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Applications of Schrödinger equation in general Besov spaces

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In this paper, we obtain solution of Schrödinger equation in general Besov spaces. Precise results on L_p and general Besov estimates of the maximal function of the solutions to the Schrodinger equation are given. The obtained results improve some recent results. Further, we shall consider estimates of general L_2-norm and the general Besov type norm of integrals of this kind by means of the general Besov norm of the function f, and give L_p-estimates of their maximal functions.

Key words: Maximal function, general Besov norm, Schrödinger equation.

INTRODUCTION

The Schrödinger equation was formulated in 1926 by Austrian physicist Erwin Schrödinger. Used in physics, specifically quantum mechanics, it is an equation that describes how the quantum state of a physical system changes in time. The Schrödinger equation takes several different forms, depending on the physical situation. Now, we present the equation for the general case and for the simple case encountered in many textbooks for a general quantum system.

In the standard interpretation of quantum mechanics, the quantum state, also called a wave function or state vector, is the most complete description that can be given to a physical system. Solutions to Schrödinger's equation describe not only molecular, atomic and subatomic systems, but also macroscopic systems, possibly even the whole universe. The most general form is the time-dependent Schrödinger equation, which gives a description of a system evolving with time. For systems in a stationary state (that is, where the Hamiltonian is not explicitly dependent on time), the time-independent Schrödinger equation is sufficient. Approximate solutions to the time-independent Schrödinger equation are commonly used to calculate the energy levels and other properties of atoms and molecules.

Schrödinger's equation can be mathematically transformed into Werner Heisenberg's matrix mechanics, and into Richard Feynman's path integral formulation. The Schrödinger equation describes time in a way that is inconvenient for relativistic theories, a problem which is not as severe in matrix mechanics and completely absent in the path integral formulation. It is well-known that the solution is Schrödinger equation (Almeida et al., 2013; Cowling, 1983; Furioli and Terraneo, 2003).

$$\frac{\partial u}{\partial t} = -i\Delta u\,,\;\; u(0,x) = f(x), \quad (x \in R^n, t \in R) \tag{1}$$

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is given by:

$$u(x, t) = c_n \iint e^{i(x-y)\xi + it|\xi|^2} f(y)\, d\xi\, dy.$$

In this paper we shall consider estimates of the general Besov type norm of integrals of this kind by means of the general Besov norm of the function "f" (Taoka, 2005), then we will give L^p estimates of their maximal functions for more information on Schrödinger equation (Almeida et al., 2013; Carbery, 1985; Jing-Wei et al., 2013; Cowling, 1983; Muramatu and Taoka, 2004; Müller-Kirsten, 2006; Furioli and Terraneo, 2003; Fukuma and Muramatu, 1999; Polelier, 2011; Michael et al., 2012; Shankar, 1994; Taoka, 2005). Our motivations in the present work are slightly different from what has previously been done. Firstly, we aim at a better understanding of the recent construction of self-similar solutions for Equation (1). A self-similar solution is by definition invariant by the scaling Equation (2), and therefore cannot be obtained by these aforementioned results in Sobolev spaces.

RESULTS

Our first result is the following theorem:

Theorem 1

Let σ be a positive number, I = (0, 1), γ > 1 and let $1 < p, q < \infty$

Assume that h(t, ξ), h*(t, ξ) are real-valued, measurable, and C^∞ in t and the inequality

$$\alpha \frac{\partial^k h(t,\xi)}{\partial t^k} + \beta \frac{\partial^k h^*(t,\xi)}{\partial t^k} \le \lambda_k (\alpha+\beta)(1+|\xi|^2) \quad (2)$$

Holds for any positive integer k, where λ_k is a constant independent of t and ξ and α and β are positive constants. Then the operator

$$T_1\big(f(t,x)+g(t,x)\big) = \lambda_n \left[\alpha \iint_{R^n} e^{i(x-y)\xi + ih(t,\xi)} f(y) d\xi dy + \beta \iint_{R^n} e^{i(x-y)\xi + ih^*(t,\xi)} g(y) d\xi dy \right] \quad (3)$$

Where $\lambda_n = \frac{1}{(2\pi)^n}$ is bounded from $B^{\sigma\gamma}_{2,q}(R^n)$ to $B^{\sigma}_{2,q}(I; L^2 R^2)$

Proof

First, consider the case where *q = 2* and σ is a non-negative integer *m*. Notice that $B^m_{2,2} = H^m$. Then,

$$\begin{aligned}
&\partial_t^k T_1\big(f(t,x)+g(t,x)\big) \\
&= \lambda_n \alpha \iint_{R^n} e^{i(x-y)\xi + ih(t,\xi)} H_k(t,\xi) f(y) d\xi dy \\
&+ \lambda_n \beta \iint_{R^n} e^{i(x-y)\xi + ih^*(t,\xi)} H_k(t,\xi) g(y) d\xi dy
\end{aligned}$$

With $|H_k(t\xi)| \le \lambda_k (1+|\xi|^k)$. Hence using Parseval's formula we obtain that

$$\begin{aligned}
&\| T_1(f+g) \|^2_{L_2(I;L_2(R^n))} \\
&\le \lambda_0 \alpha \int_0^1 \| f \|^2_{L_2} dt + \lambda_0 \beta \int_0^1 \| g \|^2_{L_2} dt \\
&= \lambda_0 \big(\alpha \| f \|^2_{L_2} + \beta \| g \|^2_{L_2} \big).
\end{aligned}$$

Also,

$$\begin{aligned}
&\| \partial_t^k T_1(f+g) \|^2_{L_2(I;L_2(R^n))} \\
&\le \lambda_k \alpha \int_0^1 dt \ \| (1+|\xi|^{k\gamma}) \hat{f}(\xi) \|^2_{L_2(R^n)} \\
&+ \lambda_k \beta \int_0^1 dt \ \| (1+|\xi|^{k\gamma}) \hat{g}(\xi) \|^2_{L_2(R^n)} \\
&\le \lambda_k \left[\alpha \| f \|^2_{H^{k\gamma}(R^n)} + \beta \| g \|^2_{H^{k\gamma}(R^n)} \right]
\end{aligned}$$

Hence, we deduce that

$$\| T_1(f+g) \|^2_{H^m(I;L_2(R^n))} \le \lambda_m \big[\alpha \| f \|_{H^{m\gamma}(R^n)} + \beta \| g \|_{H^{k\gamma}(R^n)} \big]$$

Since, the Besov spaces are identical with the real interpolation of the Sobolev spaces:

$$(L_2(\Omega; X) H^m(\Omega; X))_{p,q} = B^{mp}_{2,q}(\Omega; X)$$

where X is a Banach space and $(.,.)_{p,q}$ denotes the real interpolating spaces. Therefore, the conclusion of the theorem follows from interpolation of linear operators and the fact that T_1 is bounded from:

$H^{m\gamma}(R^n)$ to $H^m(I; L_2(R^n))$ for any non-negative integer m. Our proof is therefore completed. For the next result we will need the following lemma.

Lemma 1

Let I = (0, 1), $1 < q < p < \infty$, $0 \le s < \infty$ and let σ be a positive number. Then, the general Besov space $B^{\sigma}_{p,q,s}(I; L_p(R^n))$ is continuously imbedded in the space $L_p(R^n; B^{\sigma}_{p,q,s}(I))$.

Proof

The proof of this Lemma is similar to Lemma 1, so we will omit it.

Theorem 2

Assume that h(t, ξ), h*(t, ξ) are real-valued, measurable satisfying condition of Equation (2). Then the operator T_1 as defined by Equation (3) will satisfy the following inequality:

$$\iint_{R^n} \| T_1(f+g) \|^2_{L_\infty(I)} dx \le C(\alpha \| f \|_{B^{\gamma/2}{}_{2,1}(R^n)} + \beta \| g \|_{B^{\gamma/2}{}_{2,1}(R^n)})^2$$

where C is a constant independent of t and ξ and α and β are positive constants.

Proof

To get L_2 maximal estimates for the operator of Equation (3), from Lemma 1 and the imbedding theorem $B^{1/2}_{2,1,s}(I) \subset L_\infty(I)$, it follows that $B^{1/2}_{2,1,s}(I; L_2(R^n)) \subset L_2(R^n; B^{1/2}_{2,1,s}(I)) \subset L_2(R^n; L_\infty(I))$ with continuous inclusions, which, combined with Theorem 1, completes the proof of Theorem 2.

Theorem 3

Let X, Y and Z be Hilbert spaces, and T and S be operators defined by

$$T(f(x)+g(x)) = \lambda_n \left[\alpha \iint_{R^n} e^{i(x-y)\xi} \hat{K}(\xi) f(y) d\xi dy + \beta \iint_{R^n} e^{i(x-y)\xi} \hat{K}(\xi) g(y) d\xi dy \right] \tag{4}$$

And

$$S(f_1+g_1)(x) = \lambda_n \left[\alpha \iint_{R^n} e^{i(x-y)\xi} \hat{H}(\xi) f(y) d\xi dy + \beta \iint_{R^n} e^{i(x-y)\xi} \hat{H}(\xi) g(y) d\xi dy \right] \tag{5}$$

where $\hat{K}(\xi)$ and $\hat{H}(\xi)$ are $L(X,Y)-$valued functions of $\xi \in R^n$ with $\sup_{\xi} \| \hat{K}(\xi) \|_{L(X,Y)} < \infty$, and $\sup_{\xi} \| \hat{H}(\xi) \|_{L(X,Y)} < \infty$ Then the operator T^* is bounded operator from $L_2(R^n; Y)$ to $L_2(R^n; X)$ and T^* is defined by

$$T^*(f+g)(x) = \lambda_n \alpha \iint_{R^n} e^{i(x-y)\xi} (K \hat{\xi}))^* f(y) d\xi dy + \lambda_n \beta \iint_{R^n} e^{i(x-y)\xi} (K \hat{\xi}))^* g(y) d\xi dy \tag{6}$$

and ST is the bounded operator from $L_2(R^n; X)$ to $L_2(R^n; Z)$ and is defined by the formula

$$T^*(f+g)(x) = \lambda_n \alpha \iint_{R^n} e^{i(x-y)\xi} \hat{H}(\xi) \hat{K}(\xi) f(y) d\xi dy + \lambda_n \beta \iint_{R^n} e^{i(x-y)\xi} \hat{H}(\xi) \hat{K}(\xi) g(y) d\xi dy \tag{7}$$

where α and β are positive constants.

Proof

Let $f, g \in S(R^n; X)$ and $f_1, g_1 \in S(R^n; Y)$. Then, we have

$$\begin{aligned}
&< f_1 + g_1,\ T(f+g) >_{L_2(R^n; y)} \\
&= \lambda_n \alpha \iint_{R^n} < (f_1+g_1)(x), e^{ix\xi} f^\wedge(\xi) K^\wedge(\xi) >_Y dx\, d\xi \\
&+ \lambda_n \beta \iint_{R^n} < (f_1+g_1)(x), e^{ix\xi} g^\wedge(\xi) K^\wedge(\xi) >_Y dx\, d\xi \\
&= \lambda_n \alpha \int < \int e^{-ix\xi} (f_1+g_1)(x) dx,\ f^\wedge(\xi) K^\wedge(\xi) >_Y d\xi \\
&+ \lambda_n \beta \int < \int e^{-ix\xi} (f_1+g_1)(x) dx,\ g^\wedge(\xi) K^\wedge(\xi) >_Y d\xi \\
&= \lambda_n \alpha \int < (K^\wedge(\xi))^* (f_1+g_1)^\wedge(\xi),\ f^\wedge(\xi) >_X d\xi \\
&+ \lambda_n \beta \int < (K^\wedge(\xi))^* (f_1+g_1)^\wedge(\xi),\ g^\wedge(\xi) >_X d\xi
\end{aligned}$$

$$= \lambda_n \alpha \int < (K^\wedge(\xi))^* (f_1 + g_1)^\wedge(\xi), \int e^{-ixt} f(x) dx >_X d\xi$$
$$+ \lambda_n \beta \int < (K^\wedge(\xi))^* (f_1 + g_1)^\wedge(\xi), \int e^{-ixt} g(x) dx >_X d\xi$$
$$= \lambda_n \alpha \int < \int e^{ix\xi} (K^\wedge(\xi))^* (f_1 + g_1)^\wedge(\xi), f(x) dx >_X dx$$
$$+ \lambda_n \beta \int < \int e^{ix\xi} (K^\wedge(\xi))^* (f_1 + g_1)^\wedge(\xi), g(x) dx >_X dx.$$

Therefore, we have

$$T^*(f_1 + g_1)(x)$$
$$= \lambda_n \alpha \int e^{ix\xi} K^\wedge(\xi)(f_1 + g_1)^\wedge(\xi)\, dx$$
$$+ \lambda_n \beta \int e^{ix\xi} K^\wedge(\xi)(f_1 + g_1)^\wedge(\xi)\, dx$$
$$= \lambda_n \alpha \int e^{i(x-y)\xi} (K^\wedge(\xi))^* (f_1 + g_1)^\wedge(\xi)\, d\xi$$
$$+ \lambda_n \beta \int e^{i(x-y)\xi} (K^\wedge(\xi))^* (f_1 + g_1)^\wedge(\xi)\, d\xi.$$

Since,

$$T^\wedge(f_1 + g_1)(\xi) = K^\wedge(\xi)(f_1 + g_1)^\wedge(\xi)$$

It follows that

$$ST(f + g)(x)$$
$$= \lambda_n \alpha \int e^{ix\xi} H^\wedge(\xi) T^\wedge(f + g)(\xi)\, d\xi$$
$$+ \lambda_n \beta \int e^{ix\xi} H^\wedge(\xi) T^\wedge(f + g)(\xi)\, d\xi$$
$$= \lambda_n \alpha \int e^{i(x-y)\xi} H^\wedge(\xi) K^\wedge(\xi)(f + g)(y)\, d\xi\, dy$$
$$+ \lambda_n \beta \int e^{i(x-y)\xi} H^\wedge(\xi) K^\wedge(\xi)(f + g)(y)\, d\xi\, dy.$$

This completes the proof of our theorem.

Remark

It should be remarked that theorems 1, 2, and 3 generalizing the corresopnding results in Fukuma and Muramatu (1999).

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