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Full Length Research Paper

Effect of internal heat generation on Marangoni convection in a superposed fluid-porous layer with deformable free surface

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Linear stability analysis is applied to investigate the effect of internal heat generation on Marangoni convection in a two-layer system comprising an incompressible fluid-saturated porous layer over which lies a layer of the same fluid. The upper free surface is deformable and is subject to a general thermal condition, while the lower boundary is rigid and is fixed at a constant temperature (isothermal) or a constant heat flux. The Beavers-Joseph condition is employed at the interface and the Forchheimer-extended-Darcy equation is employed to describe the flow regime in the porous medium. The linear stability theory and the normal mode analysis are applied and the resulting eigenvalue problem is solved exactly. For both the upper and lower boundaries fixed at a constant heat flux, the analytical asymptotic solution of long wavelength is obtained using regular perturbation technique. It is observed that the critical Marangoni number decreases with an increase in the dimensionless heat source strength. However, an increase of the Bond number and the decrease of the Darcy number will help to slow the process of destabilizing the system.

Key words: Marangoni convection, heat generation, porous layer.

INTRODUCTION

The problem of convection due to temperature-dependent surface tension forces in a horizontal system has been studied extensively by many researchers. This type of convection instability is referred to as the Marangoni instability and was first theoretically analysed by Pearson (1958). The effect of the internal heat generation on the Benard-Marangoni instability of a horizontal liquid layer with a deformable upper free surface was investigated by Char and Chiang (1994). The stability analysis was based on the linear stability theory and the resulting eigenvalue problem was solved by employing the fourth order Runge-Kutta-Gill method. Wilson (1997) used a combination of analytical and numerical techniques to analyze the effect of uniform internal heat generation on the onset of steady Marangoni convection in a horizontal layer of quiescent fluid heated from below. He gave a comprehensive description of the stability characteristics of the layer both when the lower boundary is conducting and when it is insulating to temperature perturbations. Gasser and Kazimi (1976) studied the onset of convection in a porous medium with internal heat generation by employing a rigid lower surface with a free upper surface and isothermal conditions at the upper and lower surfaces. The combination of the critical Rayleigh numbers presented in their paper was expected to hold true for a bed with a rigid isothermal upper boundary as well as a free isothermal surface upper boundary.

The convective instability of a fluid overlying a porous region saturated with the same fluid has been first considered by Sun (1973). By using the shooting method to solve the linear stability equations, he showed that the critical Rayleigh number in the porous layer decreased

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continuously as the thickness of the fluid increased. Nield (1977) formulated the problem of convection in a fluid layer overlying a porous medium when the upper free surface was assumed to be deformable. He used a Darcy law in formulating the equations in porous medium. Somerton and Catton (1982) added the Brinkman term in the equation of motion to solve the problem using the Galerkin method. The problem of salinity gradients and temperature in a superposed liquid and porous layer were considered by Chen and Chen (1988). Taslim and Narusawa (1989) considered the thermal stability for different systems of superposed porous and fluid regions. The effect of throughflow on the onset of thermal convection in a fluid layer overlying a porous layer has been investigated by Chen (1990). Chen et al. (1991) studied the problem with an isotropic permeability and thermal diffusivity in the porous layer. The linearized stability equations were solved using the shooting method. McKay (1998) examined the onset of buoyancy convection in a layer of fluid on top of a saturated porous layer. Straughan (2001, 2002) investigated a fundamental model for convection in a porous-fluid layer system and obtained the eigenvalue and eigenfunction numerically by utilizing the Chebyshev tau method. The penetrative convection in a two-layer system in which a layer of fluid overlay and saturated a porous medium was simulated via internal heating has been studied by Carr (2004). The behaviour of the results were explained and illustrated with a range of streamlines.

Shivakumara et al. (2006) studied the onset of Marangoni convection in a composite porous-layer system. At the contact surface between the fluid-saturated porous medium and the adjacent bulk fluid, they employed both the Beavers-Joseph and Jones conditions. The effect of variation of different physical parameters on the onset of Marangoni convection in a two-layer system was investigated in detail by Shivakumara and Chavaraddi (2007). They found that the ratio of the thickness of the fluid to the porous layer has a profound effect on the stability of the system. The influence of Rayleigh effect combined with Marangoni effect on the onset of convection in a liquid overlying a porous layer was studied by Liu et al. (2008). The eigenvalue problem has been solved by means of a Chebyshev tau method. Very recently, Shivakumara et al. (2011) investigated the onset of surface tension driven convection in a fluid layer overlying a layer of an anisotropic porous medium. They showed that decreasing the mechanical anisotropy parameter and increasing the thermal anisotropy parameter led to the stabilization of the system.

The purpose of this paper is to study the effect of the internal heat generation on the Marangoni convection in a two-layer system comprising an incompressible fluid saturated porous layer over which lies a layer of the same fluid when the upper boundary is assumed to be deformable. The linear stability theory and the normal mode analysis are applied and the resulting eigenvalue problem is solved exactly. For both the upper and lower boundaries fixed at a constant heat flux, the analytical asymptotic solution of long wavelength is obtained using regular perturbation technique.

PROBLEM FORMULATION

Consider an infinite horizontal incompressible fluid-saturated porous layer of thickness d_p underlying a layer of the same fluid of thickness d, heated from below as shown in Figure 1.

The bottom boundary is rigid while the upper free surface at which the surface tension acts is assumed to be deformable. The temperatures of the lower and upper boundaries are taken to be uniform and equal to T_l and T_u, respectively, with $T_l > T_u$.

The governing equations for the fluid layer are:

$$\nabla \cdot \boldsymbol{V} = 0 , \tag{1}$$

$$\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{V} = -\frac{1}{\rho_0}\nabla p + g + \nu\nabla^2 \boldsymbol{V} , \tag{2}$$

$$\frac{\partial T}{\partial t} + (\boldsymbol{V} \cdot \nabla)T = \kappa\nabla^2 T + q , \tag{3}$$

and for the porous layer are:

$$\nabla_p \cdot \boldsymbol{V}_p = 0 , \tag{4}$$

$$\frac{\rho_0}{\phi}\frac{\partial V_p}{\partial \mathrm{t}} + \frac{\rho_0 C}{\sqrt{K}}|\boldsymbol{V}_p|\boldsymbol{V}_p = -\nabla_p p_p + \rho_0 g - \frac{\mu}{\mathrm{K}}\boldsymbol{V}_p , \tag{5}$$

$$(\rho_0 c)_p \frac{\partial T}{\partial t} + (\rho_0 c_p)_l (\boldsymbol{V}_p \cdot \nabla_p)T = \kappa_p \nabla_p^2 T + q_p , \tag{6}$$

where V and V_p are the velocity vectors in the fluid layer and porous layer, respectively, T is the temperature, q and q_p are the uniformly distributed volumetric internal heat generations in the fluid layer and porous layer, respectively, p is the pressure, ν is the kinematic viscosity, ρ_0 is the fluid density, K is the permeability of the porous medium, κ is the thermal diffusivity of the fluid, ϕ is the porosity of the porous medium, C is the drag coefficient, c is the specific heat and the subscripts p and l refer to the porous medium and liquid medium, respectively.

Under the steady-state conditions, we seek the form of $(u,\ v,\ w,\ p,\ T) = [0,\ 0,\ 0,\ p_b(z),\ T_b(z)]$ in the fluid layer and $(u_p, v_p, w_p, p_p, T_p) = [0,\ 0,\ 0,\ p_{pb}(z_p),\ T_{pb}(z_p)]$ in

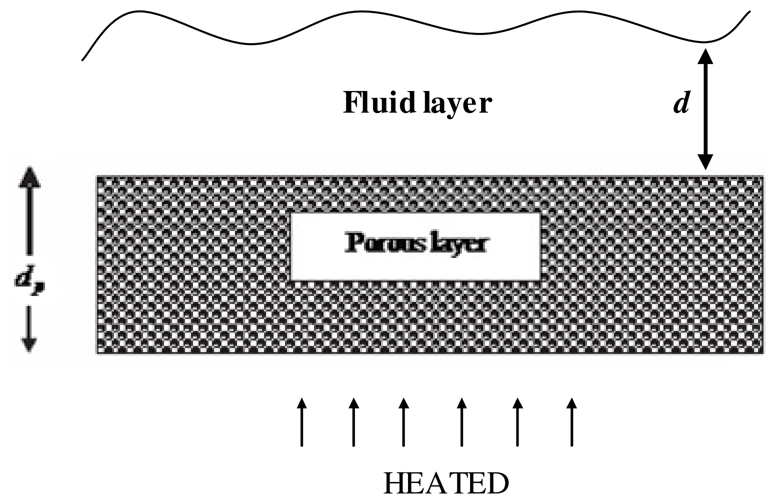


Figure 1. The saturated porous layer heated from below.

the porous layer, where the subscript *b* denotes the basic state. The temperature distributions $T_b(z)$ and $T_{pb}(z_p)$ are found to be

$$T_b(z) = T_0 - \left[\left(\frac{H(T_0 - T_u)}{k + Hd} - \frac{qd}{2\kappa} \right) z + \frac{q}{2\kappa} z^2 \right], \quad \text{at } 0 \le z \le d, \tag{7}$$

$$T_{pb}(z_p) = T_0 - \left[\left(\frac{(T_l - T_0)}{d_p} - \frac{q_p d_p}{2\kappa_p} \right) z_p + \frac{q_p}{2\kappa_p} (z_p)^2 \right], \quad \text{at } -d_p \le z_p \le 0. \tag{8}$$

where

$$T_0 = \frac{(\kappa_p T_l [k + Hd] + \kappa T_u d_p H) + d(qd + q_p d_p) d_p H}{\kappa_p [k + Hd] + \kappa d_p H}$$

is the inter-face temperature, *H* is the heat transfer coefficient and *k* is the thermal conductivity of the fluid.

In order to investigate the stability of the basic solution, infinitesimal disturbances are introduced in the form

$$(u, v, w, p, T) = [0, 0, 0, p_b(z), T_b(z)] + (u', v', w', p', T'), \tag{9}$$

as in the fluid layer and in the porous layer we have;

$$(u_p, v_p, w_p, p_p, T_p) = [0, 0, 0, p_{pb}(z_p), T_{pb}(z_p)] + (u'_p, v'_p, w'_p, p'_p, T'_p) \tag{10}$$

where the primed quantities are the perturbed ones over their equilibrium counterparts.

Equations (9) and (10) are substituted into Equations (1) to (6) and they are linearized in the usual manner. The variables are then non-dimensionalized using the scales d, d^2/κ, κ/d, and $(T_0 - T_u)/d$ as the units of length, time, velocity and temperature in the fluid layer and d_p, $d^2{}_p/\kappa_p$, κ_p/d_p, and $(T_l - T_0)/d_p$ as the corresponding characteristic quantities in the porous layer. The perturbed governing equations of the liquid layer and the porous layer in the non-dimensional form can be obtained as:

$$\frac{1}{\Pr} \frac{\partial}{\partial t} (\nabla^2 w) - \nabla^4 w = 0, \tag{11}$$

$$\frac{\partial T}{\partial t} + [Q(1 - 2z) - 1] w = \nabla^2 T, \tag{12}$$

$$\frac{Da}{\Pr_p} \frac{\partial}{\partial t} (\nabla_p^2 w_p) + (\nabla_p^2 w_p) = 0, \tag{13}$$

$$S \frac{\partial T_p}{\partial t} + [Q_p (1 - 2z_p) - 1] w_p = \nabla_p^2 T_p. \tag{14}$$

For the fluid layer, $\Pr = \nu/\kappa$ is the Prandtl number, $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the Laplacian operator and for the porous layer, $\Pr_p = \nu/\phi\, \kappa_p$ is the Prandtl number, $Da = K/d_p^2$ is the Darcy number, *S* is the ratio of heat capacities and $\nabla_p^2 = \partial^2/\partial x_p^2 + \partial^2/\partial y_p^2 + \partial^2/\partial z_p^2$. *Q* and Q_p are the dimensionless heat source strengths in the fluid and porous media, respectively, which are defined as $Q = \frac{qd^2}{2\kappa \Delta T}$ and $Q_p = \frac{q_p d_p{}^2}{2\kappa_p \Delta T}$. The dimensionless pertubed boundary conditions on the upper deformable free surface of the fluid layer (z = 1) at which convection driven by temperature-dependent surface tension forces is allowed are given by:

$$w = \frac{\partial \eta}{\partial t}, \tag{15}$$

$$\frac{\partial T}{\partial z} + \mathrm{Bi}\,(T - \eta) = 0, \tag{16}$$

$$\left(\frac{\partial^2}{\partial z^2} - \nabla_h^2\right) w + Ma\, \nabla_h^2 (\eta - T) = 0,. \tag{17}$$

$$\mathrm{Cr}\left[\frac{1}{\mathrm{Pr}}\frac{\partial}{\partial t} + \left(\frac{\partial^2}{\partial z^2} + 3\nabla_h^2\right)\right]\frac{\partial w}{\partial z} + \left(\mathrm{Bo} - \nabla_h^2\right)\nabla_h^2 \eta = 0. \tag{18}$$

At the interface (z = 0), the normal component of velocity, temperature and heat flux are continuous, which yield

$$w = \frac{\zeta}{\varepsilon} w_p, \tag{19}$$

$$T = \frac{\varepsilon}{\zeta} T_p, \tag{20}$$

$$\frac{\partial T}{\partial z} = \frac{\partial T_p}{\partial z_p}, \tag{21}$$

$$\frac{\partial^2 w}{\partial z^2} + \frac{1}{\sqrt{Da}}\left[\frac{\alpha \zeta^3}{\varepsilon}\right]\frac{\partial w_p}{\partial z_p} = \frac{\alpha \zeta}{\sqrt{Da}}\frac{\partial w}{\partial z}, \tag{22}$$

$$\left(3\nabla_h^2 + \frac{\partial^2}{\partial z^2}\right)\frac{\partial w}{\partial z} + \frac{\zeta^4}{\varepsilon\, Da}\left(\frac{\partial w_p}{\partial z_p}\right) = 0. \tag{23}$$

The lower rigid boundary of the porous layer (z_p = -1) is fixed with a constant heat flux or is isothermal, respectively, so that we have:

$$w_p = 0, \tag{24}$$

$$\frac{\partial T_p}{\partial z_p} = 0, \tag{25}$$

$$\text{or } T_p = 0, \tag{26}$$

where $\mathrm{Bi} = \dfrac{Hd}{k}$ is the Biot number, $Ma = \dfrac{\sigma d^2 H (T_0 - T_u)/(k + Hd)}{\mu \kappa}$ is the Marangoni number, $\mathrm{Cr} = \dfrac{\mu\kappa}{\sigma d}$ is the Crispation number, $\mathrm{Bo} = \dfrac{\rho_0 g d^2}{\sigma}$ is the Bond number, σ is the rate of change of surface tension with temperature, α is the slip parameter, $\varepsilon = \kappa/\kappa_p$ is the ratio of thermal diffusivities and $\zeta = d/d_p$ is the depth ratio.

The perturbation quantities in a normal mode form are:

$$(w, T) = [W(z), \theta(z)] \exp\left[i\left(\ell x + my\right) + \omega t\right], \tag{27}$$

in the fluid layer and in the porous layer.

$$(w_p, T_p) = \left[W_p(z_p),\ \theta_p(z_p)\right] \exp\left[i\left(\tilde{\ell} x_p + \tilde{m} y_p\right) + \omega_p t\right], \tag{28}$$

Here, ℓ and m are the wave numbers in the x and y directions, respectively, in the fluid layer, while $\tilde{\ell}$ and $\tilde{m}$ are the corresponding wave numbers in the porous layer and $\omega = \omega_r + i\omega_i \left(= \varepsilon / \varsigma^2 \omega_p\right)$ is the growth parameter of the disturbances where ω_r and ω_i are the growth rate instability and frequency, respectively. Substituting Equations (27) and (28) into Equations (11) to (26), matching the solutions in the fluid and porous layers with $\zeta = a/a_p$ and setting $\omega = 0$ to obtain the equation relevant to the neutral stability, we will have:

$$\left(D^2 - a^2\right)^2 W = 0, \tag{29}$$

$$\left(D^2 - a^2\right)\theta = [Q(1 - 2z) - 1] W, \tag{30}$$

$$\left(D_p^2 - a_p^2\right) W_p = 0, \tag{31}$$

$$\left(D_p^2 - a_p^2\right)\theta_p = \left[Q_p(1 - 2z_p) - 1\right] W_p. \tag{32}$$

The boundary conditions at the upper free surface (z = 1) are:

$$W = 0, \tag{33}$$

$$D\theta + \mathrm{Bi}\,(\theta - \chi) = 0, \tag{34}$$

$$\left(D^2 + a^2\right) W + Ma\,(\theta - \chi) a^2 = 0, \tag{35}$$

$$-\mathrm{Cr}\left(D^2 - 3a^2\right) DW + \left(\mathrm{Bo} + a^2\right) a^2 \chi = 0, \tag{36}$$

at the interface (z = 0)

$$W - \frac{\zeta}{\varepsilon} W_p = 0, \tag{37}$$

$$D\theta - D\theta_p = 0, \tag{38}$$

$$\theta - \frac{\varepsilon}{\zeta}\theta_p = 0, \tag{39}$$

$$D^2W - \frac{\alpha\zeta}{\sqrt{Da}}DW + \frac{\alpha\zeta^3}{\varepsilon\sqrt{Da}}DW_p = 0\,, \tag{40}$$

$$D^3W - 3a^2DW + \frac{\zeta^4}{\varepsilon\, Da}DW_p = 0\,, \tag{41}$$

and those at the bottom boundary (z_p = -1) are

$$W_p = 0\,, \tag{42}$$

$$D\theta_p = 0, \tag{43}$$

for lower rigid boundary is fixed with a constant heat flux and

$$\theta_p = 0, \tag{44}$$

for lower rigid boundary is isothermal.

METHOD OF SOLUTION

The resulting eigenvalue problem is solved exactly for rigid lower boundary and isothermal or fixed with a constant heat flux. The analytical asymptotic solution for the critical Marangoni number *Ma* is also obtained by regular perturbation method with wave number *a* as a perturbation parameter when both boundaries are fixed with a constant heat flux.

Exact solution

The resulting eigenvalue problem is solved exactly, in general, with Ma as an eigenvalue. Since Equations (29) and (31) are independent of θ and θ_p, they can be directly solved to get the general solution in the form

$$W = A_1\cosh(az) + A_2\sinh(az) + A_3 z\cosh(az) + A_4 z\sinh(az), \tag{45}$$

$$W_p = A_{p1}\sinh(a_p z_p) + A_{p2}\cosh(a_p z_p)\,, \tag{46}$$

where A_1 – A_4, A_{p1} and A_{p2} are the constants to be determined. Using the boundary conditions (33), (37), (40), (41) and (42), we have obtained

$$W = A\left[\cosh(az) + \Delta_1\sinh(az) + \Delta_2 z\cosh(az) + \Delta_3 z\sinh(az)\right], \tag{47}$$

$$W_p = A\frac{\varepsilon}{\varsigma}\left[\coth(a_p)\sinh(a_p z_p) + \cosh(a_p z_p)\right], \tag{48}$$

Where:

$$\Delta_1 = \frac{\varsigma^3 a_p\coth(a_p)}{2a^3 Da}\,,$$

$$\Delta_2 = \frac{-2a\left[\cosh(a) + \Delta_1\sinh(a)\right] - \sinh(a)\left[-a^2 + \frac{a\varsigma\alpha\Delta_1}{\sqrt{Da}} - \frac{2a^3\alpha\Delta_1\sqrt{Da}}{\varsigma}\right]}{2a\cosh(a) + \frac{\alpha\varsigma\sinh(a)}{\sqrt{Da}}},$$

$$\Delta_3 = \frac{\frac{-\alpha\varsigma}{\sqrt{Da}}\left[\cosh(a) + \Delta_1\sinh(a)\right] - \cosh(a)\left[a^2 - \frac{a\varsigma\alpha\Delta_1}{\sqrt{Da}} + \frac{2a^3\alpha\Delta_1\sqrt{Da}}{\varsigma}\right]}{2a\cosh(a) + \frac{\alpha\varsigma\sinh(a)}{\sqrt{Da}}}.$$

The heat Equations (30) and (32) have to be solved defining their right-hand side expressions given by (47) and (48), respectively. Thus, we obtain θ and θ_p equations for both lower boundaries conditions as:

$$\theta = \frac{A}{4a^2}\left[\delta_1\sinh(a\,z) + \vartheta_1\cosh(a\,z) + \delta(z)\right], \tag{49}$$

$$\theta_p = A\left[\left(\frac{a\delta_1 - \lambda_4}{a_p} - \frac{\varepsilon z_p}{2\zeta a_p}\right)\sinh(a_p z_p) + \left(\frac{\varepsilon\vartheta_1}{\zeta} - \frac{\varepsilon\coth(a_p z_p)}{2\zeta a_p}\right)\cosh(a_p z_p)\right], \tag{50}$$

when the lower rigid boundary is fixed with a constant heat flux and

$$\theta = \frac{A}{4a^2}\left[\delta_2\sinh(a\,z) + \vartheta_2\cosh(a\,z) + \delta(z)\right], \tag{51}$$

$$\theta_p = A\left[\left(\frac{a\delta_2 - \lambda_5}{a_p} - \frac{\varepsilon z_p}{2\zeta a_p}\right)\sinh(a_p z_p) + \left(\frac{\varepsilon\vartheta_2}{\zeta} - \frac{\varepsilon\coth(a_p z_p)}{2\zeta a_p}\right)\cosh(a_p z_p)\right], \tag{52}$$

when the lower rigid boundary is isothermal, where

$$\delta_i = \frac{\lambda_1\cosh(a) + \lambda_2\sinh(a) - a\vartheta_i\sinh(a)}{a\cosh(a)}\,,$$

$$\vartheta_1 = \frac{\varepsilon\left[\left(\lambda_3 - \lambda_4\cosh(a_p)\right)\cosh(a) + \left(\lambda_1\cosh(a) + \lambda_2\sinh(a)\right)\cosh(a_p)\right]}{a_p\zeta\cosh(a)\cosh(a_p) + a\varepsilon\sinh(a_p)\sinh(a)},$$

$$\vartheta_2 = \frac{\lambda_5\cosh(a) + \varepsilon\sinh(a_p)\left[\left(\lambda_1 - \lambda_4\right)\cosh(a) + \lambda_2\sinh(a)\right]}{a_p\zeta\cosh(a)\cosh(a_p) + a\varepsilon\sinh(a_p)\sinh(a)},$$

$$\delta(z) = \left[(\Delta_3 - 2a)z - a\Delta_2 z^2 + Q\left\{\left(2a + 2\Delta_1 - \frac{2}{a}\Delta_2 - \Delta_3\right)z - (2a - a\Delta_2 - 2\Delta_3)z^2 - \frac{4}{3}a\Delta_2 z^3\right\}\right]\sinh(az)$$
$$+\left[(-2a\Delta_1 + \Delta_2)z - a\Delta_3 z^2 + Q\left\{\left(2 + 2a\Delta_1 - \Delta_2 - \frac{2}{a}\Delta_3\right)z - (2a\Delta_1 - 2\Delta_2 - a\Delta_3)z^2 - \frac{4}{3}a\Delta_3 z^3\right\}\right]\cosh(az),$$

$$\lambda_1 = \left[2a^2 + 2a\Delta_1 + (a^2 - 1)\Delta_2 + a\Delta_3\right] + Q\left[-2 + \left(\frac{a^2}{3} - 1\right)\Delta_2 + \left(a + \frac{2}{a}\right)\Delta_3\right],$$

$$\lambda_2 = \left[2a + 2a^2\Delta_1 + a\Delta_2 + (a^2 - 1)\Delta_3\right] + Q\left[-2\Delta_1 + \left(a + \frac{2}{a}\right)\Delta_2 + \left(\frac{a^2}{3} - 1\right)\Delta_3\right],$$

$$\lambda_3 = \frac{2a^2\varepsilon}{\zeta\, a_p\sinh(a_p)}\left[\sinh^2(a_p) - Q_p\cosh^2(a_p) - 2\right],$$

$$\lambda_4 = \left[2a\Delta_1 - \Delta_2\right] - Q\left[2 + 2a\Delta_1 - \Delta_2 - \frac{2}{a}\Delta_3\right] + Q_p\left[\frac{2\varepsilon a^2}{(a_p)^2\varsigma}\right],$$

$$\lambda_5 = \frac{2a^2\varepsilon^2}{\varsigma a_p\sinh(a_p)}\left[\left\{\cosh(a_p)\sinh(a_p) - a_p\right\} + Q_p\left\{2a_p - \cosh(a_p)\sinh(a_p)\right\}\right].$$

Substituting Equations (34) and (36) into the boundary condition (35) and let Bi = 0, we obtain an analytically expression for the Marangoni number, *Ma* which can be conveniently written as:

$$Ma_i = \frac{-8a\left[\left(a + a\Delta_2 + \Delta_3\right)\cosh(a) + \left(a\Delta_1 + \Delta_2 + a\Delta_3\right)\sinh(a)\right]}{\delta_i \sinh(a) + \vartheta_i \cosh(a) + \delta(1) + \dfrac{8a^3 \mathrm{Cr}\left[\left(1+\Delta_2\right)\sinh(a) + \left(\Delta_1 + \Delta_3\right)\cosh(a)\right]}{a^2 + \mathrm{Bo}}} \tag{53}$$

where *i* = 1 refers to the lower rigid boundary is fixed with a constant heat flux and *i* = 2 refers to the lower rigid boundary is isothermal.

Regular perturbation method

It is known that, as a fluid is subjected to a uniform heat flux from below ($D\theta = 0$) and above ($Bi = 0$), the onset cellular convection corresponds to a vanishingly small wave number (Nield, 1977). To find the analytical asymptotic solutions using the regular perturbation method with wave number a as a perturbation parameter, the dependent variables in both the fluid and porous layers are now expanded in powers of a^2 in the form

$$(W, \Theta, \eta) = \sum_{i=0}^{N} a^{2i} (W_i, \Theta_i, \eta_i), \tag{54}$$

$$(W_p, \Theta_p) = \sum_{i=0}^{N} \left(\frac{a_p^2}{\zeta^2}\right)^i (W_{pi}, \Theta_{pi}). \tag{55}$$

Substituting Equations (54) and (55) into Equations (29) to (43) and collecting coefficients of like powers of a^2 yields the following set of equations:

The equations of order zero are

$$D^4 W_0 = 0, \tag{56}$$

$$D^2 \Theta_0 = \left[Q(1-2z) - 1\right] W_0, \tag{57}$$

$$D^2 W_{p0} = 0, \tag{58}$$

$$D^2 \Theta_{p0} = \left[Q_p (1 - 2z_p) - 1\right] W_{p0}, \tag{59}$$

at $z = 1$:

$$W_0 = D\Theta_0 = D^2 W_0 = 0, \tag{60}$$

at $z = 0$:

$$W_0 = \frac{\zeta}{\varepsilon} W_{p0}, \tag{61}$$

$$\Theta_0 = \frac{\varepsilon}{\zeta} \Theta_{p0}, \tag{62}$$

$$D\Theta_0 = D\Theta_{p0}, \tag{63}$$

$$D^2 W_0 - \frac{\alpha\zeta}{\sqrt{\mathrm{Da}}} DW_0 = -\frac{\alpha\zeta^3}{\varepsilon\sqrt{\mathrm{Da}}} DW_{p0}, \tag{64}$$

$$D^3 W_0 = -\frac{\zeta^4}{\varepsilon\,\mathrm{Da}} DW_{p0}, \tag{65}$$

and at $z_p = -1$:

$$W_{p0} = D\Theta_{p0} = 0. \tag{66}$$

The solution of order zero is given by:

$$W_0 = 0, \ \Theta_0 = \frac{\varepsilon}{\zeta}, \ W_{p0} = 0, \ \Theta_{p0} = 1. \tag{67}$$

The equations of order a^2 are

$$D^4 W_1 = 0, \tag{68}$$

$$D^2 \Theta_1 - \left[Q(1-2z) - 1\right] W_1 = \frac{\varepsilon}{\zeta}, \tag{69}$$

$$D^2 W_{p1} = 0, \tag{70}$$

$$D^2 \Theta_{p1} - \left[Q_p (1 - 2z_p) - 1\right] W_{p1} = 1, \tag{71}$$

at $z = 1$:

$$W_1 = D\Theta_1 = 0, \tag{72}$$

$$\Pr D^2 W_1 + Ma\left(\Pr \frac{\varepsilon}{\zeta} - \eta\right) = 0, \tag{73}$$

$$\Pr \mathrm{Cr}\, D^3 W_1 - \mathrm{Bo}\,\eta = 0, \tag{74}$$

at $z = 0$:

$$W_1 = \frac{1}{\zeta\,\varepsilon} W_{p1}, \tag{75}$$

$$\Theta_1 = \frac{\varepsilon}{\zeta^3} \Theta_{p1}, \tag{76}$$

$$D\Theta_1 = \frac{1}{\zeta^2} D\Theta_{p1}, \tag{77}$$

$$D^2 W_1 - \frac{\alpha\zeta}{\sqrt{\mathrm{Da}}} DW_1 = -\frac{\alpha\zeta}{\varepsilon\sqrt{\mathrm{Da}}} DW_{p1}, \tag{78}$$

$$D^3 W_1 = -\frac{\zeta^2}{\varepsilon\,\mathrm{Da}} DW_{p1}, \tag{79}$$

Table 1. Comparison of critical Marangoni number, Ma_{1c} for different values of Cr and ζ with Q = 0 and Q_p=0 (that is, in the absence of heat generation).

ζ	Shivakumara and Chavaraddi (2007)			Present		
	Cr =10^{-6}	Cr = 10^{-4}	Cr = 10^{-1}	Cr =10^{-6}	Cr = 10^{-4}	Cr = 10^{-1}
0.2	337.752	225.796	0.673	337.7519	225.7963	0.6730
0.4	208.303	159.286	0.668	208.3031	159.2864	0.6679
0.6	156.159	126.844	0.666	156.1588	126.8439	0.6661
0.8	129.456	108.622	0.665	129.4562	108.6223	0.6649
1.0	113.301	97.012	0.664	113.3013	97.0117	0.6641

and at $z_p = -1$,

$$W_{p1} = 0, \tag{80}$$

$$D\Theta_{p1} = 0. \tag{81}$$

Using the symbolic algebra package MAPLE 12, after long and tedious algebra, we have obtained the critical Mrangoni number; Ma_c as given sebusequently,

$$Ma_c = \frac{-240\,\zeta(2\,\psi_1 + Q\psi_2)(1+\varepsilon\,\zeta)}{\varepsilon[2(\alpha\zeta + 3\sqrt{\mathrm{Da}}) + Q(\alpha\zeta + 2\sqrt{\mathrm{Da}})]\ [(2\sqrt{\mathrm{Da}} + \alpha\zeta) - 5\,\zeta\,\psi_4]}, \tag{82}$$

where

$$\psi_1 = \zeta^4\alpha^2 + 3\,\zeta\,\mathrm{Da}\left(3\zeta + \alpha^2[1+\zeta]\right) + 3\alpha\sqrt{\mathrm{Da}}\left(2\zeta^3 + 3\,\mathrm{Da}\,[1+\zeta]\right),$$

$$\psi_2 = \zeta^4\alpha^2 + 3\,\zeta\,\mathrm{Da}\left(2\zeta + \alpha^2[1+\zeta]\right) + \alpha\sqrt{\mathrm{Da}}\left(5\zeta^3 + 6\mathrm{Da}[1+\zeta]\right),$$

$$\psi_3 = Q\zeta\left(60\mathrm{Da} - \zeta^3\right) + 60\varepsilon\,\mathrm{Da}\left(5Q_p - 3\right) - 360\frac{\zeta^3\mathrm{Cr}(1+\varepsilon\,\zeta)}{\varepsilon\,\mathrm{Bo}},$$

$$\psi_4 = \zeta^2\left[\alpha\zeta^2 + 6\sqrt{\mathrm{Da}}\left(2\alpha\sqrt{\mathrm{Da}} + \zeta\right)\right] + 2\,\mathrm{Da}\left(2\,\alpha\,\zeta + 3\sqrt{\mathrm{Da}}\right).$$

RESULTS AND DISCUSSION

In the absence of internal heat generation (that is, $Q=0$), the Equations (53) and (82) reduce to the expression given by Shivakumara and Chavaraddi (2007). For each case investigated in this paper, we use Equation (53) to obtain the critical Marangoni number. The critical Marangoni number, Ma_c depends on the depth ratio ζ, Crispation number Cr, Bond number Bo, Darcy number Da, slip parameter β and heat source strength in fluid Q and porous medium Q_p. The critical Marangoni numbers Ma_{1c} and Ma_{2c} correspond to the lower rigid boundary which is fixed with a constant heat flux and is isothermal, respectively.

The critical values of Marangoni number, Ma_{1c} when $\varepsilon = 0.725$, $\mathrm{Da} = 4\times10^{-6}$, $\mathrm{Bo} = 0.1$, $\alpha = 1$, $Q = 0$ and Q_p = 0 are shown in Table 1. Results obtained by Shivakumara and Chavaraddi (2007) are also included in this table. It is seen that the present results are in good agreement with those of Shivakumara and Chavaraddi (2007). Table 2 presents the critical values of Marangoni number, Ma_{1c} for different values of Cr, Q and ζ with $\varepsilon = 0.725$, $\mathrm{Da} = 4\times10^{-6}$, $\mathrm{Bo} = 0.1$ and Q_p = 0. Different values of ζ have been chosen in the sense that they represent the different values of depth ratio for the two layers system. From Table 2, it is shown that the Ma_{1c} decreases with an increase in the value of the heat generation, Q. Table 3 shows the critical Marangoni number, Ma_{2c} for different values of Cr and Q with $\varepsilon = 0.725$, $\mathrm{Da} = 4\times10^{-6}$, $\mathrm{Bo} = 0.1$ and Q_p = 0. Table 3 also illustrates the comparison of selected values. As seen in this table, the agreement between the present results and the results from Shivakumara and Chavaraddi (2007) is very good. Also, as seen in Tables 2 and 3, we found that Ma_{1c} and Ma_{2c} decrease as the value of Cr increases. Consequently, the degree of allowing the free surface to deform is a destabilizing effect on the system. However, decreasing ζ has the effect of increasing the critical Marangoni number and hence, can help to delay the Marangoni convection.

The comparison and the present results for the critical values of the Marangoni number, Ma_{2c} for different values of Cr and ζ are tabulated in Table 4. We realize that the results carried out from this analysis when ζ >> 1 which in turns to the fluid case, are closed similar with those values obtained by Char and Chiang (1994). We proceed by substituting ζ = 0.9, 0.8, 0.7 and surprisingly we found that the Ma_c values increase with the decreasing of ζ. These results are very contrast when the lower rigid boundary is fixed at a constant heat flux and it might be due to the effect of behaviour of insulating and conducting materials when they are in contact with the source of heat at the bottom of the system. As expected, further increase of Cr will decrease the values of Ma_c and thus, making the system unstable.

Figure 2 illustrates the Marangoni number, Ma_1 and Ma_2 as a function of wave number, a, for different values of the effect of Cr with Da = 0.004, $\alpha = 0.1$, $\varepsilon = 0.725$,

Table 2. Critical values of Marangoni number, Ma_{1c} for different values of Cr, Q and ζ with $Q_p = 0$.

ζ	$Q = 3$			$Q = 5$		
	Cr =10^{-6}	**Cr = 10^{-4}**	**Cr = 10^{-1}**	**Cr =10^{-6}**	**Cr = 10^{-4}**	**Cr = 10^{-1}**
0.2	275.6509	196.2403	0.6727	245.5518	180.4899	0.6725
0.4	133.4985	111.5073	0.6668	107.7114	92.9249	0.6660
0.6	98.5168	85.9807	0.6644	79.0612	70.7794	0.6633
0.8	81.3684	72.6162	0.6629	65.2179	59.4726	0.6616
1.0	71.1073	64.3282	0.6618	56.9646	52.5299	0.6603

Table 3. Critical values of Marangoni number, Ma_{2c} for different values of Q and Cr with $\zeta = 5$ and $Q_p = 0$.

Cr	Shivakumara and Chavaraddi (2007)	Present			
	$Q = 0$	$Q = 0$	$Q = 1$	$Q = 5$	$Q = 10$
10^{-6}	79.591	79.5788	56.8941	36.1152	20.8221
10^{-5}	79.541	79.5690	56.8850	36.1093	20.8186
10^{-4}	79.444	79.4720	56.7993	36.0493	20.7832
10^{-3}	66.668	66.7252	33.4058	16.6450	8.3346
10^{-2}	6.667	6.6703	3.3365	1.6675	0.8383
10^{-1}	0.667	0.6666	0.3336	0.1667	0.0835

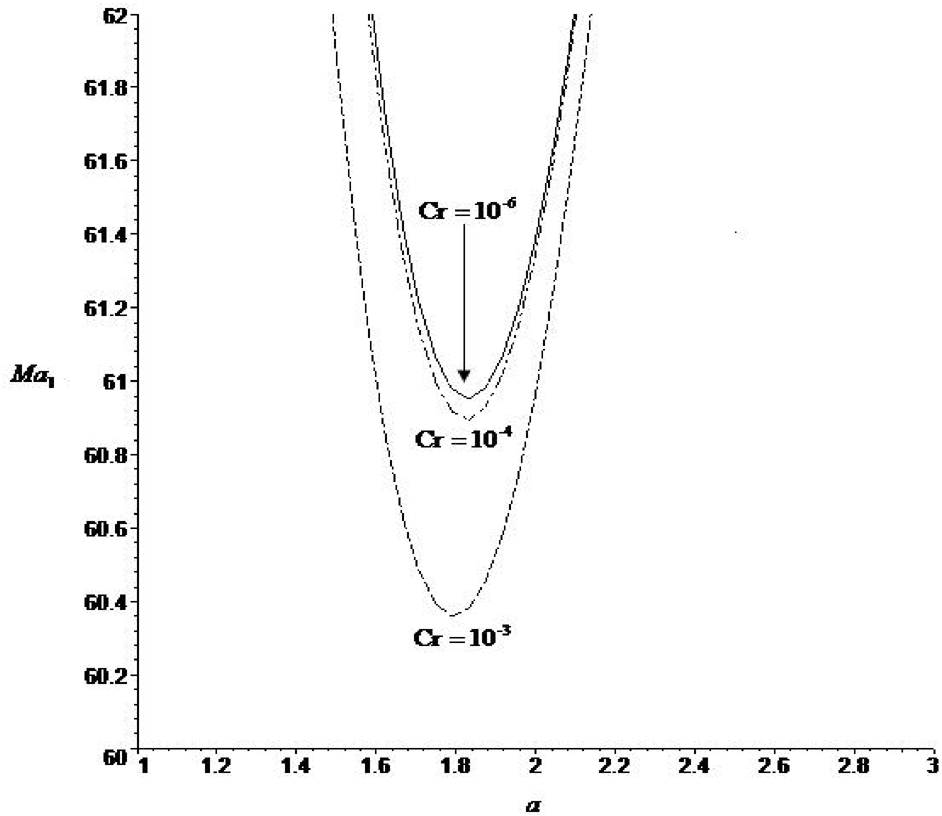
Table 4. Critical values of Marangoni number, Ma_{2c} for different values of Cr and ζ with $Q = 5$ and $Q_p = 0$.

Cr	Char and Chiang (1994)	Present			
	Fluid	$\zeta >> 1$	$\zeta = 0.9$	$\zeta = 0.8$	$\zeta = 0.7$
0	26.426	26.4217	25.3722	25.3595	25.3337
10^{-6}	26.425	26.4213	25.3716	25.3589	25.3331
10^{-5}	26.421	26.4168	25.3661	25.3534	25.3275
10^{-4}	26.376	26.3722	25.3103	25.2974	25.2714
10^{-3}	11.111	11.0800	11.1736	11.1450	11.2426
10^{-2}	1.111	1.0952	1.1152	11.1144	1.1194
10^{-1}	0.111	0.1107	0.1114	0.11135	0.1117

Bo = 0.1, $Q_p = 0$, $Q = 1$ and $\zeta = 0.8$. In Figure 2 we see that Ma_1 and Ma_2 decrease with an increase in the value of the Cr and hence make the system more unstable. The basic cause of this behaviour is that higher values of Cr correspond to lower rigidity of the free upper surface of the fluid layer which makes the system unstable. The critical Marangoni numbers, Ma_{1c} and Ma_{2c} as functions of heat generation, Q for different values of Cr with Da=0.003, $\alpha = 0.1$, $\varepsilon = 0.725$, $Q_p = 1$, $\zeta = 0.7$ and Bo = 1 are shown in Figure 3. Clearly, increasing Q has the effect of decreasing Ma_{1c} and Ma_{2c} but further decrease of Cr will increase the critical Marangoni number. Next, we look at Figure 4 for the influence of the Darcy number, Da, on the critical Marangoni numbers, Ma_{1c} and Ma_{2c} with $\alpha = 0.1$, $\varepsilon = 0.725$, Bo = 1, $Q_p = 1$, $Q = 2$ and $\zeta = 0.9$. As expected, the critical Marangoni numbers Ma_{1c} and Ma_{2c} increase with the decreasing of the Darcy number, Da. Physically, a larger porosity provides more flow area and results in the stronger fluid flow and heat transfer rate. A medium with small porosity which refers to small Da can help to delay the Marangoni number as can be seen clearly in Figure 4. The critical Marangoni numbers obtained by regular perturbation method are also shown in Figure 4(a) when the lower rigid boundary is fixed with a constant heat flux. It is now demonstrated that the exact solutions of the complete set of Equations (29) to (36) are in excellent agreement with the asymptotic solutions.

To examine the effect of the Bond number Bo, it is

(a)



(b)

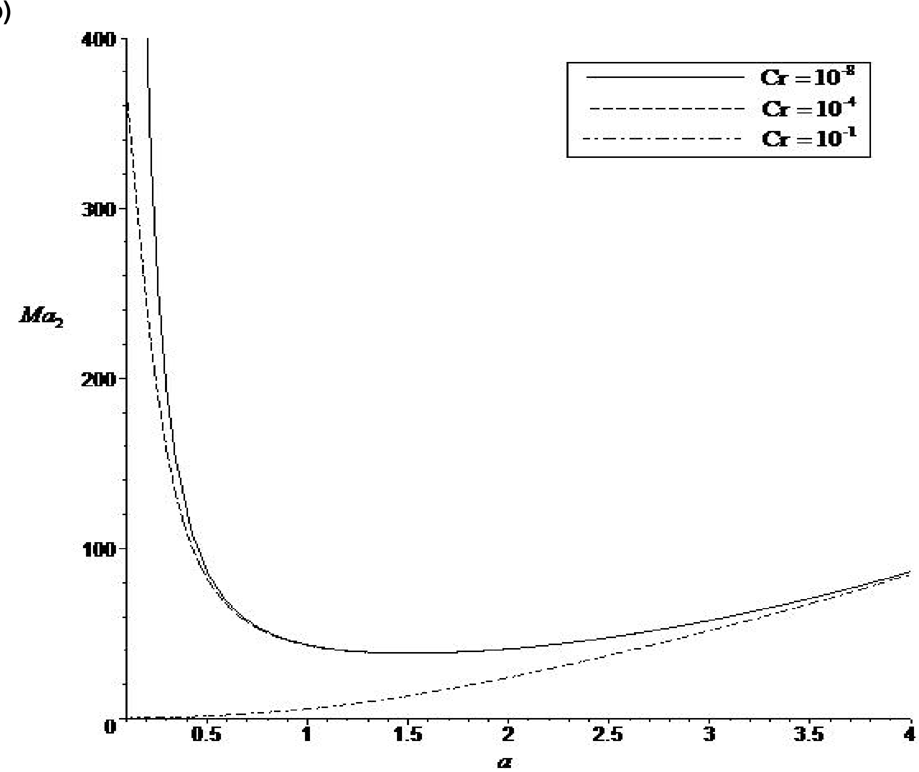


Figure 2. Marangoni number (a) Ma_1 and (b) Ma_2 as a function of wave number *a* for different values of Cr in the case of Da = 0.004, $\alpha = 0.1$, $\varepsilon = 0.725$, Bo = 0.1, Q_p = 0, Q = 1 and ζ = 0.8.

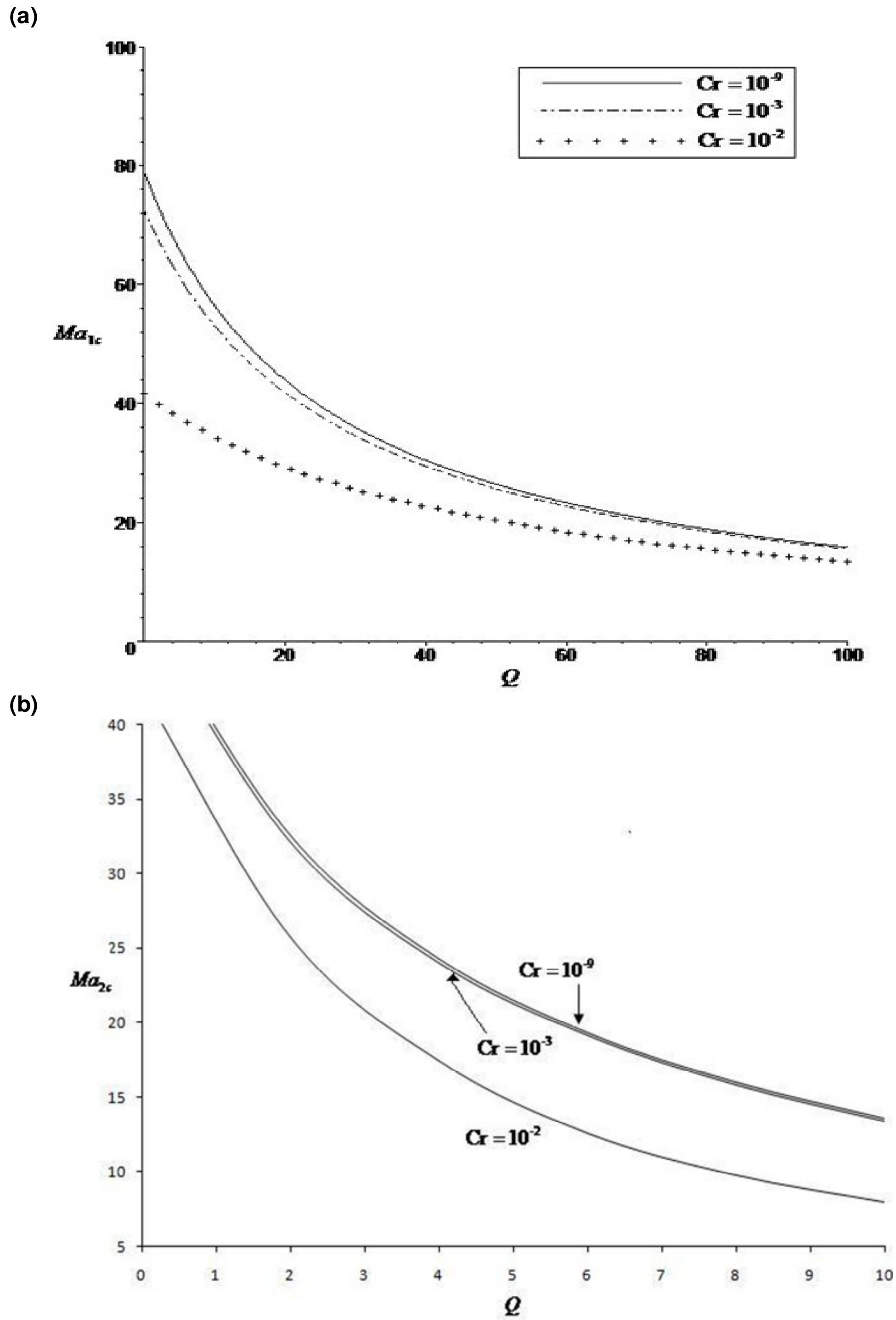


Figure 3. Critical Marangoni number (a) Ma_{1c} and (b) Ma_{2c} as a function of Q for different values of Cr in the case of Da = 0.003, $\alpha = 0.1$, $\varepsilon = 0.725$, $Q_p = 1$, $\zeta = 0.7$ and Bo = 1.

shown in Figure 5 the critical Marangoni numbers, Ma_{1c} and Ma_2 as functions of depth ratio ζ with α = 1, ε = 0.725, Da = 4 × 10^{-6}, Q_p = 0.1, Cr = 0.001 and Q = 10. It is shown that as the value of Bo increases, the system becomes more stable. We may conclude that further increase of Bo will increase in the gravity effect, which keeps the free surface flat against the effect of the Cr and the system becomes more stable. Figure 6 shows the

(a)

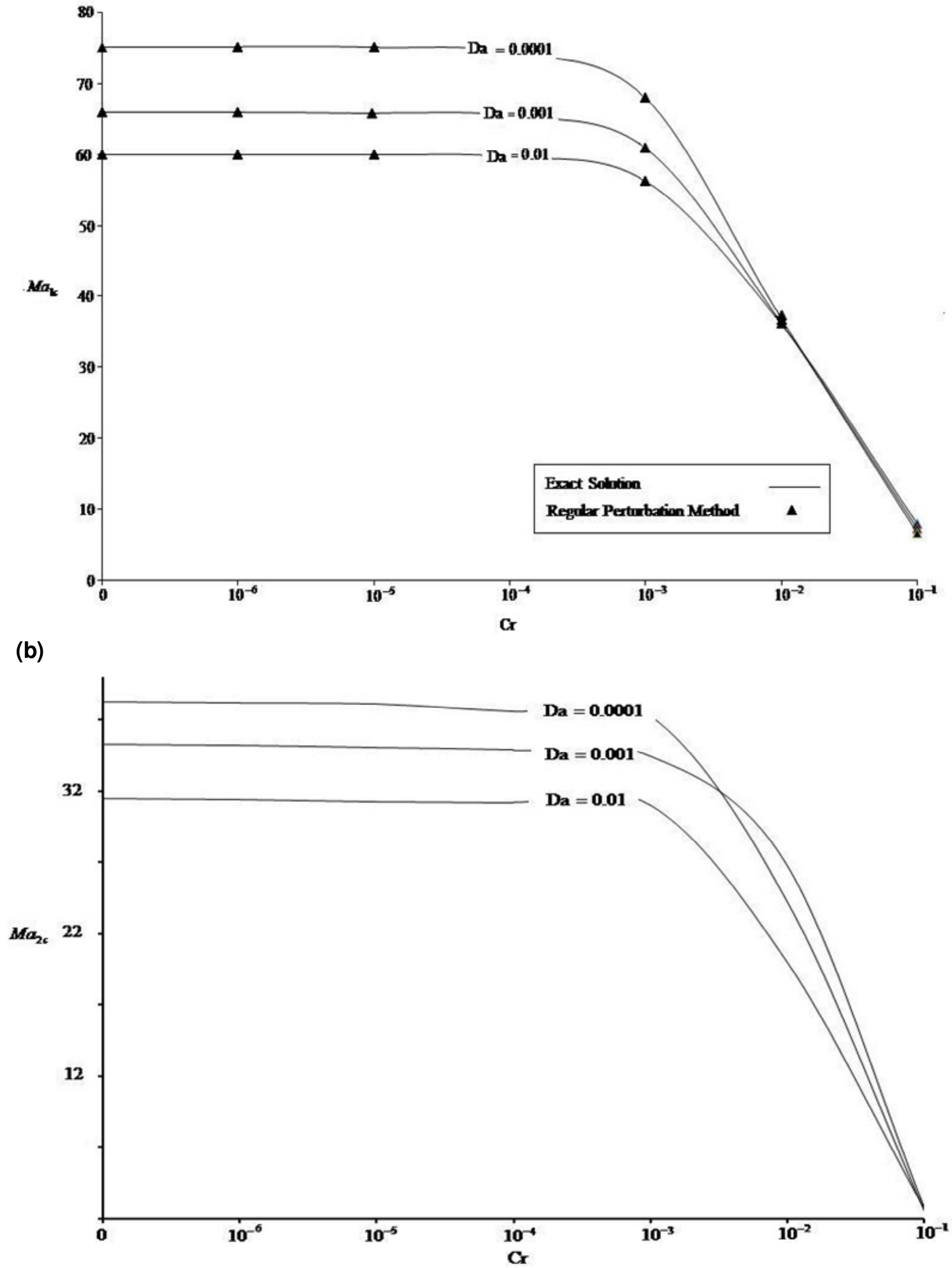


Figure 4. Critical Marangoni number (a) Ma_{1c} and (b) Ma_{2c} as a function of Cr for different values of Da in the case of α = 0.1, ε = 0.725, Bo = 1, Q_p = 1, Q = 2 and ζ = 0.9.

critical Marangoni numbers Ma_{1c} and Ma_{2c} as functions of depth ratio ζ for different values of Cr with α = 1, ε = 0.725, Da = 4 × 10^{-6}, Q_p = 0.1, Bo=0.2 and Q = 5. For the lower rigid boundary is fixed with a constant heat flux, it is noted that the Ma_{1c} increases for approximately 0.01 < ζ < 0.11 at which the maximum values of Ma_{1c} are reached and then decreasing for ζ > 0.11. As $\xi \to \infty$, Cr $\to 0$, the critical Marangoni number, Ma_{1c} attains a constant value 24, which is the exact value known for the case of a single fluid layer in the absence of surface deflection at the free surface with heat generation (Wilson, 1997). However, Ma_{2c}

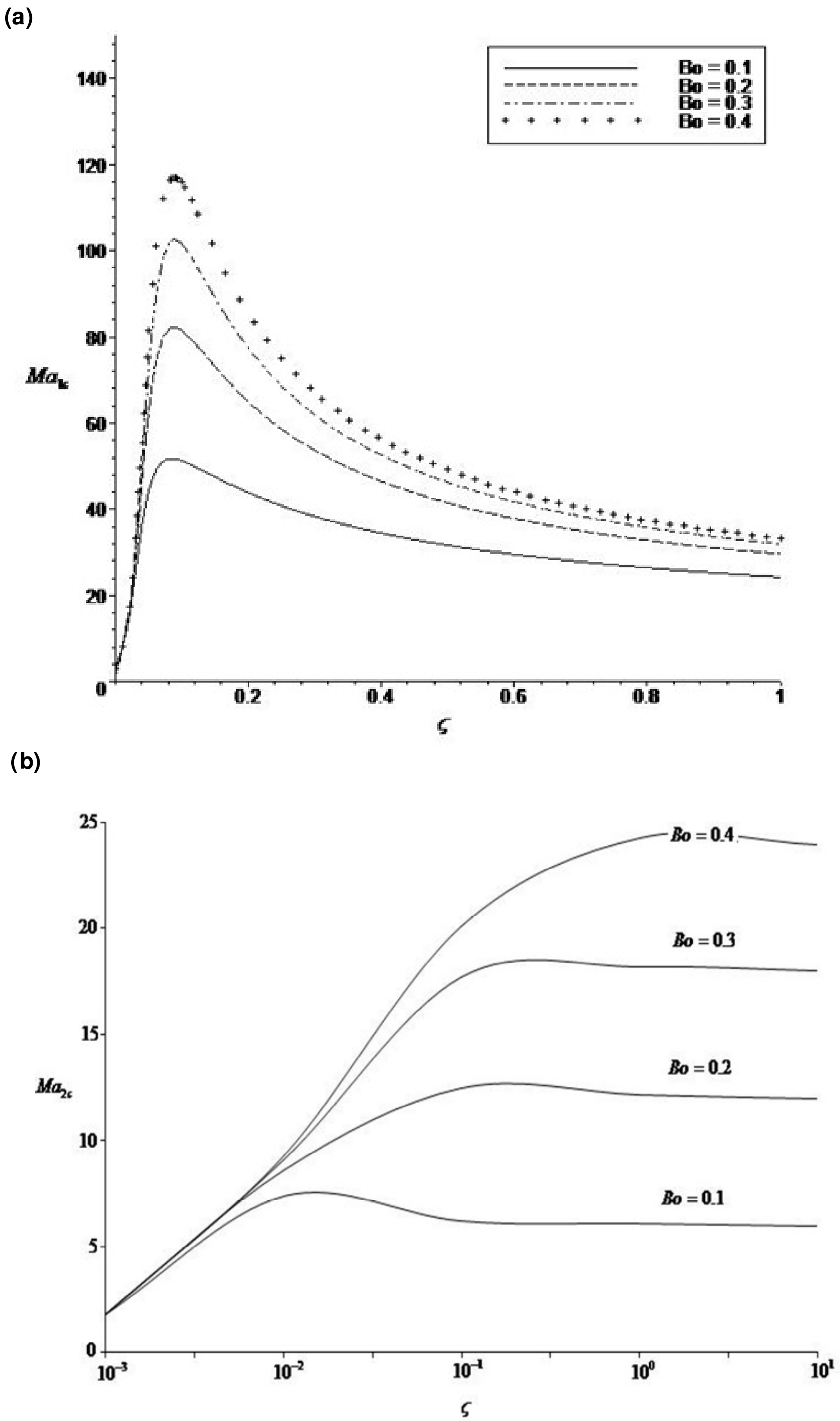


Figure 5. Critical Marangoni number (a) Ma_{1c} and (b) Ma_{2c} as a function of ζ for different values of Bo in the case of $\alpha = 1$, $\varepsilon = 0.725$, Da = 4×10^{-6}, $Q_p = 0.1$, Cr = 0.001 and $Q = 10$.

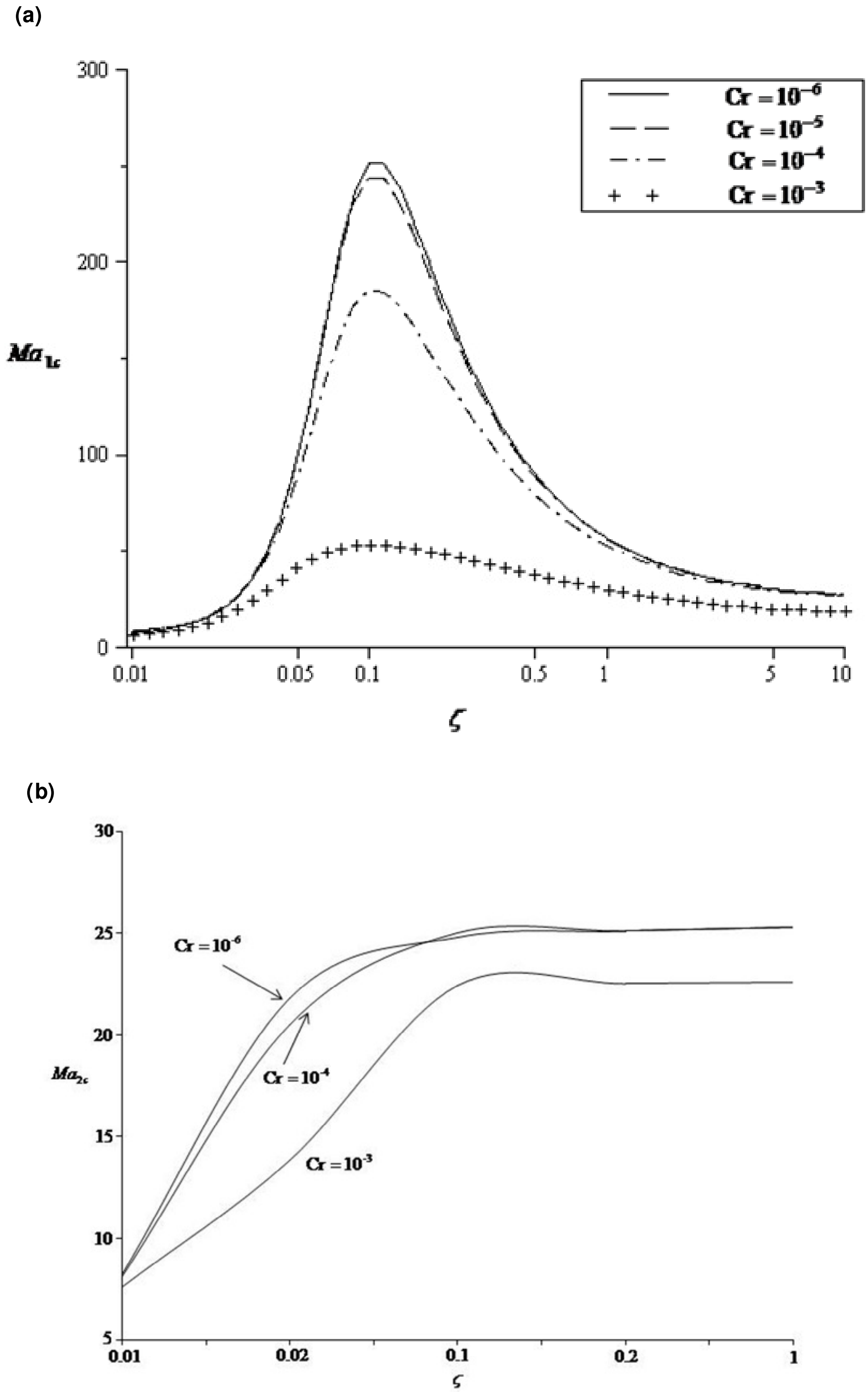


Figure 6. Critical Marangoni number (a) Ma_{1c} and (b) Ma_{2c} as a function of ζ for different values of Cr in the case of $\alpha = 1$, $\varepsilon = 0.725$, Da = 4×10^{-6}, $Q_p = 0.1$, Bo=0.2 and $Q = 5$.

monotonically increases with ζ and ultimately attains an asymptotic value for large ζ without showing any decreasing trend as observed in the insulating case. This may be due to the asymmetric temperature boundary conditions imposed at the boundaries of the composite system.

CONCLUSIONS

The stability analysis of the Marangoni convection in a two-layer system with internal heat generation and when the upper surface is assumed to be deformable is investigated theoretically and the following results are obtained:

1. The internal heat generation in the two-layer saturated porous matrix is clearly a destabilizing factor to make the system more unstable, however the decrease in the Crispation number, Cr delayed the process and the onset of convection.
2. The parameter ζ, which is the ratio of the thickness of fluid layer to porous layer has an important effect on the stability of the system.
3. The decrease of the Darcy number and an increase in the Bond number will help to slow the process of the onset of destabilizing the system.

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