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Full Length Research Paper

Some numerical methods for solving Burgers equation

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In this paper, we construct and suggest some new, efficient, and accurate numerical algorithms based on the spectral methods for solving the Burgers' equation. Spectral methods using Chebyshev and Legendre polynomials are developed. Test problems are used to validate these methods that are found to be accurate and efficient. Also, the results show that despite the fact that Chebyshev polynomials are the most widely used ones; the use of Legendre polynomials gives more accurate results. In addition the present study includes numerical solution of Burgers' equation by means of the Adomian decomposition method. Comparison with other methods is also given.

Key words: Burgers' equation, spectral, Adomian, Tanh methods.

INTRODUCTION

There has been a great interest in approximating the solution of initial boundary value problems using spectral methods as a truncated series of globally defined smooth functions in each space variable, which can be considered as a development of the known weighted residual method. The choice of trial functions distinguishes spectral methods from finite difference and finite element methods. For problems with smooth solutions this method is highly accurate and in many other equations (especially nonlinear problems) gives a good approximation to the solution. The trial functions considered here are the well-known Chebyshev and Legendre polynomials, which are defined as:

i) Chebyshev polynomial $T_n(x)$ are defined on (-1,1)

with respect to the weight function $w(x)=\frac{1}{\sqrt{1-x^2}}$ as

$$T_n(x)=\cos(n\cos^{-1}x),\ n\geq 0$$

and the recurrence relation:

$$T_{n+1}(x)=2xT_n(x)-T_{n-1}(x), T_0(x)=1, T_1(x)=x, \tag{1}$$

Where,

$$\int_{-1}^{1}\frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}dx=c_n\delta_{nm}, c_0=2, c_n=1, n>0, \delta_{nm}=\begin{cases}1, & if\, n=m\\ 0, & if\, n\neq m\end{cases} \tag{2}$$

Actually,

$$T_n(x)=\frac{n}{2}\sum_{m=0}^{\frac{n}{2}}(-1)^m\frac{(n-m-1)!}{m!(n-2m)!}(2x)^{n-2m}, \tag{3}$$

ii) Legendre polynomials $p_n(x)$ on (-1,1) with the weight function w(x) = 1 are defined with the recurrence relation:

$$(n+1)p_{n+1}(x)=(2n+1)xp_n(x)-np_{n-1}(x), p_0(x)=1, p_1(x)=x, \tag{4}$$

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And,

$$\int_{-1}^{1} p_n(x) p_m(x) dx = \frac{2}{2n+1}\delta_{nm}, p_n(x) = \frac{1}{2n}\sum_{m=0}^{\frac{n}{2}}(-1)^m \binom{n}{m}\binom{2n-m}{n} x^{n-2m} \quad (5)$$

Kaya (2004), Kaya et al. (2003), Adomian (1994), Wazwaz (2000), Ismail et al. (2004) and Khater et al. (2002) have used the Adomian decomposition method (ADM) for solving large nonlinear partial differential equations (coupled system). In this paper, we use the Adomian decomposition method for solving the Burgers' equation, which has been suggested and developed by Adomian (1994). Some numerical examples are tested to illustrate the efficiency of the proposed algorithm. It is an interesting and open problem to extend this technique for solving the variational inequalities associated with the obstacle, unilateral, contact and free boundary value problems. Noor (2004, 2009) and Noor et al. (1993), developed applications, formulation and other numerical methods for solving variational inequalities and related optimization problems.

Burgers equation and the Tanh-function method

We consider the Berger equation of the following type:

$$u_t + \varepsilon u u_x - \nu u_{xx} = 0, \quad a \le x \le b, \quad (6)$$

Where ε *and* ν are positive parameters and the subscripts t and x denote differentiation. Appropriate boundary conditions will be chosen from the following:

$$u(a,t) = \beta_1, u(b,t) = \beta_2 \quad \forall t > 0, \quad (7)$$

and the initial condition to be used is described in "the initial state" (Khate et al., 2002).

In this study, we find the exact solution for Burgers' equation using the recent Tanh-function method (Parkes et al., 1997, 1998; Khater et al., 2002; Fan, 2000, 2001; Evans and Raslan, 2005a, b). For this purpose, suppose we use the transformation:

$$u(x,t) = f(\xi), \quad (8)$$

Where $\xi = c(x - \lambda t)$.

Based on this fact, we consider the following changes:

$$\frac{\partial}{\partial t}(.) = -c\lambda\frac{d}{d\xi}(.), \frac{\partial}{\partial x}(.) = c\frac{d}{d\xi}(.), \frac{\partial^2}{\partial x^2}(.) = c^2\frac{d^2}{d\xi^2}(.), \quad (9)$$

Using (9), Burgers' Equation 6 can be represented by the ordinary differential equation:

$$-c\lambda\frac{df(\xi)}{d\xi} + \varepsilon\, c\, f(\xi)\frac{df(\xi)}{d\xi} - c^2\nu\frac{d^2 f(\xi)}{d\xi^2} = 0. \quad (10)$$

Integration (10), we have:

$$-c\lambda f(\xi) + \frac{\varepsilon\, c}{2} f^2(\xi) - c^2\nu\frac{df(\xi)}{d\xi} = 0. \quad (11)$$

We introduce a new independent variable:

$$y = \tanh(\xi), \quad (12)$$

which leads to the change of derivative:

$$\frac{d}{d\xi}(.) = (1-y^2)\frac{d}{dy}(.), \quad (13)$$

We now introduce the following Tanh series:

$$f(\xi) = s(y) = \sum_{i=0}^{m} a_i\, y^i, \quad (14)$$

Where m is a positive integer.

From (13) and (14), we get:

$$-c\lambda s + \frac{c\varepsilon}{2}s^2 - c^2\lambda\,\nu(1-y^2)\frac{ds}{dy} = 0. \quad (15)$$

To determine the parameter m, we balance the linear term of highest order in (15) with the highest order nonlinear term. This in turn gives $m = 1$, so we get:

$$s(y) = a_0 + a_1 y. \quad (16)$$

From (15) and (16), we obtain the following system of algebraic equations in:

$a_0, a_1, c,$ and λ

$$\begin{aligned} y^0 &: -c^2\nu\ a_1 + 0.5\,c\varepsilon\ a_0^2\varepsilon - c\,\lambda\ a_0 = 0, \\ y^1 &: a_0\, a_1 c\,\varepsilon - a_1 c\,\lambda = 0, \\ y^2 &: 0.5c\,\varepsilon\, a_1^2 + c^2\nu\,\varepsilon\, a_1{}^2 = 0. \end{aligned} \tag{17}$$

With the aid of Mathematical, we find:

$$\begin{aligned} &a_0 = \frac{-2c\,\nu}{\varepsilon}, a_1 = \frac{-2c\,\nu}{\varepsilon}, \lambda = -2c\,\nu, \\ &, a_0 = \frac{2c\,\nu}{\varepsilon}, a_1 = \frac{-2c\,\nu}{\varepsilon}, \lambda = 2c\,\nu, \end{aligned} \tag{18}$$

Thus, we obtain:

$$\begin{aligned} u(x,t) &= \frac{2c\nu}{\varepsilon}(-1 - Tanh[c(x - 2c\nu t]), \\ u(x,t) &= \frac{2c\nu}{\varepsilon}(1 - Tanh[c(x - 2c\nu t]) \end{aligned}, \tag{19}$$

Which are solutions of (6). In a similar way, the following solutions are obtained:

$$\begin{aligned} u(x,t) &= \frac{2c\nu}{\varepsilon}(-1 - Coth[c(x - 2c\nu t]), \\ u(x,t) &= \frac{2c\nu}{\varepsilon}(1 - Coth[c(x - 2c\nu t]) \end{aligned}. \tag{20}$$

We now consider some numerical methods for finding the analytical solutions of the Berger equation using the Chebyshev and Legendre polynomials.

NUMERICAL SCHEMES

In this study, we consider several analytical methods for solving the Bergers equation of the type (6) and this is the main motivation of this paper. For simplicity and to convey the basic ideas of these schemes, we assume that the solution of (6) is considered in the following form:

$$u(x,t) = \sum_{n=0}^{N} A_n(t)\,\Phi_n(x)\,, \tag{21}$$

Where $A_n(t)$ are time dependent unknown coefficients and $\Phi_n(x)$ are the trial functions in the space variable x. Form 21 and 6, we obtain:

$$\sum_{n=0}^{N}\dot{A}_n(t)\Phi_n(x) + \varepsilon\sum_{n=0}^{N}A_n(t)\Phi_n(x)\sum_{n=0}^{N}A_n(t)\Phi_n'(x) - \gamma\sum_{n=0}^{N}A_n(t)\Phi_n''(x) = 0, \tag{2}$$

Where • and ‘ are differentiation with respect to time and space respectively. We consider $\Phi_n(x)$ as Chebyshev and Legendre polynomials.

Spectral method using Chebyshev polynomials

We introduce a formula for the first and second derivative of an infinitely differentiable function in terms of Chebyshev polynomials. We can see that:

$$\frac{1}{n+1}T'_{n+1}(x) - \frac{1}{n-1}T'_{n-1}(x) = 2T_n(x)\ . \tag{23}$$

We suppose that we are given f(x) as an infinitely differentiable function on (-1,1) as:

$$f(x) = \sum_{n=0}^{\infty} A_n\, T_n(x)\,, \tag{24}$$

So the (k-1)[th] derivative of f(x) is:

$$f^{(k-1)}(x) = \sum_{n=0}^{\infty} A_n\, T_n^{(k-1)}(x)\,, \tag{25}$$

Rewrite (25) in the form:

$$f^{(k-1)}(x) = \sum_{n=0}^{\infty} A_n^{(k-1)}\, T_n\ (x)\,,$$

Then

$$f^{(k)}(x) = \sum_{n=0}^{\infty} A_n^{(k)}\, T_n\ (x)\,, \tag{26}$$

It can be easily shown that:

$$A_n^{(1)} = \frac{2}{c_n}\sum_{\substack{p=n+1 \\ p+n\,odd}}^{\infty} p\, A_p\ , \tag{27}$$

And

$$A_n^{(2)} = \frac{1}{c_n}\sum_{\substack{p=n+2 \\ p+n\,even}}^{\infty} p^2(p^2 - n^2)\, A_p\ , \tag{28}$$

Where $c_0 = 2, c_n = 1, n > 0$. Let the numerical solution be of the form:

$$u(x,t) = \sum_{n=0}^{N} A_n T_n(x).$$

Differentiation and substituting in (6)

$$\sum_{n=0}^{N} \dot{A}_n T_n(x) + \sum_{l=0}^{N} A_l T_l(x) \sum_{kl=0}^{N-1} A_k T'_k(x) - \gamma \sum_{n=0}^{N-2} A_n T''_n(x) = 0, \quad (29)$$

$T_n(x)$ are polynomials of degree n. Here $T_n'(x)$ and $T_n''(x)$ are of degrees (n-1) and (n-2) respectively.

Our main purpose is to express the derivatives in terms of $T_n(x)$. Let:

$$\begin{aligned} u_x(x,t) &= \sum_{n=0}^{N-1} A_n^{(1)} T_n(x), \\ u_{xx}(x,t) &= \sum_{n=0}^{N-2} A_n^{(2)} T_n(x), \end{aligned} \quad (30)$$

Expressing $A_n^{(1)}(t)$ and $A_n^{(2)}(t)$ in terms of $A_n(t)$'s (27) and (28), we obtain from (29):

$$\sum_{n=0}^{N} \dot{A}_n(t) T_n(x) + \sum_{l=0}^{N} \sum_{k=0}^{N-1} A_l(t) A_k(t) T_l(x) T'_k(x) - \gamma \sum_{n=0}^{N-2} A_n(t) T''_n(x) = 0. \quad (31)$$

It is known that Chebyshev polynomials satisfy:

$$T_l(x) T_k(x) = 0.5(T_{|l+k|}(x) + T_{|l-k|}(x)). \quad (32)$$

From (32) and (31), we have:

$$\sum_{n=0}^{N} \dot{A}_n(t) T_n(x) + 0.5 \sum_{l=0}^{N} \sum_{k=0}^{N-1} A_l(t) A_k^{(1)}(t) (T_{|l+k|}(x) + T_{|l-k|}(x)) - \gamma \sum_{n=0}^{N-2} A_n^{(2)}(t) T_n(x) = 0 \quad (33)$$

Multiplying Equation 33 by $\dfrac{T_m(x)}{\sqrt{1-x^2}}$ and integrating with respect to x over (-1,1) and considering the orthogonal property of Chebyshev polynomials, we get:

$$\begin{aligned} &\sum_{n=0}^{N} \dot{A}_n(t) \left(\frac{\pi}{2} c_n \delta_{nm}\right) + 0.5 \sum_{l=0}^{N} \sum_{k=0}^{N-1} A_l(t) A_k^{(1)}(t) \left(\frac{\pi}{2} c_n \delta_{n|l+k|} + \frac{\pi}{2} c_n \delta_{n|l-k|}\right) \\ &\qquad - \gamma \sum_{n=0}^{N-2} A_n^{(2)}(t) \frac{\pi}{2} c_n \delta_{nm} = 0. \end{aligned} \quad (34)$$

Thus:

$$\dot{A}_n(t) + 0.5 \sum_{\substack{l,k=0 \\ |l+k|=n}}^{N,N-1} A_l(t) A_k(t) + 0.5 \sum_{\substack{l,k=0 \\ |l-k|=n}}^{N,N-1} A_l(t) A_k^{(1)}(t) - \gamma A_n^{(2)}(t) = 0 \quad (35)$$

Which is a first order ODE that can be solved using a suitable finite difference technique for $\dot{A}_n(t)$ and Equation 35 can be solved, hence the solution at different time levels.

Spectral method using Legender polynomials

Considering the recurrence relation:

$$(n+1) p_{n+1}(x) = (2n+1) x p_n(x) - n p_{n-1}(x), \; p_0(x) = 1, \; p_1(x) = x. \quad (36)$$

In a similar way to Chebyshev polynomials, we can deduce that if f(x) is expressed as $f(x) = \sum_{n=0}^{\infty} A_n P_n(x)$, then $f^{(k)}(x) = \sum_{n=0}^{\infty} A_n^{(k)} P_n(x)$ and it can be shown that:

$$A_n^{(1)} = (2n+1) \sum_{\substack{p=n+1 \\ p+n\ odd}}^{\infty} A_p, \quad (37)$$

And

$$A_n^{(2)} = (n+0.5) \sum_{\substack{p=n+2 \\ p+n\ even}}^{\infty} [p(p+1) - n(n+1)] A_p, n \geq 0, \quad (38)$$

Now, assume that the solution of (6) in the form:

$$u(x,t) = \sum_{n=0}^{N} A_n(t) P_n(x), \quad (39)$$

Where $P_n(x)$ is the well known Legendre polynomials satisfying (4).

Now, $A_n(t)$ are chosen such that u(x,t) satisfies Burger's Equation 6. Differentiating (39) with respect to x and substituting in (4):

$$\sum_{n=0}^{N}\dot{A}_n P_n(x)+\sum_{l=0}^{N}A_l P_l(x)\sum_{k=0}^{N-1}A_k P_k'(x)-\gamma\sum_{n=0}^{N-2}A_n P_n''(x)=0. \quad (40)$$

We can express P_n' and P_n'' in terms of $P_n(x)$ as:

$$u_x(x,t)=\sum_{n=0}^{N-1}A_n^{(1)}P_n(x),$$
$$u_{xx}(x,t)=\sum_{n=0}^{N-2}A_n^{(2)}P_n(x), \quad (41)$$

It shows that:

$$A_n^{(1)}=(2n+1)\sum_{\substack{p=n+1\\ p+n\,odd}}^{N}A_p\ , \quad (42)$$

And

$$A_n^{(2)}=(n+0.5)\sum_{\substack{p=n+2\\ p+n\,even}}^{N}[p(p+1)-n(n+1)]A_p, n\geq 0, \quad (43)$$

Then Equation 40 with (41), (42) leads to:

$$\sum_{n=0}^{N}\dot{A}_n(t)P_n(x)+\sum_{l=0}^{N}\sum_{k=0}^{N-1}A_l(t)A_k^{(1)}(t)T_l(x)T_k(x)-\gamma\sum_{n=0}^{N-2}A_n^{(2)}(t)T_n(x)=0, \quad (44)$$

The product of two polynomials is given in (44) as:

$$P_l(x)P_k(x)=\sum_{i=0}^{\min(l,k)}\frac{(l+k+0.5-2i)}{(l+k+0.5-i)}\frac{B_i B_{k-i}B_{l-i}}{B_{l+k-i}}P_{l+k-2i}, \quad (45)$$

Where $B_i=\dfrac{\Gamma(i+0.5)}{\Gamma(0.5)\,i!}$, and Γ is the well known Gamma function.

From (45) and (44), we get:

$$\sum_{n=0}^{N}\dot{A}_n(t)P_n(x)+\sum_{l=0}^{N}\sum_{k=0}^{N-1}\sum_{i=0}^{\min(l,k)}A_l(t)A_k^{(1)}(t)\frac{(l+k+0.5-2i)}{(l+k+0.5-i)}\frac{B_i B_{k-i}B_{l-i}}{B_{l+k-i}}P_{l+k-2i}(x)$$
$$-\gamma\sum_{n=0}^{N-2}A_n^{(2)}(t)P_n(x)=0 \quad (46)$$

Multiplying both sides of (46) by $P_m(x)$ and integrating with respect to x over the interval (-1,1), Equation 46 becomes:

$$\sum_{n=0}^{N}\dot{A}_n(t)\int_{-1}^{1}P_n(x)P_m(x)dx+\sum_{l=0}^{N}\sum_{k=0}^{N-1}\sum_{i=0}^{\min(l,k)}A_l(t)A_k^{(1)}(t)\frac{(l+k+0.5-2i)}{(l+k+0.5-i)}\frac{B_i B_{k-i}B_{l-i}}{B_{l+k-i}}$$
$$\int_{-1}^{1}P_{l+k-2i}(x)P_m(x)dx-\gamma\sum_{n=0}^{N-2}A_n^{(2)}(t)\int_{-1}^{1}P_n(x)P_m(x)dx=0 \quad (47)$$

Applying the orthogonal property, we have:

$$\dot{A}_n(t)+\sum_{l=0}^{N}\sum_{k=0}^{N-1}\sum_{i=0}^{\min(l,k)}A_l(t)A_k^{(1)}(t)\frac{(l+k+0.5-2i)}{(l+k+0.5-i)}\frac{B_i B_{k-i}B_{l-i}}{B_{l+k-i}}-\gamma A_n^{(2)}(t)=0 \quad (48)$$

This is a first order ordinary differential equations which can be treated and solved using suitable technique leading to a nonlinear algebraic system of equations.

The initial state

At time level zero, the analytical solutions are given as:

$$u(x,0)=\sum_{n=0}^{N}A_n(0)\Phi_n(x). \quad (49)$$

We divide the interval (-1,1) into elements of equal length; and let $-1=x_0\leq\ldots\leq x_n=1$, be partition of (-1,1) by the knots x_i, thus for (N+1) different values of x_i we have the following system:

$$u(x_i,0)=\sum_{n=0}^{N}A_n(0)\Phi_n(x_i), i=0,1,\ldots,N. \quad (50)$$

We can obtain initial values for the expansion coefficients by solving the linear system using Gauss elimination with partial pivoting.

The Adomian decomposition method

Consider the partial differential equation written in the form:

$$L_t u(x,t) = Lu(x,t) + N(u), \tag{51}$$

With initial condition given by:

$$u(x,0) = g(x), \tag{52}$$

Where L_t, L are partial differential operators, N is a nonlinear operator.

Applying the inverse operator L_t^{-1} to the equation and using the initial condition, then:

$$u(x,t) = g(x) + L_t^{-1} Lu(x,t) + L_t^{-1} N(u), \tag{53}$$

The Adomian decomposition method assumes the unknown function $u(x,t)$ to be expressed as infinite series of the form:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{54}$$

And the nonlinear term $N(u)$ can be expressed hn terms of the Adomian polynomials as:

$$N(u) = \sum_{n=0}^{\infty} A_n, \tag{55}$$

Where A_n are the Adomian polynomials that can be generated for all forms of nonlinearly (1,4,21,22). So we get:

$$\sum_{n=0}^{\infty} u_n(x,t) = g(x) + L_t^{-1}(L(u)) + L_t^{-1}(\sum_{n=0}^{\infty} A_n), \tag{56}$$

Identifying the zero component u_0 by all terms that arise from the initial condition and as a result the remaining components $u_n, n \geq 1$ can be determined using the recurrence relation:

$$\begin{aligned} u_0 &= g(x), \\ u_{n+1} &= L_t^{-1}(L(\sum_{n=0}^{\infty} u_n) + L_t^{-1}(\sum_{n=0}^{\infty} A_n), n \geq 1, \end{aligned} \tag{57}$$

Knowing u_0 we find $u_1, u_2, \ldots u_n$, and hence can find $u(x,t)$.

Example

We consider Burgers equation in the form:

$$u_t + \varepsilon u u_x - \nu u_{xx} = 0, \quad 0 \leq x \leq 30, \tag{58}$$

With boundary conditions u (30, t) = 0, u (0, t) = 1, and initial condition:

$$u(x,0) = 0.5(1 - \tanh\{\frac{1}{4\nu}(x - 15\}), \tag{59}$$

Which has the exact solution:

$$u(x,t) = 0.5(1 - \tanh\{\frac{1}{4\nu}(x - 15 - 0.5t\}), t \geq 0. \tag{60}$$

Using the transformation $X = \frac{2x - 30}{30}$, the domain (0,30) leads to (-1,1) and consequently the solution to the problem becomes:

$$u(x,t) = 0.5(1 - \tanh\{\frac{1}{4\nu}(x - 0.5t\}), t \geq 0, -1 \leq x \leq 1, \tag{61}$$

u(-1,t) = 1, u(1,t) = 0, and the equation is:

$$u_t + \frac{1}{15} u u_x - \frac{\nu}{225} u_{xx} = 0. \tag{62}$$

Numerical results for test problems are given in the following cases:

RESULTS

Numerical results using Chebyshev polynomials

In Table 1 we give a list of error norms for different truncated numbers, using spectral method with Chebyshev polynomials. We used a subroutine given by Powell's based on Newton's method for solving the nonlinear equations for each level time.

Numerical results using Legendre polynomials

In Table 2 we give a list of error norms for different

Table 1. The error for time = 1, time step = 0.01.

	Error	N = 10	N = 18	N = 24
ν = 1	L_2-error	2.18E-2	5.79E-3	5.81E-3
	$L\infty$-error	2.39E-2	1.21E-2	1.17E-2
ν = 1.4	L_2-error	9.13E-3	4.14E-3	5.34E-3
	$L\infty$-error	1.02E-3	4.73E-3	7.60E-3

Table 2. The error for time = 1, time step = 0.01.

	Error	N = 10	N = 18	N = 24
ν =1	L_2-error	1.62E-2	5.79E-3	5.70E-3
	$L\infty$-error	1.72E-2	1.24E-2	1.20E-2
ν =1.4	L_2-error	4.75E-3	1.03E-3	1.78E-3
	$L\infty$-error	5.35E-3	1.35E-3	3.94E-3

truncated numbers, using spectral method with Legendre polynomials. We used a subroutine given by Powell's based on Newton's method for solving the nonlinear equations for each level time. Our results show that despite the fact that Chebyshev polynomials are the most widely used ones, the use of Legendre polynomials give more accurate results.

The Adomian decomposition method for Burgers' equation

Consider the following problem: Find a function u(x, t) satisfying the Burgers' equation with the initial:

$$u(x,t) = 0.5(1-\tanh\{\frac{1}{4\nu}x), -1 \le x \le 1, \qquad (63)$$

And as in "the Adomian decomposition method" we get the recurrence relations:

$$\begin{aligned} &u_0(x,t) = u(x,0)\ , \\ &u_{n+1}(x,t) = \nu(u_n)_{xx} - \varepsilon A_n\ ,\ n \ge 0 \end{aligned} \qquad (64)$$

Where A_n , $n \ge 0$ are Adomian polynomials that represent the nonlinear terms.

Recall that these polynomials can be formulated for each nonlinear term. Here, we list sets of Adomian polynomials as:

$$A_0 = u_0 u_{0x}, A_1 = u_0 u_{1x} + u_1 u_{0x}, A_2 = u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x}, .. \qquad (65)$$

Using these polynomials and employing the appropriate recursive relation we find:

$$\begin{aligned} &u_0(x,t) = 0.5(1-\tanh\{\frac{1}{4\nu}x), \\ &u_1(x,t) = 0.0625\sec h[\frac{15x}{4}]^2, \\ &u_2(x,t) = 0.0078125t^2 \sec h[\frac{15x}{4}]^2 \tanh[\frac{15x}{4}], \end{aligned} \qquad (66)$$

and so on. The solution in a series form is given by:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + ..., \qquad (67)$$

It is necessary to make tests for these series solutions, by showing a comparison with the exact solutions of the Burgers' equation. In the program, we put $\nu = 1$, t = 2, and take only four terms of the series. The series and the exact solution results are shown in Table 3, it seems that the series solution are confirmed with the exact solution. Figures 1 and 2 shows the Adomian solution for Burgers' equation at t = 2.

Table 3. Adomian decomposition solution at t = 2.

X	ADM sol.	Exact sol.	Error
-1.0	0.999665	0.999665	-1.30E-7
-0.4	0.880828	0.880828	3.08E-5
0.0	0.622396	0.622459	-6.34E-5
0.4	0.0758605	0.0758582	2.29E-6
1.0	0.0009108	0.0009110	-1.53E-7

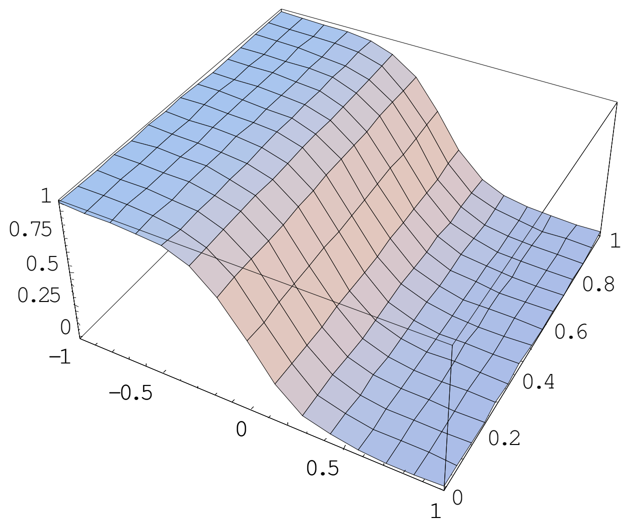

Figure 1. ADM solution, $\nu = 1, 0 \le t \le 1$.

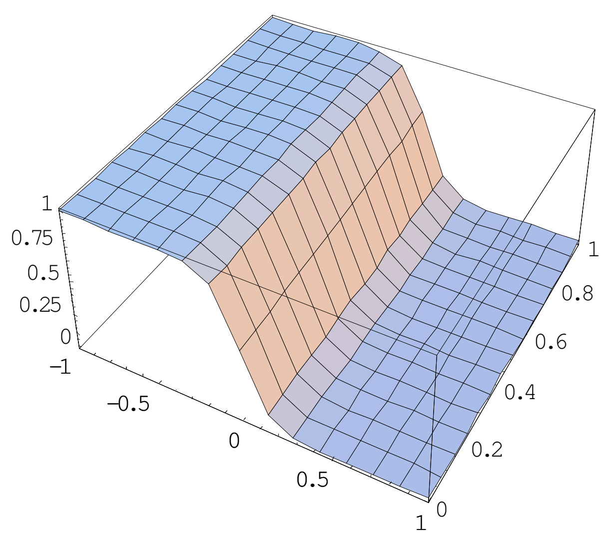

Figure 2. ADM solution, $\nu = 0.5, 0 \le t \le 1$.

Conclusion

In this work, the spectral method was applied successfully for solving the nonlinear Burgers' equation. The use of Legendre polynomials is proposed as a basis for the space of solutions and shows that it gives better results than the Chebyshev polynomials in which the weight function for the first (Legendre) is more suitable near the boundaries than the other (Chebshev). Also the applications of the recent Adomian decomposition method have been proposed and given very accurate results and show that it can be widely used for many nonlinear partial differential equations. In addition the Tanh-function method is used to get the exact solution and all numerical results are shown in a comparison with the obtained exact solution.

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