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*Full Length Research Paper*

# Construction of the Frenet-Serret frame of a curve in 4D Galilean space and some applications

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**In this work, first, we construct Frenet-Serret frame of a curve in the Galilean 4-space. As a result of this, we obtain the mentioned curve's Frenet-Serret equations. Then, we prove that tangent vector of a curve in Galilean 4-space satisfies a vector differential equation of fourth order. Additionally, some characterizations of Galilean spherical curves and an example of the main results are presented.**

**Key words:** Galilean space, Frenet-Serret frame, spherical curves, vector differential equation.

## INTRODUCTION

The discovery of the moving Frenet-Serret frame for regular curves opened a door to classical differential geometry. With the aid of the ordinary differential equations, researchers studied some aspects of the theory of the curves, spherical images, involutes-evolutes, sphere-cal curves and Bertrand curves, e.g. At the beginning of the twentieth century, Einstein's theory constructed a bridge between mathematical physics and modern differential geometry. It has been observed that Lorentz-Minkowski geometry plays an important role in the explanation of the relativistic motion of the charged particles in an electromagnetic field. For instance, a particle in special relativity means a curve with a time-like unitary tangent vector, Caltenco et al. (2002). Making use of the Frenet-Serret equations established in this new area, researchers extend some of classical differential geometry topics to Lorentz-Minkowski spaces. There exists an extensive literature on the subject (Ali and Turgut, 2010; Barros et al., 2001; İlarslan and Boyacıoğlu, 2008; López, 2003, 2008, 2001.

In recent years, researchers have begun to investigate curves and surfaces in the Galilean space and thereafter pseudo-Galilean space. The theory of the curves in Galilean space is extensively studied in Röschel (1986). In this space we refer; about spherical curves in $G_3$, Ergüt and Öğrenmiş (2009), Ogrenmis et al. (2007); on Bertrand curves Öğrenmiş et al. (2009). It is safe to report that a good amount of researches have also been done in pseudo-Galilean space by the aid of the interesting paper by Divjak (1998); and thereafter classical differential geometry papers Divjak and Milin-Šipuš (2003), Divjak and Milin-Šipuš (2008) and Öğrenmiş and Ergüt (2009).

In this work, in the light of the existing literature we extend aspects of classical differential geometry topics to the Galilean 4-space. We first construct Frenet-Serret frame of a curve and in terms of this frame, we obtain Frenet-Serret equations in the space $G_4$. Besides, we prove that tangent vector of a curve in $G_4$ satisfies a vector differential equation of fourth order. Moreover, we express some characterizations of Galilean spherical curves.

## PRELIMINARIES

The study of mechanics of plane-parallel motions reduces to the study of a geometry of three dimensional space with coordinates $\{x, y, t\}$ is given by the motion formula Yaglom (1979).

$$\begin{cases} x' = (\cos\alpha)x + (\sin\alpha)y + (v\cos\beta)t + a, \\ y' = -(\sin\alpha)x + (\cos\alpha)y + (v\sin\beta)t + b, \\ t' = t + d. \end{cases}$$

This geometry can be called three-dimensional Galilean Geometry. Yaglom (1979) stressed that four-dimensional Galilean Geometry, which studies all properties invariant under motions of objects in space, is even more complex. Yaglom (1979) also stated this geometry can be described more precisely as the study of those properties of four-dimensional space with coordinates that are invariant under the general Galilean transformations.

$$\begin{cases} x' = (\cos\beta\cos\alpha - \cos\gamma\sin\beta\sin\alpha)x + (\sin\beta\cos\alpha - \cos\gamma\cos\beta\sin\alpha)y \\ \quad + (\sin\gamma\sin\alpha)z + (v\cos\delta_1)t + a, \\ y' = -(\cos\beta\sin\alpha + \cos\gamma\sin\beta\cos\alpha)x + (-\sin\beta\sin\alpha + \cos\gamma\cos\beta\cos\alpha)y \\ \quad + (\sin\gamma\cos\alpha)z + (v\cos\delta_2)t + b, \\ z' = (\sin\gamma\sin\beta)x - (\sin\gamma\cos\beta)y + (\cos\gamma)z + (v\cos\delta_3)t + c, \\ t' = t + d, \end{cases}$$

with $\cos^2\delta_1 + \cos^2\delta_2 + \cos^2\delta_3 = 1.$

In the light of the motion equations above, we observe that the geometry of four-dimensional Galilean geometry roughly restricted. In this work, we intend to study theory of curves in the Galilean 4-space.

The basic elements of the theory of the curves in the Galilean space are presented in Röschel (1986), Öğrenmiş et al. (2009). First we extended the classical elements of the theory of the curves expressed in Röschel (1986) and Öğrenmiş et al. (2009) to the Galilean 4-space.

Let $\alpha : I \subset IR \to G_4$ be a curve given by

$$\alpha(t) = (x(t), y(t), z(t), w(t)),$$

Where $x(t), y(t), z(t), w(t) \in C^4$ (the set of three times continuously differentiable functions) and $t$ run through in a real interval. Let $\alpha$ be a curve in $G_4$, parameterized by arclength $t = s,$ given in coordinate form

$$\alpha(s) = (s, y(s), z(s), w(s))$$

In affine coordinates the Galilean scalar product between two points

$P_i = (x_{i1}, x_{i2}, x_{i3}, x_{i4})$ , $i = 1,2$ is defined by

$$g(P_1, P_2) = \begin{cases} |x_{21} - x_{11}|, & \text{if} \quad x_{11} \neq x_{21}, \\ \sqrt{(x_{22} - x_{12})^2 + (x_{23} - x_{13})^2 + (x_{24} - x_{14})^2}, & \text{if} \quad x_{11} = x_{21}. \end{cases}$$

We define the Galilean cross product in $G_4$ for the vectors $a = (a_1, a_2, a_3, a_4)$, $b = (b_1, b_2, b_3, b_4)$ and $c = (c_1, c_2, c_3, c_4)$ as follows:

$$a \wedge b \wedge c = \begin{vmatrix} 0 & e_2 & e_3 & e_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix},$$

Where $e_i$ are the standard basis vectors.

In this paper, we shall denote the inner product of two vectors $a,b$ in the sense of Galilean by the notation $< a,b >_G$ . The Galilean sphere of the space $G_4$ is defined by

$$S_G^3(m,r) = \{\varphi - m \in G_4 :< \varphi - m, \varphi - m >_G = \pm r^2\}.$$

## Construction of the Frenet-Serret frame

Let $\alpha(s) = (s, y(s), z(s), w(s))$ be a curve parameterized by arclength $s$ in $G_4$. Here, we denote differentiation with respect to $s$ by a dash. The first vector of the Frenet-Serret frame, namely the tangent vector of $\alpha$ is defined by

$$t = \alpha'(s) = (1, y'(s), z'(s), w'(s)).$$

Since $t$ is a unit vector, so we may express

$$< t, t >_G = 1. \tag{1}$$

Differentiating the formula (1) with respect to $s$ , we have $< t', t >_G = 0.$ The vector function $t'$ gives us the rotation measurement of the curve $\alpha$ . The real valued function

$$\kappa(s) = \|t'\| = \sqrt{(y'')^2 + (z'')^2 + (w'')^2}$$

is called the first curvature of the curve $\alpha$ . In the rest of the paper, we shall suppose $\kappa(s) \neq 0$ at everywhere. Similar to space $G_3$ , we define the principal vector

$$n(s) = \frac{t'(s)}{\kappa(s)}$$

or another words

$$n(s) = \frac{1}{\kappa}(0, y''(s), z''(s), w''(s)). \tag{2}$$

By the aid of the differentiation of the principal normal vector (2), we define the second curvature function as

$$\tau(s) = \|n'(s)\|_G. \quad (3)$$

This real valued function is called "torsion" of the curve $\alpha$. The third vector field, namely binormal vector field of the curve $\alpha$ is defined by

$$b(s) = \frac{1}{\tau(s)}\left(0, \left(\frac{1}{\kappa(s)}y''\right)', \left(\frac{1}{\kappa(s)}z''\right)', \left(\frac{1}{\kappa(s)}w''\right)'\right).$$

(4)

Therefore the vector $b(s)$, is both perpendicular to $t$ and $n$. The fourth unit vector is defined by

$$e = \mu \quad t \wedge n \wedge b. \quad (5)$$

Here the coefficient $\mu$ is taken $\pm 1$ to make $+1$ the matrix $[t, n, b, e]$. We define the third curvature of the curve $\alpha$ by the inner product

$$\sigma = < b', e >_G . \quad (6)$$

Here, as well-known, the set $\{t, n, b, e, \kappa, \tau, \sigma\}$ is called the Frenet-Serret apparatus of the curve $\alpha$. And here, we know that the vectors $\{t, n, b, e\}$ are mutually orthogonal vectors satisfying

$$< t,t >_G = < n,n >_G = < b,b >_G = < e,e >_G = 1,$$
$$<t,n>_G = <t,b>_G = <t,e>_G = <n,b>_G = <n,e>_G = <b,e>_G = 0.$$

## The Frenet-Serret equations

Let $\alpha(s) = (s, y(s), z(s), w(s))$ be a curve parameterized by arclength $s$ in $G_4$. Considering the definitions above, first we know that

$$t' = \kappa(s)n(s). \quad (7)$$

It is possible to define the vector $n'$ according to frame $\{t, n, b, e\}$ by

$$n' = \delta_1(s)t(s) + \delta_2(s)n(s) + \delta_3(s)b(s) + \delta_4(s)e(s),$$

$\delta_i \in IR$, for $1 \le i \le 4$. Multiplying both sides by the vectors $\{t, n, b, e\}$ and considering equation (1), we have, respectively

$$\begin{cases} \delta_1 = < n', t >_G = 0, \\ \delta_2 = < n', n >_G = 0, \\ \delta_3 = < n', b >_G = \tau(s). \end{cases}$$

By the formulas (3) and (4), we easily obtain $n'$ according to standard frame

$$n'(s) = \left(0, \left(\frac{1}{\kappa(s)}y''\right)', \left(\frac{1}{\kappa(s)}z''\right)', \left(\frac{1}{\kappa(s)}w''\right)'\right). \quad (8)$$

So (8) and the definition of cross product yields

$$\delta_4 = < n', e >_G = 0.$$

Since, we immediately arrive at

$$n' = \tau(s)b(s). \quad (9)$$

In order to compute the vector function $b'$, let us decompose

$$b' = \beta_1(s)t(s) + \beta_2(s)n(s) + \beta_3(s)b(s) + \beta_4(s)e(s),$$

Where the functions $\beta_i \in IR$ for $1 \le i \le 4$.

Similar to $n'$, we express

$$\begin{cases} \beta_1 = < b', t >_G = 0, \\ \beta_2 = < b', n >_G = -\tau(s), \\ \beta_3 = < b', b >_G = 0, \\ \beta_4 = < b', e >_G = \sigma(s). \end{cases}$$

So, we get

$$b' = -\tau(s)n(s) + \sigma(s)e(s). \quad (10)$$

In an analogous way, we write

$$e' = \gamma_1(s)t(s) + \gamma_2(s)n(s) + \gamma_3(s)b(s) + \gamma_4(s)e(s),$$

for the real valued functions $\gamma_i \in IR$ for $1 \le i \le 4$. Then, in terms (10), one can obtain

$$\begin{cases} \gamma_1 = < e', t >_G = 0, \\ \gamma_2 = < e', n >_G = 0, \\ \gamma_3 = < e', b >_G = -\sigma(s), \\ \gamma_4 = < e', e >_G = 0. \end{cases}$$

By the equation above, we obtain

$$e' = -\sigma(s)b(s).$$

Since, we write Frenet-Serret equations

$$\begin{cases} t' = \kappa(s)n(s), \\ n' = \tau(s)b(s), \\ b' = -\tau(s)n(s) + \sigma(s)e(s), \\ e' = -\sigma(s)b(s). \end{cases} \tag{11}$$

We may also write the Frenet-Serret equations in matrix form

$$\begin{bmatrix} t' \\ n' \\ b' \\ e' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ 0 & 0 & \tau & 0 \\ 0 & -\tau & 0 & \sigma \\ 0 & 0 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \\ e \end{bmatrix}.$$

Consequently, we obtained Frenet-Serret equations of the curve $\alpha$.

## Vector differential equation satisfied by the curves of $G_4$

**Theorem 1:** Let $\alpha(s) = (s, y(s), z(s), w(s))$ be a curve parameterized by arclength $s$ in $G_4$ with the Frenet-Serret equations (11). Tangent vector of a curve in $G_4$ satisfies a vector differential equation of fourth order as follows:

$$\left\{ \frac{1}{\sigma} \left[ \frac{1}{\tau} \left( \frac{t'}{\kappa} \right)' \right]' \right\}' + \left[ \frac{\tau}{\kappa\sigma} t' \right]' + \frac{\sigma}{\tau} \left( \frac{t'}{\kappa} \right)' = 0. \tag{12}$$

**Proof:** Let $\alpha(s) = (s, y(s), z(s), w(s))$ be a curve parameterized by arclength $s$ in $G_4$ with Frenet-Serret equations (11). Using $(11)_1$ in $(11)_2$ we have

$$\left( \frac{t'}{\kappa} \right)' = \tau(s)b(s). \tag{13}$$

Substituting (12) to $(11)_3$, we get

$$\left[ \frac{1}{\tau} \left( \frac{t'}{\kappa} \right)' \right]' + \frac{\tau}{\kappa} t' = \sigma(s)e(s). \tag{14}$$

Differentiating (14) and considering the formula $(11)_4$, we have (12) as desired.

## Spherical curves in $G_4$

In this section, we give some characterizations for the Galilean spherical curves by the classical differential geometry methods. Recall that, in the Euclidean space, for an arbitrary curve $\alpha = \alpha(s)$ lies on a sphere with center $c$, then $(\alpha - c)^2$ is constant (hence all of its derivatives with respect to $s$ are zero) and so we are led to the following definition of contact; Millman and Parker (1977):

**Definition 1:** Let c and $r > 0$ given and $f(s) = (\alpha - c)^2$. We say that $\alpha$ has $jth$ order spherical contact with sphere of radius $r$ and center c at $s = s_0$ if

$$f(s_0) = r^2, f'(s_0) = f''(s_0) = ... = f^{(j)}(s_0) = 0.$$

**Theorem 2:** Let $\alpha(s) = (s, y(s), z(s), w(s))$ be a curve parameterized by arclength $s$ in $G_4$ with Frenet-Serret equations (11). If $\alpha$ lies on the Galilean sphere $S_G^3$ center $c$ and radius $r$, then the center is

$$c = \alpha(s) + \rho(s)n(s) + \frac{\rho'}{\tau} b(s) + \frac{1}{\sigma} \left\{ \frac{\tau}{\kappa} + \left[ \frac{\rho'}{\tau} \right]' \right\} e(s), \tag{15}$$

where $\rho = \dfrac{1}{\kappa}$.

Proof: Let $\alpha(s) = (s, y(s), z(s), w(s))$ be a curve parameterized by arclength $s$ in $G_4$ with Frenet-Serret equations (11). In the light of the definition 2 and definition of Galilean sphere, we may write

$$f(s) = < c(s) - \alpha(s), c(s) - \alpha(s) >_G = \pm r^2. \tag{16}$$

By definition of contact, the formula (16) has to satisfy

$$f'(s) = f''(s) = f'''(s) = f^{(IV)}(s) = 0.$$

Since, we differentiate

$$< -t(s), c(s) - \alpha(s) >_G = 0. \tag{17}$$

One more differentiating of (17) gives us

$$-\kappa < n(s), c(s) - \alpha(s) >_G + 1 = 0. \tag{18}$$

We may compose the vector

$$c(s)-\alpha(s)=c_1(s)t(s)+c_2(s)n(s)+c_3(s)b(s)+c_4(s)e(s). \qquad (19)$$

By (17), we easily have $c_1 = 0$. Using (18) in (19), we also have

$$-\kappa c_2 + 1 = 0. \qquad (20)$$

So,

$$c_2 = \frac{1}{\kappa}. \qquad (21)$$

Differentiating (18), we also express

$$-\kappa' < n(s), c(s)-\alpha(s) >_G - \kappa < n'(s), c(s)-\alpha(s) >_G = 0. \qquad (22)$$

Substituting $n' = \tau(s)b(s)$ to (22) and dividing both sides by $\kappa$, we have

$$c_3 = \frac{1}{\tau}\left(\frac{1}{\kappa}\right)'. \qquad (23)$$

In terms of (22), it can written as

$$< b(s), c(s)-\alpha(s) >_G = \frac{1}{\tau}\left(\frac{1}{\kappa}\right)'. \qquad (24)$$

Differentiating (24), we may write

$$c_4 = \frac{1}{\sigma}\left\{\frac{\tau}{\kappa} + \left[\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right]'\right\}. \qquad (25)$$

Considering the obtained components (21), (23) and (25) and denoting $\rho = \frac{1}{\kappa}$, we have (15).

**Theorem 3:** Let $\alpha(s) = (s, y(s), z(s), w(s))$ be a curve parameterized by arclength $s$ in $G_4$ with Frenet-Serret equations (11). If $\alpha$ lies on the Galilean sphere $S_G^3$, then the radius of the sphere satisfies

$$r^2 = \rho^2 + \left(\frac{\rho'}{\tau}\right)^2 + \frac{1}{\sigma^2}\left[\rho\tau + \left(\frac{\rho'}{\tau}\right)'\right]^2, \qquad (26)$$

where $\rho = \frac{1}{\kappa}$.

Proof: The proof can be obtained by the taking norm of both sides of (15).

**Theorem 4:** Let $\alpha(s) = (s, y(s), z(s), w(s))$ be a curve parameterized by arclength $s$ in $G_4$ with Frenet-Serret equations (11). If $\alpha$ lies on the Galilean sphere $S_G^3$, then the curvature functions of $\alpha$ satisfy the differential equation

$$\frac{\sigma}{\tau}\rho' + \left\{\frac{1}{\sigma}\left[\rho\tau + \left(\frac{\rho'}{\tau}\right)'\right]\right\}' = 0.$$

**Proof:** The proof can be obtained by the differentiation of the formula (26).

**Example**

Let us consider the following curve

$$I \subset IR \to G_4$$

$$\alpha = \alpha(s) = \left(s,\ \frac{\sqrt{3}}{2}s,\ \arctan s - \frac{s}{2},\ -\ln\sqrt{1+s^2}\right). \qquad (27)$$

Differentiating (27), we have

$$\alpha'(s) = \left(1,\ \frac{\sqrt{3}}{2},\ \frac{1}{1+s^2} - \frac{1}{2},\ -\frac{s}{1+s^2}\right). \qquad (28)$$

Galilean inner product follows that $< \alpha', \alpha' >_G = 1$. So the curve is parameterized by arclength and the tangent vector is (28). In order to calculate the first curvature let us express

$$t' = \left(0,\ 0,\ -\frac{2}{(1+s^2)^2},\ \frac{s^2-1}{(1+s^2)^2}\right). \qquad (29)$$

Taking the norm of both sides, we have $\kappa(s) = 1$. Thereafter, we arrive at

$$n = \left(0,\ 0,\ -\frac{2}{(1+s^2)^2},\ \frac{s^2-1}{(1+s^2)^2}\right). \qquad (30)$$

One more differentiating of (30), we have

$$n' = \left(0,\ 0,\ \frac{s^2-1}{\left(1+s^2\right)^2},\ \frac{2s}{\left(1+s^2\right)^2}\right). \quad (31)$$

By the aid of the formula (31), we have the torsion function

$$\tau(s) = \frac{2}{1+s^2}$$

and binormal vector

$$b = \left(0,\ 0,\ \frac{2\left(s^2-1\right)}{\left(1+s^2\right)^3},\ \frac{4s}{\left(1+s^2\right)^3}\right). \quad (32)$$

The cross product of (28), (30) and (32) is formed by

$$t \wedge n \wedge b = \begin{vmatrix} 0 & e_2 & e_3 & e_4 \\ 1 & \frac{\sqrt{3}}{2} & \frac{1}{1+s^2} - \frac{1}{2} & -\frac{s}{1+s^2} \\ 0 & 0 & -\frac{2}{\left(1+s^2\right)^2} & \frac{s^2-1}{\left(1+s^2\right)^2} \\ 0 & 0 & \frac{2\left(s^2-1\right)}{\left(1+s^2\right)^3} & \frac{4s}{\left(1+s^2\right)^3} \end{vmatrix}.$$

Since, we have

$$e = \mu\left(0,\quad -2\frac{s^4-6s^2+1}{(1+s^2)^5},\quad 0,\quad 0\right). \quad (33)$$

In order to determine the third curvature of the curve, differentiating (33) and considering (32), we have $\sigma = 0$.

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