).}$$

The Darboux vector for the timelike curve is given by

$$w = \tau t - \kappa b \text{ (Uğurlu, 1997).}$$

(a) If $|\tau| < |\kappa|$, then *w* is a spacelike vector. In this situation, we can write

$$\begin{cases} \kappa = \|w\| \cosh\theta \\ \tau = \|w\| \sinh\theta \end{cases}, \quad \|w\|^2 = g(w, w) = \kappa^2 - \tau^2$$

and

$$c = \frac{w}{\|w\|} = \sinh\theta\, t - \cosh\theta\, b,$$

where *c* is unit vector of direction *w*.

(b) If $|\tau| > |\kappa|$, then *w* is a timelike vector. In this situation, we have

$$\begin{cases} \kappa = \|w\| \sinh\theta \\ \tau = \|w\| \cosh\theta \end{cases}, \quad \|w\|^2 = -g(w, w) = \tau^2 - \kappa^2$$

and

$$c = \frac{w}{\|w\|} = \cosh\theta\, t - \sinh\theta\, b.$$

where θ is the hyperbolic timelike angle between the unit tangent vector *t* and the normalisation of the Darboux vector $c = w/\|w\|$ as Figure 1.

Proposition

Let α be a timelike (or spacelike) curve with curvatures κ and τ. The curve α is a general helix if and only if $\frac{\tau}{\kappa} =$ constant (Barros et al., 2001).

Remark 1

We can easily see from equations of the (a) and (b) that: $\frac{\tau}{\kappa} = \tanh\theta$ or $\frac{\tau}{\kappa} = \coth\theta$, if $\theta =$ constant then α is a general helix.

Lemma 1

The natural lift $\overline{\alpha}$ of the curve α is an integral curve of the geodesic spray X if and only if α is a geodesic on M.

Proof ($\Longrightarrow$)**:** Let $\overline{\alpha}$ be an integral curve of the geodesic spray *X*. Then we have

$$X(\overline{\alpha}(t)) = \frac{d}{dt}(\overline{\alpha}(t)) \quad (5)$$

Since *X* is a geodesic spray on *TM*, we have

$$X(\overline{\alpha}(t)) = \varepsilon\, g(\overline{\alpha}(t), S(\overline{\alpha}(t)))\xi\big|_{\alpha(t)}. \quad (6)$$

From (2), (5) and (6) we get

$$\frac{d}{dt}\left(\dot{\alpha}(t)\big|_{\alpha(t)}\right) = \varepsilon\, g\left(\dot{\alpha}(t)\big|_{\alpha(t)}, S\left(\dot{\alpha}(t)\big|_{\alpha(t)}\right)\right)\xi\big|_{\alpha(t)} \quad (7)$$

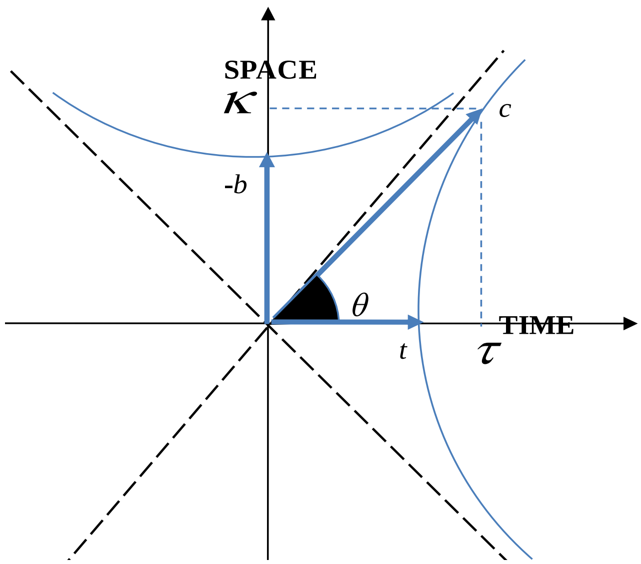

Figure 1. Hyperbolic timelike angle θ between *t* and *c*.

Since the last equation is true for all $\alpha(t)$, using (3) we found that

$$D_{\dot{\alpha}(t)}\dot{\alpha}(t)=\varepsilon\, g(\dot{\alpha}(t),S(\dot{\alpha}(t)))\xi\,.$$

Thus, from the last equation and Gauss formula we have

$$\overline{D}_{\dot{\alpha}(t)}\dot{\alpha}(t)=D_{\dot{\alpha}(t)}\dot{\alpha}(t)-\varepsilon\, g(\dot{\alpha}(t),S(\dot{\alpha}(t)))\xi=0\,,$$

where $\overline{D}$ is the Gauss connection on *M*. Hence we have seen that α is a geodesic on *M*.

($\Leftarrow$) **:** Now assume that α be a geodesic on *M*. Then

$$\overline{D}_{\dot{\alpha}(t)}\dot{\alpha}(t)=0\,.$$

Hence, by the Gauss formula we have

$$D_{\dot{\alpha}(t)}\dot{\alpha}(t)-\varepsilon\, g(\dot{\alpha}(t),S(\dot{\alpha}(t)))\xi=0\,.$$

Since *X* is a geodesic spray, we can write

$$\frac{d}{dt}\left(\dot{\alpha}((t))\big|_{\alpha(t)}\right)-X\left(\dot{\alpha}(t)\big|_{\alpha(t)}\right)=0\,,$$

$$\frac{d}{dt}\left(\dot{\alpha}(t)\big|_{\alpha(t)}\right)=X\left(\dot{\alpha}(t)\big|_{\alpha(t)}\right).$$

From the Equation (2), we found that

$$\frac{d}{dt}(\overline{\alpha}(t))=X(\overline{\alpha}(t))$$

Remark 2

Let α be a timelike curve. In this situation, its involute curve α^* must be a spacelike curve. (α,α^*) being the involute-evolute curve couple, the following lemma was obtained by Bilici and Çalışkan, Zonguldak Karaelmas University, personel communication.

Lemma 2

Let (α,α^*) be the involute-evolute curve couple. The relations between the Frenet vectors of the curve couple as follow.

(1) If w is a spacelike vector($|\kappa|>|\tau|$), then

$$\begin{bmatrix} t^* \\ n^* \\ b^* \end{bmatrix}=\begin{bmatrix} 0 & 1 & 0 \\ -\cosh\theta & 0 & \sinh\theta \\ -\sinh\theta & 0 & \cosh\theta \end{bmatrix}\begin{bmatrix} t \\ n \\ b \end{bmatrix}.$$

(2) If w is a timelike vector($|\kappa|<|\tau|$), then

$$\begin{bmatrix} t^* \\ n^* \\ b^* \end{bmatrix}=\begin{bmatrix} 0 & 1 & 0 \\ \sinh\theta & 0 & -\cosh\theta \\ -\cosh\theta & 0 & \sinh\theta \end{bmatrix}\begin{bmatrix} t \\ n \\ b \end{bmatrix}$$

Remark 3

(1) In this situation, the causal characteristics of the Frenet frames of the curves α and its involute curve α^* are {*t* timelike, *n* spacelike, *b* spacelike} and { t^* spacelike, n^* timelike, b^* spacelike}.
(2) If α is a timelike curve with timelike *w*, then the causal characteristics of the Frenet frame of the curves α and α^* must be of the form, {*t* timelike, *n* spacelike, *b* spacelike} and { t^* spacelike, n^* spacelike, b^* timelike}.

MAIN RESULTS

The natural lift of the spherical indicatrix of tangent vector of the involute curve α^*

Let α be a timelike curve with spacelike (or timelike) *w*. We will investigate how evolute curve α must be a curve satisfying the condition that $\overline{\alpha^*_{t^*}}$ is an integral curve of geodesic spray, where $\alpha^*_{t^*}$ being the spherical indicatrix of tangent vector of involute curve α^*, $\overline{\alpha^*_{t^*}}$ is the natural lift of the curve $\alpha^{*}{}_{t^*}$.

If $\overline{\alpha^{*}{}_{t^*}}$ is an integral curve of the geodesic spray, then by means of Lemma 1.

$$\overline{D}_{\dot{\alpha}^{*}{}_{t^*}}\dot{\alpha}^{*}{}_{t^*} = 0, \qquad (8)$$

where $\overline{D}$ is the connection of S_1^2 and the equation of the spherical indicatrix of tangent vector of the involute curve α^* is $\alpha^{*}{}_{t^*} = t^*$. Thus from Lemma 2. and (8) we have

$$\frac{\dot{\theta}}{\|w\|}\sinh\theta\, t - \frac{\dot{\theta}}{\|w\|}\cosh\theta\, b = 0.$$

Because of $\{t, n, b\}$ are linear independent , we have

$\theta =$ constant,

$\dfrac{\tau}{\kappa} =$ constant.

Result 1

If the curve α is a general helix, then the spherical indicatrix $\alpha^{*}{}_{t^*}$ of the involute curve α^* is a great circle (geodesic line) on the Lorentzian unit sphere S_1^2. In this case, from the Lemma1 the natural lift $\overline{\alpha^{*}{}_{t^*}}$ of $\alpha^{*}{}_{t^*}$ is an integral curve of the geodesic spray on the tangent bundle $T(S_1^2)$. In the case of a timelike curve with timelike w, similar result can be easily obtained in following same procedure.

Example

Let $\alpha(\theta) = \left(2\sinh\left(\frac{\theta}{\sqrt{3}}\right), 2\cosh\left(\frac{\theta}{\sqrt{3}}\right), \frac{\theta}{\sqrt{3}}\right)$ be a unit speed time-like helix such that

$t = \left(\frac{2}{\sqrt{3}}\cosh\left(\frac{\theta}{\sqrt{3}}\right), \frac{2}{\sqrt{3}}\sinh h\left(\frac{\theta}{\sqrt{3}}\right), \frac{1}{\sqrt{3}}\right)$, $\kappa = \frac{2}{3}$ and $\tau = -\frac{1}{3}$.

If α is a time-like curve then its involute curve is a space-like. In this situation, the involutes of the curve α can be given by the equation

$$\alpha^*(\theta) = \left(2\sinh\left(\frac{\theta}{\sqrt{3}}\right) + |c-\theta|\frac{2}{\sqrt{3}}\cosh\left(\frac{\theta}{\sqrt{3}}\right), 2\cosh\left(\frac{\theta}{\sqrt{3}}\right) + |c-\theta|\frac{2}{\sqrt{3}}\sinh\left(\frac{\theta}{\sqrt{3}}\right), \frac{c}{\sqrt{3}}\right), c \in IR.$$

The short calculations give the following equation of the spherical indicatrix of tangent vector of the involute curve α^*

$$\alpha^{*}{}_{t^*} = t^* = \left(\sinh\left(\frac{\theta}{\sqrt{3}}\right), \cosh\left(\frac{\theta}{\sqrt{3}}\right), 0\right).$$

Since

$$g\left(\alpha^{*}{}_{t^*}{}', \alpha^{*}{}_{t^*}{}'\right) = -\left(\frac{1}{\sqrt{3}}\right)^2 < 0,$$

$\alpha^{*}{}_{t^*}$ is timelike. For being $\alpha^{*}{}_{t^*}$ is a timelike curve, its spherical image which is geodesic line, lies on the Lorentzian unit sphere S_1^2. Furthermore, we can write

$$g(t, t^*) = 0.$$

The natural lift of the spherical indicatrix of principal normal vector of involute curve α^*

Let α be a timelike curve with spacelike (or timelike) *w*. We will investigate here, how α must be a curve satisfying the condition that $\overline{\alpha^{*}{}_{n^*}}$ is an integral curve of the geodesic spray, where $\alpha^{*}{}_{n^*}$ being the spherical indicatrix of principal normal vector of α^*, $\overline{\alpha^{*}{}_{n^*}}$ is the natural lift of the curve $\alpha^{*}{}_{n^*}$.

If $\overline{\alpha^{*}{}_{n^*}}$ is an integral curve of the geodesic spray, then by means of Lemma1. we have

$$\overline{\overline{D}}_{\dot{\alpha}^{*}{}_{n^*}}\dot{\alpha}^{*}{}_{n^*} = 0, \quad (\alpha^{*}{}_{n^*} = n^*)$$

$$\overline{\overline{D}}_{\dot{\alpha}^*{}_{n^*}}\dot{\alpha}^*{}_{n^*}=\Big[(-\dot{\sigma}sinh\theta-\dot{\theta}\sigma cosh\theta-\frac{\kappa}{k_n}+\|w\|k_n\,cosh\theta)t+(\frac{\dot{k}_n}{k_n{}^2})n$$
$$+(\dot{\sigma}\,cosh\theta+\dot{\theta}\sigma sinh\theta+\frac{\tau}{k_n}-\|w\|k_n\,sinh\theta)b\Big]\frac{1}{\|w\|k_n}$$

$$\sigma=\frac{\gamma_n}{k_n}$$

where $\overline{\overline{D}}$ is the connection of H_0^2, γ_n and k_n are the geodesic curvatures of the curve α with respect to S_1^2 and IR_1^3, respectively.

$$\left(\gamma_n=\frac{\dot{\theta}}{\|w\|},\,k_n=\frac{1}{\|w\|}\sqrt{\left|\dot{\theta}^2+\|w\|^2\right|}\right)$$

Since $\{t,n,b\}$ are linear independent,

$$\begin{cases}-\dot{\sigma}\,sinh\,\theta-\dot{\theta}\sigma\,cosh\,\theta-\dfrac{\kappa}{k_n}+\|w\|k_n\,cosh\,\theta=0\\ \dfrac{\dot{k}_n}{k_n{}^2}=0\\ \dot{\sigma}\,cosh\,\theta+\dot{\theta}\sigma\,sinh\,\theta+\dfrac{\tau}{k_n}-\|w\|k_n\,sinh\,\theta=0\end{cases}$$

and we get $k_n=$constant, $\gamma_n=$ constant.

Result 2

If the geodesic curvatures of the evolute curve α with respect to IR_1^3 and S_1^2 are constant, then the spherical indicatrix $\alpha^*{}_{n^*}$ is a geodesic line on the hyperbolic unit sphere H_0^2. In this case, the natural lift $\overline{\alpha^*{}_{n^*}}$ of $\alpha^*{}_{n^*}$ is an integral curve of the geodesic spray on the tangent bundle $T\left(H_0^2\right)$. In particular, if the evolute curve α is a timelike curve with timelike *w*, then the similar result can be easily obtained by taking S_1^2 instead of H_0^2.

The natural lift of the spherical indicatrix of the binormal vector of involute curve α^*

Let α be a timelike curve with spacelike or timelike *w*. We will investigate how α must be a curve satisfying the condition that $\overline{\alpha^*{}_{b^*}}$ is an integral curve of the geodesic spray, where $\alpha^*{}_{b^*}$ being the spherical indicatrix of binormal vector of α^*, $\overline{\alpha^*{}_{b^*}}$ is the natural lift of the curve $\alpha^*{}_{b^*}$.

If $\overline{\alpha^*{}_{b^*}}$ is an integral curve of the geodesic spray, then by means of Lemma 1.

$$\overline{\overline{D}}_{\dot{\alpha}^*{}_{n^*}}\dot{\alpha}^*{}_{n^*}=0,\qquad(\alpha^*{}_{b^*}=b^*)$$

that is, $\dfrac{\|w\|}{\dot{\theta}}n=0$.

Since $\{t,n,b\}$ are linear independent, we have $\|w\|$= 0.

Thus we get $\kappa=0$, $\tau=0$. So we can give the following result.

Result 3

The spherical indicatrix $\alpha^*{}_{b^*}$ of the involute curve α^* can not be a geodesic line on the hyperbolic unit sphere H_0^2, because, the evolute curve α whose curvature and torsion are equal to 0 is a straight line. In this case, $\left(\alpha,\alpha^*\right)$ can not occur in the involute-evolute curve couple. Thus, the natural lift $\overline{\alpha^*{}_{b^*}}$ of the curve $\alpha^*{}_{b^*}$ can never be an integral curve of the geodesic spray on the tangent bundle $T\left(H_0^2\right)$.

DISCUSSION

In this research, the natural lift curves of the spherical indicatrices of tangent, principal normal and binormal vectors of the spacelike involute curve α^* have been given in the Minkowski 3-space IR_1^3. Furthermore, some interesting results about the timelike evolute curve α were obtained, depending on the assumption that the natural lift curves $\overline{\alpha^*{}_{t^*}}$, $\overline{\alpha^*{}_{n^*}}$ and $\overline{\alpha^*{}_{b^*}}$ of the spherical indicatrices $\alpha^*{}_{t^*}$, $\alpha^*{}_{n^*}$ and $\alpha^*{}_{b^*}$ of the involute curve α^* should be the integral curve on the tangent bundle $T\left(S_1^2\right)$ or $T\left(H_0^2\right)$. It's expected that these results will be helpful to mathematicians who are specialized on mathematical modelling.

ACKNOWLEDGMENTS

The author would like to thank the referees for their valuable suggestions which improved the first version of the paper.

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