International Journal of the Physical Sciences Vol. 6(13), pp. 3246-3254, 4 July, 2011
Available online at http://www.academicjournals.org/IJPS
DOI: 10.5897/IJPS11.042

Full Length Research Paper

A non-polynomial spline method for solving linear fourth-order boundary-value problems

Omotayo A. Taiwo[1]* and Oladotun M. Ogunlaran[2]

[1]Department of Mathematics, Faculty of Science, University of Ilorin, Ilorin, Nigeria.
[2]Department of Mathematics and Statistics, Faculty of Science and Science Education, Bowen University, Iwo, Nigeria.

Accepted 7 March, 2011

In this paper, we use a non-polynomial spline method to develop a numerical technique for solving linear fourth-order boundary-value problems which are first reduced to a system of second-order boundary-value problems. Three numerical examples are considered to demonstrate the usefulness of the method and to show that the method converges with sufficient accuracy to the exact solutions.

Key words: Cubic spline, non polynomial spline, boundary-value problems, system of equations.

INTRODUCTION

We use a non-polynomial spline approximation to develop a new method for obtaining smooth approximations to the solutions of fourth-order boundary-value problems of the form:

$$u^{iv}(x)+a_1(x)u'''(x)+a_2(x)u''(x)+a_3(x)u'(x)+a_4(x)u(x)=f(x),\ a\le x\le b \qquad (1)$$

along with the boundary conditions

$$\left.\begin{aligned} u(a)&=\alpha_0, & u(b)&=\alpha_1,\\ u''(a)&=\beta_0, & u''(b)&=\beta_1 \end{aligned}\right\} \qquad (2)$$

Where $\alpha_i,\ \beta_i,\ i=0,1$ are arbitrary finite real constants and $a_1(x),\ a_2(x),\ a_3(x),\ a_4(x)$ and $f(x)$ are continuous on $[a,b]$. The analytical solution of problem (1) with boundary conditions (2) cannot be determined for any arbitrary choice of $a_1(x),\ a_2(x),\ a_3(x),\ a_4(x)$ and $f(x)$. We therefore employ numerical methods for obtaining approximate solution to the problem. Papamichael et al. (1981) considered and developed a cubic-spline method for the solution of the following fourth-order boundary-value problem:

$$y^{iv}(x)+f(x)y(x)=g(x), \quad a\le x\le b \qquad (3)$$

$$y(a)=\alpha_0, \qquad y(b)=\alpha_1 \qquad (4)$$

$$y'(a)=\beta_0, \qquad y'(b)=\beta_1 \qquad (5)$$

where $\alpha_i,\ \beta_i,\ i=0,1$ are finite real constants and the functions $f(x)$ and $g(x)$ are continuous in $[a,b]$.

Alberg and Ito (1975) used a collocation method for obtaining the solution of second-order boundary-value problems. Usmani (1978) presented three finite-difference techniques, of order 2, 4 and 6 respectively, for the solution of (3) to (5). Siddiqi et al. (2008) used quintic spline to solve the same boundary-value problems. Taiwo et al. (2008) derived polynomial cubic-spline methods to solve (1) and (2). Usmani (1980) derived cubic-, quartic- and sextic-spline methods for the numerical solution of the non-linear second-order boundary-value problems. Sixth-degree B-spline was developed for the solution of fifth-order boundary-value problems by Caglar et al. (1999). Al-Said (2008) and Rashidinia et al. (2007) derived quadratic- and cubic-spline techniques for the solution of fourth-order obstacle problems, respectively; while Siraj-ul-Islam et al. (2006) presented a quadratic non-polynomial spline method for the solution of second-order obstacle problems.

*Corresponding author. E-mail: oataiwo2002@yahoo.com

In order to solve (1) with the associated boundary conditions (2), we reduce (1) to a system of second-order differential equations as:

$$\left.\begin{aligned} &v''(x)+a_1(x)v'(x)+a_2(x)v(x)+a_3(x)u'(x)+a_4(x)u(x)=f(x)\\ &u''(x)-v(x)=0 \end{aligned}\right\} \tag{6}$$

with the boundary conditions

$$\left.\begin{aligned} &u(a)=\alpha_0, \qquad u(b)=\alpha_1\\ &v(a)=\beta_0, \qquad v(b)=\beta_1 \end{aligned}\right\} \tag{7}$$

DESCRIPTION OF NON-POLYNOMIAL SPLINE METHOD

To derive the non-polynomial spline approximation S to (6) with boundary conditions (7), we discretize the interval $[a,b]$ using equally spaced knots $x_i=a+ih,\ i=0,1,\ldots,n.\ x_0=a,\ x_n=b$ and $h={(b-a)}/{n}$ where n is any arbitrary positive integer.

A function S of class $C^2[a,b]$, which interpolates $u(x)$ at the mesh points $x_i,\ i=0,1,\ldots,n$ depends on a parameter k, and reduces to the normal cubic spline $S(x)$ in $[a,b]$ as $k\to 0$, is termed as non-polynomial spline function.

For each segment $[x_i,x_{i+1}],\ i=0,1,\ldots,n-1,$ the non-polynomial spline $S_i(x)$ has the form

$$S_i(x)=a_i+b_i(x-x_i)+c_i\sin k(x-x_i)+d_i\cos k(x-x_i),\ i=0,1,\ldots,n-1 \tag{8}$$

where a_i,b_i,c_i and d_i are constants and k is a free parameter.

Let u_i be an approximation to $u(x_i)$, obtained by the segment $S_i(x)$ of the mixed spline function passing through the points (x_i,u_i) and (x_{i+1},u_{i+1}). $S_i(x)$ is required to satisfy the interpolatory conditions at x_i and x_{i+1}, the boundary conditions (7) and the continuity condition of first derivatives at the common nodes (x_i,u_i).

In order to obtain the coefficients in expression (8) in terms of $u_i,\ u_{i+1},\ M_i$ and M_{i+1}, we define:

$$S_i(x_i)=u_i,\ S_i(x_{i+1})=u_{i+1},\ S_i''(x_i)=M_i,\ S_i''(x_{i+1})=M_{i+1} \tag{9}$$

After some algebraic manipulations, we obtain

$$\left.\begin{aligned} &a_i=u_i+\frac{M_i}{k^2},\ b_i=\frac{1}{h}(u_{i+1}-u_i)+\frac{1}{k\theta}(M_{i+1}-M_i)\\ &c_i=\frac{1}{k^2\sin\theta}(M_i\cos\theta-M_{i+1}),\quad d_i=\frac{-M_i}{k^2} \end{aligned}\right\}$$

$$\text{where } \theta=kh,\ i=0,1,\ldots,n-1. \tag{10}$$

Using the continuity condition of the first derivative at (x_i,u_i), that is $S_{i-1}'(x_i)=S_i'(x_i)$, we obtain the following consistency relation

$$\frac{1}{h^2}(u_{i-1}-2u_i+u_{i+1})=\left(\frac{1}{\theta\sin\theta}-\frac{1}{\theta^2}\right)M_{i-1}+2\left(\frac{1}{\theta^2}-\frac{\cos\theta}{\theta\sin\theta}\right)M_i+\left(\frac{1}{\theta\sin\theta}-\frac{1}{\theta^2}\right)M_{i+1} \tag{11}$$

$$i=1,2,\ldots,n-1.$$

For the purpose of simplicity, we write (11) as

$$\frac{1}{h^2}(u_{i-1}-2u_i+u_{i+1})=\alpha M_{i-1}+2\beta M_i+\alpha M_{i+1}$$

$$\text{Where } \alpha=\frac{1}{\theta\sin\theta}-\frac{1}{\theta^2},\ \beta=\frac{1}{\theta^2}-\frac{\cos\theta}{\theta\sin\theta} \tag{12a}$$

In a similar manner, we get for $v(x)$

$$\frac{1}{h^2}(v_{i-1}-2v_i+v_{i+1})=\alpha N_{i-1}+2\beta N_i+\alpha N_{i+1} \tag{12b}$$

Now the corresponding truncation error associated with (12a) is

$$\mathrm{T}=u(x_{i-1})-2u(x_i)+u(x_{i+1})-h^2\left(\alpha u''(x_{i-1})+2\beta u''(x_i)+\alpha u''(x_{i+1})\right)$$

Applying Taylor's theorem and simplifying, we obtain

$$\mathrm{T}=h^2(1-2\alpha-2\beta)u_i''+\tfrac{h^4}{12}(1-12\alpha)u_i^{iv}+\tfrac{h^6}{360}(1-30\alpha)u_i^{vi}+\ldots\ldots$$

$$i=1,2,\ldots,n. \tag{13}$$

Hence, for arbitrary choice of α and β satisfying the condition $1-2\alpha-2\beta=0$, implies that the method (12a) is second-order convergent. The method will be fourth-order convergent if $1-2\alpha-2\beta=0$ and $\alpha=1/12$.

APPLICATION OF NON-POLYNOMIAL SPLINE

To illustrate the application of the numerical method developed in the previous section, we discretize (6) at mesh points (x_i,u_i) and (x_i,v_i). So we have

$$\left.\begin{aligned} &v_i''+a_1(x_i)v_i'+a_2(x_i)v_i+a_3(x_i)u_i'+a_4(x_i)u_i=f(x_i)\\ &u_i''-v_i=0 \end{aligned}\right\} \tag{14}$$

Substituting $M_i = u_i''$ and $N_i = v_i''$ in (14) and rewriting, we obtain,

$$\left.\begin{aligned} N_i &= f(x_i) - a_1(x_i)v_i' - a_2(x_i)v_i - a_3(x_i)u_i' - a_4(x_i)u_i \\ M_i &= v_i \end{aligned}\right\} \tag{15}$$

We use the following $O(h^2)$ finite-difference approximations for the first derivative of v and u in Equation (15)

$$\left.\begin{aligned} u_i' &\cong \frac{u_{i+1} - u_{i-1}}{2h}, & v_i' &\cong \frac{v_{i+1} - v_{i-1}}{2h} \\ u_{i+1}' &\cong \frac{3u_{i+1} - 4u_i + u_{i-1}}{2h}, & v_{i+1}' &\cong \frac{3v_{i+1} - 4v_i + v_{i-1}}{2h} \\ u_{i-1}' &\cong \frac{-u_{i+1} + 4u_i - 3u_{i-1}}{2h}, & v_{i-1}' &\cong \frac{-v_{i+1} + 4v_i - 3v_{i-1}}{2h} \end{aligned}\right\} \tag{16}$$

Substituting (16) into (15), we obtain the following:

$$N_i = f(x_i) - a_1(x_i)\frac{v_{i+1} - v_{i-1}}{2h} - a_2(x_i)v_i - a_3(x_i)\frac{u_{i+1} - u_{i-1}}{2h} - a_4(x_i)u_i \tag{17}$$

$$\begin{aligned} N_{i+1} = f(x_{i+1}) - a_1(x_{i+1})\frac{3v_{i+1} - 4v_i + v_{i-1}}{2h} - a_2(x_{i+1})v_{i+1} \\ - a_3(x_{i+1})\frac{3u_{i+1} - 4u_i + u_{i-1}}{2h} - a_4(x_{i+1})u_{i+1} \end{aligned} \tag{18}$$

$$\begin{aligned} N_{i-1} = f(x_{i-1}) - a_1(x_{i-1})\frac{-v_{i+1} + 4v_i - 3v_{i-1}}{2h} - a_2(x_{i-1})v_{i-1} \\ - a_3(x_{i-1})\frac{-u_{i+1} + 4u_i - 3u_{i-1}}{2h} - a_4(x_{i-1})u_{i-1} \end{aligned} \tag{19}$$

$$M_i = v_i \tag{20}$$

$$M_{i+1} = v_{i+1} \tag{21}$$

$$M_{i-1} = v_{i-1} \tag{22}$$

Now, substituting Equations (17) to (19) into (12b) and simplifying, we obtain

$$\begin{aligned} &\left(\frac{3\alpha a_1(x_{i-1})}{2h} - \alpha a_2(x_{i-1}) + \frac{2\beta a_1(x_i)}{2h} - \frac{\alpha a_1(x_{i+1})}{2h} - \frac{1}{h^2}\right)v_{i-1} + \left(\frac{-4\alpha a_1(x_{i-1})}{2h} - 2\beta a_2(x_i) + \frac{4\alpha a_1(x_{i+1})}{2h} + \frac{2}{h^2}\right)v_i \\ &+\left(\frac{\alpha a_1(x_{i-1})}{2h} - \frac{2\beta a_1(x_i)}{2h} - \frac{3\alpha a_1(x_{i+1})}{2h} - \alpha a_2(x_{i+1}) - \frac{1}{h^2}\right)v_{i+1} + \left(\frac{3\alpha a_3(x_{i-1})}{2h} - \alpha a_4(x_{i-1}) + \frac{2\beta a_3(x_i)}{2h} - \frac{\alpha a_3(x_{i+1})}{2h}\right)u_{i-1} \\ &+\left(\frac{-4\alpha a_3(x_{i-1})}{2h} - 2\beta a_4(x_i) + \frac{4\alpha a_3(x_{i+1})}{2h}\right)u_i + \left(\frac{\alpha a_3(x_{i-1})}{2h} - \frac{2\beta a_3(x_i)}{2h} - \frac{3\alpha a_3(x_{i+1})}{2h} - \alpha a_4(x_{i+1})\right)u_{i+1} \\ &= -\alpha f(x_{i-1}) - 2\beta f(x_i) - \alpha f(x_{i+1}) \end{aligned} \tag{23}$$

For the purpose of simplicity, we rewrite (23) as

$$X_{1i}v_{i-1} + Y_{1i}v_i + Z_{1i}v_{i+1} + X_{2i}u_{i-1} + Y_{2i}u_i + Z_{2i}u_{i+1} = -\alpha f(x_{i-1}) - 2\beta f(x_i) - \alpha f(x_{i+1}) \tag{24}$$

$$i = 1, 2, \ldots, n-1.$$

where

$$X_{2i} = \frac{3\alpha a_3(x_{i-1})}{2h} - \alpha a_4(x_{i-1}) + \frac{2\beta a_3(x_i)}{2h} - \frac{\alpha a_3(x_{i+1})}{2h}$$

$$Y_{2i} = \frac{-4\alpha a_3(x_{i-1})}{2h} - 2\beta a_4(x_i) + \frac{4\alpha a_3(x_{i+1})}{2h}$$

and

$$Z_{2i} = \frac{\alpha a_3(x_{i-1})}{2h} - \frac{2\beta a_3(x_i)}{2h} - \frac{3\alpha a_3(x_{i+1})}{2h} - \alpha a_4(x_{i+1})$$

Similarly, substituting Equations (20) to (22) into (12a) and simplifying, we get

$$X_{3i}v_{i-1} + Y_{3i}v_i + Z_{3i}v_{i+1} + X_{4i}u_{i-1} + Y_{4i}u_i + Z_{4i}u_{i+1} = 0$$
$$i = 1, 2, \ldots, n-1,$$

$$\text{Where } X_{3i} = Z_{3i} = \alpha,\ Y_{3i} = 2\beta,\ X_{4i} = Z_{4i} = -\frac{1}{h^2},\ Y_{4i} = \frac{2}{h^2} \qquad (25)$$

Therefore, Equations (24), (25) and the boundary conditions (7) give a complete system of 2(n +1) linear equations in 2(n +1) unknowns.

NUMERICAL RESULTS

We solve three boundary-value problems using step lengths $h = 1/5$ and $h = 1/10$ to test the performance of the method developed. The numerical results are summarized in Tables 1 to 3 while, Figures 1 to 3 give additional information about the results.

Example 1

Consider the following problem.

$$u^{iv}(x) - u(x) = 0$$

subject to the boundary conditions

$$\left.\begin{array}{l} u(0) = u''(0) = 1 \\ u(1) = u''(1) = 0 \end{array}\right\}$$

with $u(x) = \frac{1}{2\sinh(1)}(e^{1-x} - e^{x-1})$ as its exact solution.

Example 2

$$u^{iv}(x) - 2u''(x) + u(x) = -8e^x$$

subject to the boundary conditions

$$\left.\begin{array}{l} u(0) = u(1) = 0 \\ u''(0) = 0, \quad u''(1) = -4e \end{array}\right\}$$

The exact solution is $u(x) = x(1-x)e^x$.

Example 3

$$u^{iv}(x) - 2u'''(x) + u''(x) = 0$$

subject to the boundary conditions

$$\left.\begin{array}{l} u(0) = u''(0) = 1 \\ u(1) = u''(1) = e \end{array}\right\}$$

The exact solution is $u(x) = e^x$.

Our numerical results on the test problems show an improvement in the results when $h = 1/10$ is used compared to when $h = 1/5$ is used for each of u_i, u_i' and u_i'' in the three cases considered, as reflected in reduction of absolute errors in Tables 1 to 3 and Figures 1 to 3.

The maximum absolute errors in u_i in Example 1 are 3.531E-7 and 2.209E-8 for $h = 1/5$ and $h = 1/10$ respectively, and are reached at $x = 0.4$; while the maximum absolute errors in u_i in Example 2 are 3.568E-5 and 2.235E-6 for $h = 1/5$ and $h = 1/10$, respectively, which are attained at $x = 0.6$. In Example 3, the maximum absolute errors are 1.434E-4 and 3.564E-5 for $h = 1/5$ and $h = 1/10$, respectively, and are reached at $x = 0.6$. Therefore, maximum absolute errorin u_i is attained when x is around the middle of the interval $[a, b]$. It is equally observed from the figures that absolute errors in u_i'' exhibit similar pattern of behaviour as absolute errors in u_i over the interval $[a, b]$.

The proposed method is of order $O(h^4)$, as established

Table 1. Observed absolute errors in Example 1.

x_i	u_i		u_i'		u_i''	
	$h=1/5$	$h=1/10$	$h=1/5$	$h=1/10$	$h=1/5$	$h=1/10$
0	0	0	5.920E-4	7.852E-5	0	0
0.2	2.723E-7	1.704E-8	4.372E-4	5.873E-5	2.723E-7	1.704E-8
0.4	3.531E-7	2.209E-8	3.028E-4	4.131E-5	3.531E-7	2.209E-8
0.6	3.027E-7	1.894E-8	1.769E-4	2.557E-5	3.027E-7	1.894E-8
0.8	1.709E-7	1.069E-8	6.005E-5	1.088E-5	1.709E-7	1.069E-8
1.0	0	0	5.415E-5	3.370E-6	0	0

Table 2. Observed absolute errors in Example 2.

x_i	u_i		u_i'		u_i''	
	$h=1/5$	$h=1/10$	$h=1/5$	$h=1/10$	$h=1/5$	$h=1/10$
0	0	0	3.398E-3	3.772E-4	0	0
0.2	1.960E-5	1.228E-6	5.058E-3	5.700E-4	3.858E-5	2.416E-6
0.4	3.211E-5	2.012E-6	7.390E-3	8.411E-4	6.271E-5	3.927E-6
0.6	3.568E-5	2.235E-6	1.062E-2	1.218E-3	6.887E-5	4.313E-6
0.8	2.683E-5	1.681E-6	1.505E-2	1.735E-3	5.106E-5	3.197E-6
1.0	0	0	1.568E-2	2.105E-3	0	0

Table 3. Observed absolute errors in Example 3.

x_i	u_i		u_i'		u_i''	
	$h=1/5$	$h=1/10$	$h=1/5$	$h=1/10$	$h=1/5$	$h=1/10$
0	0	0	3.097E-4	1.848E-5	0	0
0.2	8.075E-5	2.008E-5	5.387E-4	1.735E-5	6.618E-4	1.635E-4
0.4	1.357E-4	3.373E-5	9.278E-4	8.798E-5	1.214E-3	2.996E-4
0.6	1.434E-4	3.564E-5	1.447E-3	1.850E-4	1.485E-3	3.661E-4
0.8	9.408E-5	2.339E-5	2.027E-3	2.895E-4	1.211E-3	2.982E-4
1.0	0	0	1.148E-3	8.753E-5	0	0

in expression (13), if $\alpha = 1/12$, $1\text{-}2\alpha\text{-}2\beta=0$ and $a_1(x)=a_2(x)=0$. However, if either $a_1(x)\neq 0$ or $a_2(x)\neq 0$ or both $a_1(x)\neq 0$ and $a_2(x)\neq 0$, the technique produces an order $O(h^2)$ result because the finite-difference approximations of first derivative of u and v used are of $O(h^2)$. This accounts for the reason why when h is reduced by a factor of 2 it leads to a reduction of absolute error in u_i by a factor of 16 in Examples 1 and 2; while the same reduction produces a reduction of absolute error in u_i by a factor of 4 in Example 3, as can be observed in Tables 1 to 3. Richardson's extrapolation method can be applied in conjunction with the proposed method to further enhance its accuracy.

CONCLUSION

A non-polynomial spline method has been considered for

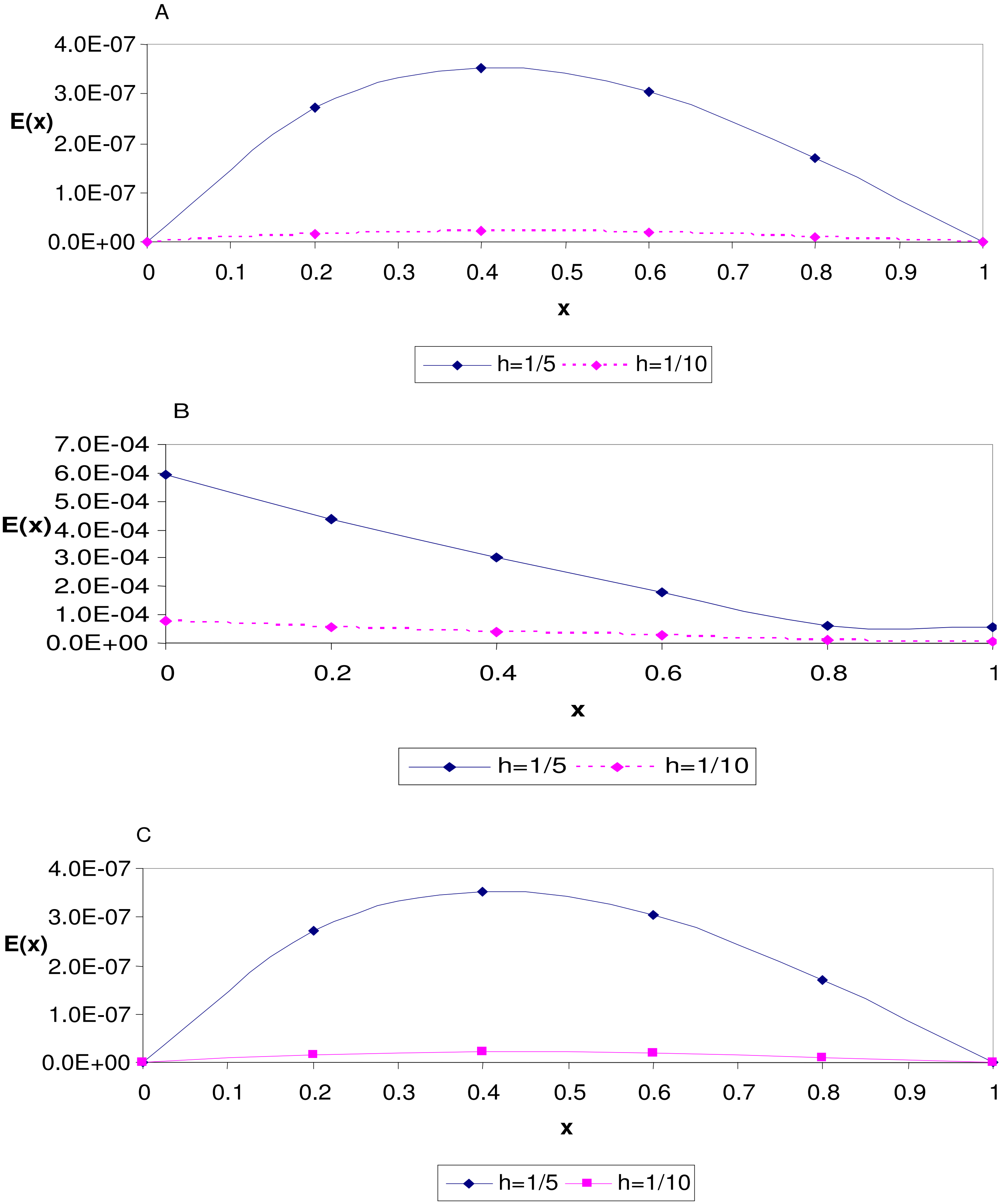

Figure 1. Comparison of absolute errors in (a) u, (b) u' and (c) u'' for two step sizes in Example 1.

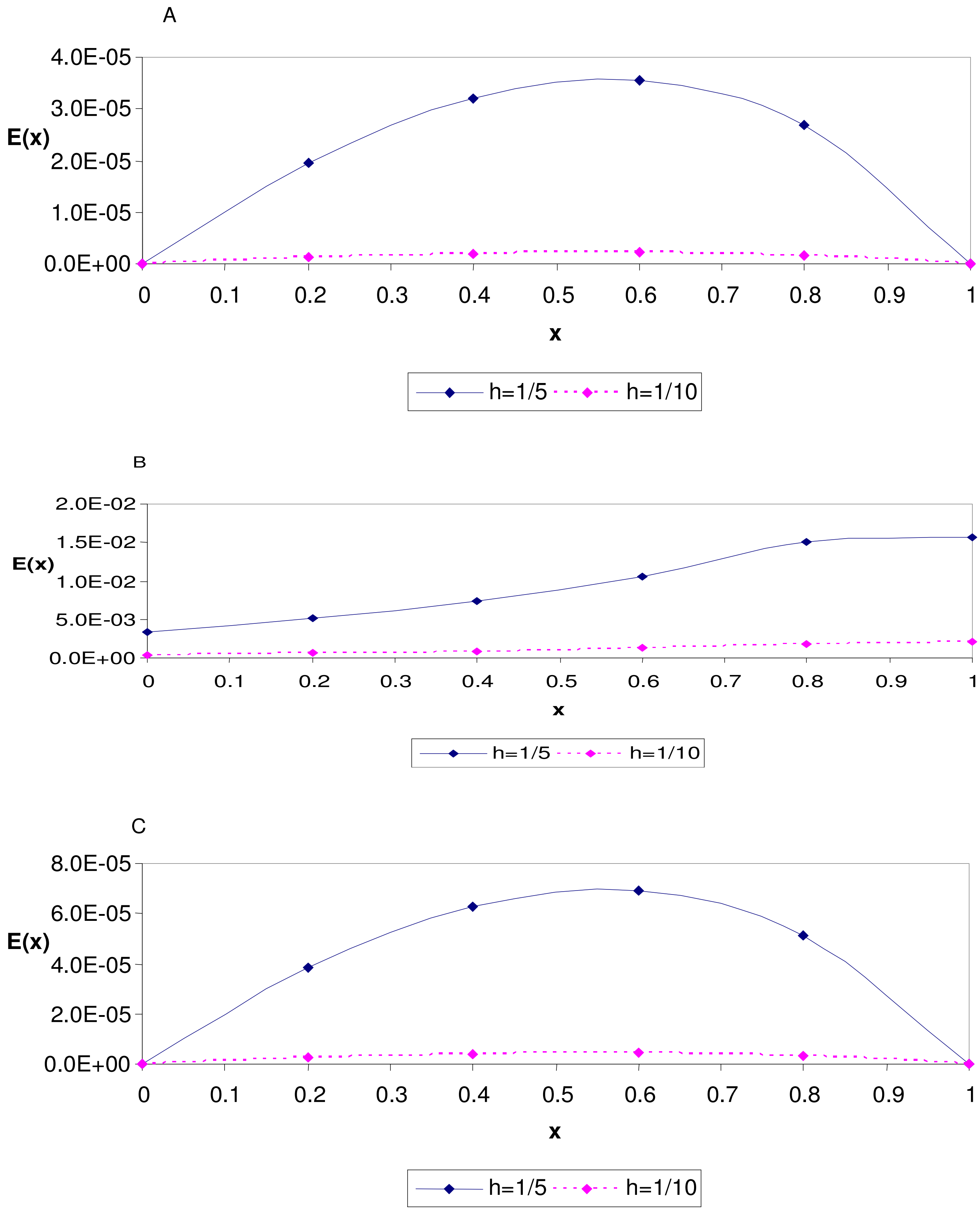

Figure 2. Comparison of absolute errors in (a) u, (b) u', (c) u'' for two step sizes in Example 2.

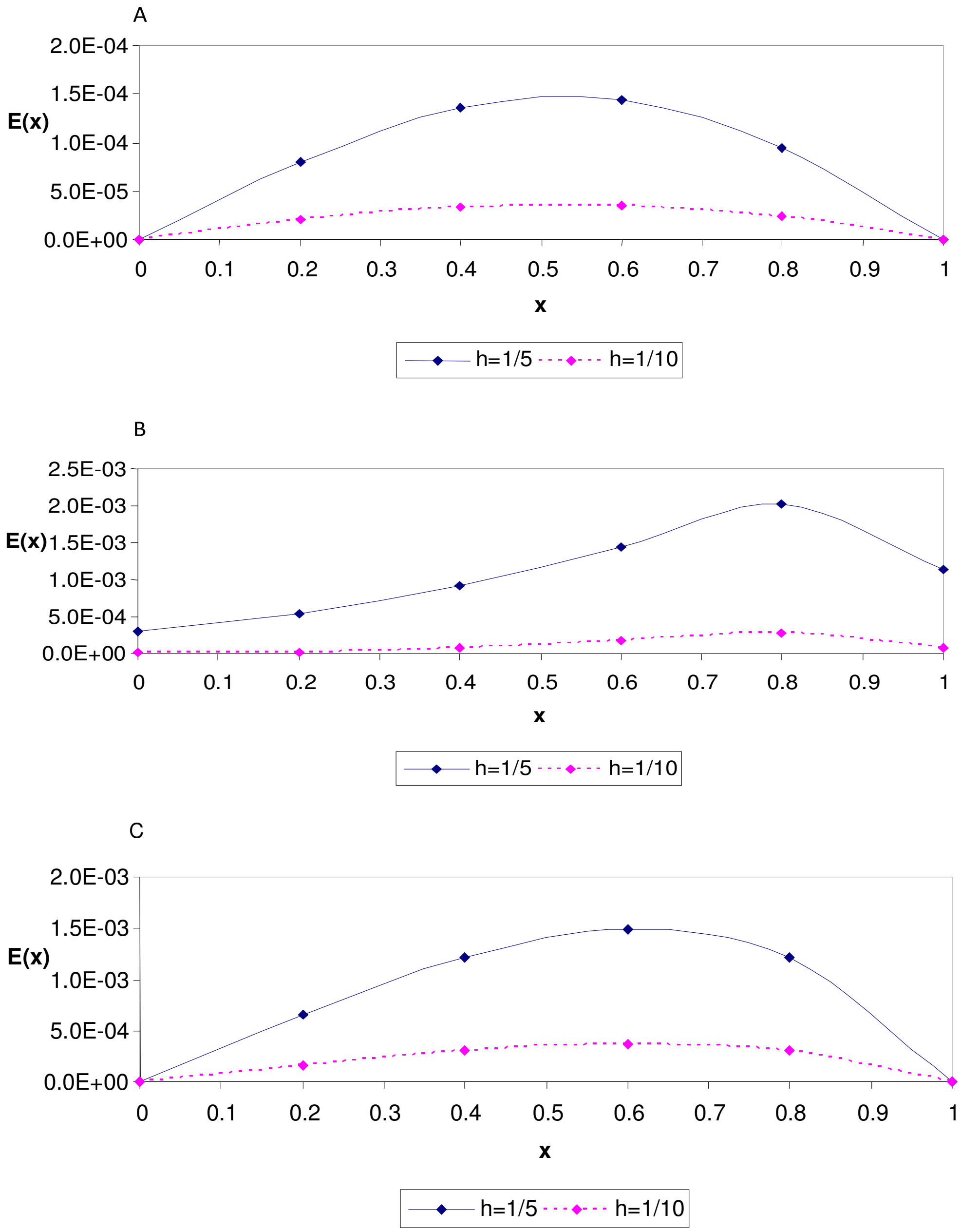

Figure 3. Comparison of absolute errors in (a) u, (b) u', (c) u'' for two step sizes in Example 3.

the numerical solution of fourth-order boundary value problems. The method is tested on three problems and the results obtained are very encouraging. The method is simple and easy to apply.

REFERENCES

Alberg JH, Ito T (1975). A collocation method for two-point boundary-value problems. Maths. Comput., 29: 761–776.

Al-Said EA (2008). Quadratic spline method for solving fourth-order obstacle problems. Appl. Math. Sci., 2: 1137–1144.

Caglar N, Caglar H (2009). B-Spline method for solving linear system of second-order boundary-value problems. Comput. Maths. Applicat., 57: 757–762.

Caglar HN, Caglar SH (1999). The numerical solution of fifth-order boundary-value problems with sixth-degree B-spline functions. Appl. Maths. Letter, 12: 25–30.

Papamichael N, Worsey AJ (1981). A cubic spline method for the solution of a linear fourth-order boundary-value problems. J. Comput. Appl. Math., 7: 187–189.

Rashidinia J, Mohammadi R, Jalilian R, Ghasemi M (2007). Convergence of cubic spline approach to the solution of a system of boundary-value problems. Appl. Math. Comp., 192: 319 – 331.

Siddiqi SS, Akram G (2008). Quintic spline solution of fourth-order

boundary-value problems. Internat. J. Numer. Anal. Model., 5: 101–111.
Siddiqi SS, Akram G, Elahi A (2008). Quartic spline solution of fifth-order boundary-value problems. Appl. Maths. Comp., 196: 214-220.
Siraj-ul-Islam, Noor MA, Tirmizi IA, Khan MA (2006). Quadratic non-polynomial spline approach to the solution of a system of second-order boundaryvalue problems. Appl. Maths. Comp., 179: 153–160.
Taiwo OA, Ogunlaran OM (2008). Numerical solution of fourth-order linear ordinary differential equations by cubic-spline collocation Tau method. J. Maths. Stat., 4: 264-268.
Usmani RA (1978). Discrete-variable methods for a boundary-value problem with engineering applications. Math. Comput., 32: 1087–1096.
Usmani RA (1980). Spline solutions for non-linear two-point boundary-value problems. Internat. J. Math. Math. Sci., 3: 151–167.