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Full Length Research Paper

A new method for solving a system of fifth-order obstacle boundary value problems

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In this paper, we use the variation of parameters method for solving a system of fifth-order boundary value problems associated with obstacle problems. The results are compared with the exact solution over the whole domain and error estimates are calculated. An example is given to illustrate the efficiency and implementation of the variation of parameters method.

Key words: Variation of parameters method, system of fifth-order obstacle boundary value problems.

INTRODUCTION

In recent years, much attention has been given for solving systems of even and odd order boundary value problems associated with obstacle and contact problems. It has been shown that the obstacle, free, moving and contact problems can be studied in the general frame of variational inequalities. This equivalent formulation has been used to study the existence and uniqueness of the solution of these obstacle problems under suitable conditions. It is known that the finite difference and other methods may not be extended for solving these obstacle problems. Lewy and Stampacchia (1969) have proved that the obstacle problems can be characterized by the system of variational equations using the penalty function technique. This characterization has been used by Lewy and Stampcchia (1969) to study the regularity of the solution of these obstacle problems. It is worth mentioning that if the obstacle is known that then the variational equations are equivalent to the system of boundary value problems, which can be solved using the finite difference and other numerical techniques. This aspect has been exploited by Noor et al. (2000, 2004) to developed several finite and spline technique for solving the systems of two, three and fourth orders boundary value problems associated with obstacle and contact problems. To the best of our knowledge, system of fifth-order boundary value problems have not considered. In this paper, first of all, we consider the system of fifth-order boundary value problems in the general setting. We developed the general variation of parameters methods for solving the general nonlinear equations. We show that the variation of parameters method can be extended for solving the system of the fifth-order obstacle boundary value problems. This is main motivation of this paper. To be more precise, we consider the system of fifth-order boundary value problems of the type:

$$u^{(v)}(x)=\begin{cases} f(x), & a\le x<c, \\ f(x)+u(x)g(x)+r, & c\le x<d, \\ f(x), & d\le x\le b, \end{cases} \tag{1}$$

with boundary conditions $u(a)=u(b)=\alpha_1,\ u(c)=u(d)=\alpha_2,\ u'(b)=\alpha_3,$ and continuity conditions of $u(x), u'(x), u''(x), u'''(x)$ and $u^{(iv)}(x)$ at internal points c and d of the interval $[a,b]$. Here r and $\alpha_i,\ i=1\ldots3$ are real and finite constants. $f(x)$ and $g(x)$ are continuous functions on $[a,b]$. Such type of problems arise in the study of obstacle, contact, unilateral and equilibrium problems arising in transportation, optimization, viscoelastic flows

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and some other branches of engineering and applied sciences, see Noor (1998, 1993, 2000, 2004, 2009) and references therein. Several techniques have been developed and used for solving systems of even and odd order boundary value problems associated with obstacle, contact and unilateral problems. Noor et al. (1986) applied finite difference method for solving unilateral problems, Khalifa and Noor (1990) applied quintic spline method for the contact problems, Noor et al. (1994) applied quartic spline method for odd-order obstacle problems. Noor et al. (2004) used quartic spline method for solving forth-order obstacle problems.

Noor et al. (2010, 2011) and Mohyud-Din et al. (2008, 2009) have used the variation of parameters method for solving a wide classes of higher orders initial and boundary value problems. We would like to point out that the variation of parameters method removes the successive applications of integral and the higher order derivative term from its iterative schemes and hence reduces the repeated computation and calculation of unneeded terms. This shows that this method is better than the other methods. The variation of parameters method makes the solution procedure simple while still maintaining the higher level of accuracy. In the present paper, we again use the variation of parameters method for solving a system of fifth-order boundary value problems associated with obstacle, unilateral and contact problems. Using the technique of Lewy and Stampacchia (1969), we show that the obstacle problem can be characterized by the system of variational equations using the penalty function. This system can be written as a system of fifth-order boundary problem for a known obstacle, We use the variation of parameters methods, developed thus, for finding the approximate solution of the system of the fifth-order boundary value problem associated with the obstacle problem. An example is given to illustrate the efficiency and implementation of this method. Results obtained in this paper stimulate further research in this fast growing and dynamic field. The interested reader is advised to discover novel and new applications of the obstacle and unilateral problems in different fields of pure and applied sciences.

Variation of parameters method

To illustrate the basic concept of the variation of parameters method for differential equations, we consider the general differential equation in operator form.

$$Lu(x)+Ru(x)+Nu(x)=g(x), \tag{2}$$

where L is a higher order linear operator, R is a linear operator of order less than L, N is a nonlinear operator and g is a source term. Using variation of parameters method, see Noor et al. (2010, 2011), we have following general scheme for finding the approximate solution of (2):

$$u(x)=\sum_{i=0}^{n-1}\frac{B_i x^i}{i!}+\int_0^x \lambda(x,s)\left(-Nu(s)-Ru(s)+g(s)\right)ds, \tag{3}$$

where n is an order of given differential equation and $B_i's$ are unknowns which can be determined by initial/boundary conditions. Here $\lambda(x,s)$ is multiplier, which can be obtained with the help of Wronskian technique. This multiplier removes the successive application of integrals in iterative scheme and it depends upon the order of equation. Mohyud-Din et al. (2009, 2010) and Noor et al. (2010, 2011) have obtained the following formula for finding the multiplier $\lambda(x,s)$ as

$$\lambda(x,s)=\sum_{i=1}^{n}\frac{s^{i-1}x^{n-i}(-1)^{i-1}}{(i-1)!(n-i)!}. \tag{4}$$

For the fifth-order order problems, we have

$$\lambda(x,s)=\frac{x^4}{4!}-\frac{x^3 s}{3!}+\frac{x^2 s^2}{2!2!}-\frac{xs^3}{3!}+\frac{s^4}{4!},$$

From (3) and (4), we can obtain the following iterative scheme for finding the approximate solution of (2) as:

$$u_{k+1}(x)=u_k(x)+\int_0^x \lambda(x,s)\left(-Nu_k(s)-Ru_k(s)+g(s)\right)ds \qquad k=0,1,2,\ldots \tag{5}$$

It is observed that the fixed value of initial guess in each iteration provides the better approximation. For the applications and the development of the variation of parameters methods, see Mohyud-Din et al. (2009, 2010) and Noor et al. (2007, 2008, 2010, 2011).

APPLICATIONS AND NUMERICAL RESULTS

To illustrate the implementation of the variation of parameters method, we consider the problem of fifth-order obstacle boundary value problem of the type:

$$\left.\begin{array}{ll} -u^{(v)}(x)\geq f(x), & \Omega=[-1,1] \\ u(x)\geq \psi(x), & \Omega=[-1,1] \\ \left[u^{(v)}(x)-f(x)\right]\left[u(x)-\psi(x)\right]=0, & \Omega=[-1,1] \\ u(-1)=u(1)=u'(1)=0,\ u\left(-\frac{1}{2}\right)=u\left(\frac{1}{2}\right)=1, & \end{array}\right\} \tag{6}$$

where $f(x)$ is given force acting on string and $\psi(x)$ is the elastic obstacle. Problems of type (6) arise in several branches of pure and applied sciences including transportation, equilibrium, optimization, mechanics, structural analysis, fluid flow through porous media and image processing in the medical sciences, Noor (1988, 1003, 2000, 2004, 2009), Noor et al. (1993) and the references therein.

Using the idea and technique of Lewy and Stampachia (1969), the obstacle problem (6) can be characterized by the following system of variational equations.

$$u^{(v)}(x)-\varphi\{u(x)-\psi(x)\}\big(u(x)-\psi(x)\big)=f(x), \qquad -1<x<1 \tag{7}$$
$$u(-1)=u(1)=u'(1)=0, u\left(-\frac{1}{2}\right)=u\left(\frac{1}{2}\right)=1,$$

Where

$$\varphi(t)=\begin{cases}1, & t\geq 0\\ 0, & t<0\end{cases} \tag{8}$$

is a discontinuous function and is known as the penalty function.

We assume that if the obstacle function $\psi(x)$ is known and is defined as:

$$\psi(x)=\begin{cases}-1, & -1\leq x\leq -\dfrac{1}{2} \text{ and } \dfrac{1}{2}\leq x\leq 1,\\ 1, & -\dfrac{1}{2}\leq x\leq \dfrac{1}{2}.\end{cases} \tag{9}$$

Then, from (7), (6) and (9), we obtain the following system of fifth-order boundary value problems of the type (1) as

$$u^{(v)}(x)=\begin{cases}f\big(u(x)\big), & -1\leq x<-\dfrac{1}{2} \text{ and } \dfrac{1}{2}\leq x\leq 1\\ f\big(u(x)\big)+u(x)-1, & -\dfrac{1}{2}\leq x<\dfrac{1}{2},\end{cases} \tag{10}$$

with boundary conditions $u(-1)=u(1)=u'(1)=0, u\left(-\frac{1}{2}\right)=u\left(\frac{1}{2}\right)=1,$ and the conditions of continuity of $u(x), u'(x), u''(x), u'''(x)$ and $u^{(iv)}(x)$ at $x=-\dfrac{1}{2}$ and $x=\dfrac{1}{2}$.

Clearly, the system of equations associated with the obstacle problem (6) is a special case of the system of fifth-order boundary value problem (1).

Example 1: We consider the following system of Equations (11) with $f\big(u(x)\big)=0$,

$$u^{(v)}(x)=\begin{cases}0, & -1\leq x<-\dfrac{1}{2} \text{ and } \dfrac{1}{2}\leq x\leq 1\\ u(x)-1, & -\dfrac{1}{2}\leq x<\dfrac{1}{2},\end{cases} \tag{11}$$

with boundary conditions: $u(-1)=u(1)=u'(1)=0,\ u\left(-\dfrac{1}{2}\right)=u\left(\dfrac{1}{2}\right)=1.$

The exact solution of the problem (11) is

$$u(x)=\begin{cases}c_5+c_4x+c_3\dfrac{x^2}{2}+c_2\dfrac{x^3}{6}+c_1\dfrac{x^4}{24}, & -1\leq x<-\dfrac{1}{2},\\ 1+c_6e^{x}+c_7e^{\alpha_1 x}\sin(\beta_1 x)+c_8e^{\alpha_2 x}\sin(\beta_2 x) & \\ +c_9e^{\alpha_2 x}\cos(\beta_2 x)+c_8e^{\alpha_1 x}\cos(\beta_1 x), & -\dfrac{1}{2}\leq x<\dfrac{1}{2},\\ c_{15}+c_{14}x+c_{13}\dfrac{x^2}{2}+c_{12}\dfrac{x^3}{6}+c_{11}\dfrac{x^4}{24}, & \dfrac{1}{2}\leq x\leq 1,\end{cases} \tag{12}$$

where
$\alpha_1=\frac{1}{4}\left(\sqrt{5}-1\right),\ \alpha_2=-\frac{1}{4}\left(\sqrt{5}+1\right),\ \beta_1=\frac{1}{4}\sqrt{2}\left(\sqrt{5+\sqrt{5}}\right),\ \beta_2=\frac{1}{4}\sqrt{2}\left(\sqrt{5-\sqrt{5}}\right)$
and
$\{c_1=42.41920767,\ c_2=-.07991987551, c_3=-7.115891311,\ c_4=.0003920802857,$
$c_5=1.777550770,\ c_6=7.261221655,\ c_7=-17.89630990,\ c_8=-7.340369643,$
$c_9=-14.38087600,\ c_{10}=7.897258366,\ c_{11}=42.41920767,\ c_{12}=.1718446390,$
$c_{13}=-7.166673255,\ c_{14}=.01088298815,\ c_{15}=1.776345879.\}$

We will use variation of parameters method for solving the system of fifth-order boundary value problems (11) to have the following iterative scheme for finding the approximate solution of (6).

$$u_{k+1}(x)=\begin{cases}u_k(x)+\displaystyle\int_0^x \lambda(x,s)(0)\,ds, & -1\leq x<-\dfrac{1}{2},\\ u_k(x)+\displaystyle\int_0^x \lambda(x,s)(u_k-1)\,ds, & -\dfrac{1}{2}\leq x<\dfrac{1}{2},\\ u_k(x)+\displaystyle\int_0^x \lambda(x,s)(0)\,ds, & \dfrac{1}{2}\leq x\leq 1,\end{cases}$$

which implies, using the value of the parameter λ from (4),

$$u_{k+1}(x)=\begin{cases}u_k(x)+\displaystyle\int_0^x\left(\frac{x^4}{4!}-\frac{x^3s}{3!}+\frac{x^2s^2}{2!2!}-\frac{xs^3}{3!}+\frac{s^4}{4!}\right)(0)\,ds, & -1\leq x<-\dfrac{1}{2},\\ u_k(x)+\displaystyle\int_0^x\left(\frac{x^4}{4!}-\frac{x^3s}{3!}+\frac{x^2s^2}{2!2!}-\frac{xs^3}{3!}+\frac{s^4}{4!}\right)(u_k-1)\,ds, & -\dfrac{1}{2}\leq x<\dfrac{1}{2},\\ u_k(x)+\displaystyle\int_0^x\left(\frac{x^4}{4!}-\frac{x^3s}{3!}+\frac{x^2s^2}{2!2!}-\frac{xs^3}{3!}+\frac{s^4}{4!}\right)(0)\,ds, & \dfrac{1}{2}\leq x\leq 1,\end{cases}$$

Case I: $-1 \le x < -\frac{1}{2}$. In this case, we have the following approximate solution.

$$u_0 = c_1 \frac{x^4}{4!} + c_2 \frac{x^3}{3!} + c_3 \frac{x^2}{2!} + c_4 x + c_5,$$

$$u_1(x) = u_0(x) + \int_0^x \left(\frac{x^4}{4!} - \frac{x^3 s}{3!} + \frac{x^2 s^2}{2!2!} - \frac{x s^3}{3!} + \frac{s^4}{4!} \right)(0)\, ds,$$

$$= c_1 \frac{x^4}{4!} + c_2 \frac{x^3}{3!} + c_3 \frac{x^2}{2!} + c_4 x + c_5,$$

Case II: $-\frac{1}{2} \le x < \frac{1}{2}$. In this case, we have following approximations

$$u_0 = c_6 \frac{x^4}{4!} + c_7 \frac{x^3}{3!} + c_8 \frac{x^2}{2!} + c_9 x + c_{10},$$

$$u_1(x) = u_0(x) + \int_0^x \left(\frac{x^4}{4!} - \frac{x^3 s}{3!} + \frac{x^2 s^2}{2!2!} - \frac{x s^3}{3!} + \frac{s^4}{4!} \right)(u_0 - 1)\, ds,$$

$$= c_{10} + c_9 x + \frac{1}{2} c_8 x^2 + \frac{1}{6} c_7 x^3 + \frac{1}{24} c_6 x^4 + \left(\frac{1}{120} c_{10} - \frac{1}{120} \right) x^5 + \frac{1}{720} c_9 x^6$$
$$+ \frac{1}{5040} c_8 x^7 + \frac{1}{40320} c_7 x^8 + \frac{1}{362880} c_6 x^9,$$

Case III: $\frac{1}{2} \le x \le 1$. In this case, we have the following approximations:

$$u_0 = c_{11} \frac{x^4}{4!} + c_{12} \frac{x^3}{3!} + c_{13} \frac{x^2}{2!} + c_{14} x + c_{15},$$

$$u_1(x) = u_0(x) + \int_0^x \left(\frac{x^4}{4!} - \frac{x^3 s}{3!} + \frac{x^2 s^2}{2!2!} - \frac{x s^3}{3!} + \frac{s^4}{4!} \right)(0)\, ds,$$

$$= c_{11} \frac{x^4}{4!} + c_{12} \frac{x^3}{3!} + c_{13} \frac{x^2}{2!} + c_{14} x + c_{15},$$

Hence, we have the following series solution of (11)

$$u(x) = \begin{cases} c_1 \frac{x^4}{4!} + c_2 \frac{x^3}{3!} + c_3 \frac{x^2}{2!} + c_4 x + c_5, & -1 \le x < -\frac{1}{2}, \\ c_{10} + c_9 x + \frac{1}{2} c_8 x^2 + \frac{1}{6} c_7 x^3 + \frac{1}{24} c_6 x^4 + \left(\frac{1}{120} c_{10} - \frac{1}{120} \right) x^5 + \frac{1}{720} c_9 x^6 & \\ + \frac{1}{5040} c_8 x^7 + \frac{1}{40320} c_7 x^8 + \frac{1}{362880} c_6 x^9, & -\frac{1}{2} \le x < \frac{1}{2}, \\ c_{11} \frac{x^4}{4!} + c_{12} \frac{x^3}{3!} + c_{13} \frac{x^2}{2!} + c_{14} x + c_{15}, & \frac{1}{2} \le x \le 1. \end{cases} \tag{13}$$

By using boundary conditions and continuity conditions at $x = -\frac{1}{2}$ and $x = \frac{1}{2}$, we obtain a system of linear equations. Solving this system of linear equations, we find the values of the unknown constants as:

$$\begin{aligned} &c_1 = 42.4191965412754,\ c_2 = -.0799268221022021,\ c_3 = -7.11589313676993, \\ &c_4 = .00039186698950,\ c_5 = 1.77755077580429,\ c_6 = 42.6709500347331, \\ &c_7 = -.0336490093159189,\ c_8 = -7.10981062423008,\ c_9 = .0010186224376 9521, \\ &c_{10} = 1.77760404346774,\ c_{11} = 42.4191965412754,\ c_{12} = .171851588431368, \\ &c_{13} = -7.16667509872789,\ c_{14} = .0108832142996231,\ c_{15} = 1.77634588110595. \end{aligned} \tag{14}$$

By using values of unknowns from (13) into (14), we obtain the analytic solution of system of fifth-order boundary value problem associated with obstacle problem (11)

$$u(x) = \begin{cases} 1.77755077580429 + .000391866989503528\, x - 3.55794656838496\, x^2 & \\ -.0133211370170337\, x^3 + 1.76746652255314\, x^4, & -1 \le x < -\frac{1}{2}, \\ 1.77760404346774 + .00101862243769521\, x - 3.55490531211504\, x^2 & \\ -.00560816821931982 x^3 + 1.77795625144721\, x^4 + .00648003369556\, x^5 & \\ +1.41475338568 \times 10^{-6}\, x^6 - .00141067671115\, x^7 - 8.345488421596 \times 10^{-7} x^8 & \\ +1.17589699169698 \times 10^{-4} x^9, & -\frac{1}{2} \le x < \frac{1}{2}, \\ 1.77634588110595 + .0108832142996231\, x - 3.58333754936394\, x^2 & \\ +.0286419314052280\, x^3 + 1.76746652255314\, x^4, & \frac{1}{2} \le x \le 1. \end{cases}$$

Table 1 shows the comparison between the exact solution and approximate solution of system of fifth-order boundary value problem (12). Figure 1 represents the graphical representation of analytical solution of system of fifth-order boundary value problem (13) by using variation of parameters method.

CONCLUSION

In this paper, we used the variation of parameters method for solving system of fifth-order obstacle boundary value problem associated with obstacle problems. The results are compared with the exact solution and error estimates are calculated over the whole domain. The results are also represented graphically which demonstrate the nature of obstacle. We analyzed that our proposed method is well suited for such physical problems as it provides best solution in less number of iterations. Results proved in this paper may lead to further research in this field.

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Table 1. Comparison between the exact solution and approximate solution of system of fifth-order boundary value problem.

S/No.	Exact solution	Approximate solution	*Errors
-1	0	-0.000000000622900	6.229E-10
-0.8	0.230926188236807	0.230926193037040	4.800E-09
-0.6	0.728395917910566	0.728395922731540	4.821E-09
-0.4	1.254222277487480	1.254222281946720	4.459E-09
-0.2	1.638091646317750	1.638091633197580	1.312E-08
0	1.777604043467740	1.777604021000000	2.246E-08
0.2	1.638413475829980	1.638413456355570	1.947E-08
0.4	1.254447480141080	1.254447472015330	8.126E-09
0.6	0.728124610421116	0.728124614266080	3.844E-09
0.8	0.230335377469973	0.230335381040960	3.571E-09
1	0	-0.000000000150000	1.500E-10

*Errors=Exact solution-Approximate solution.

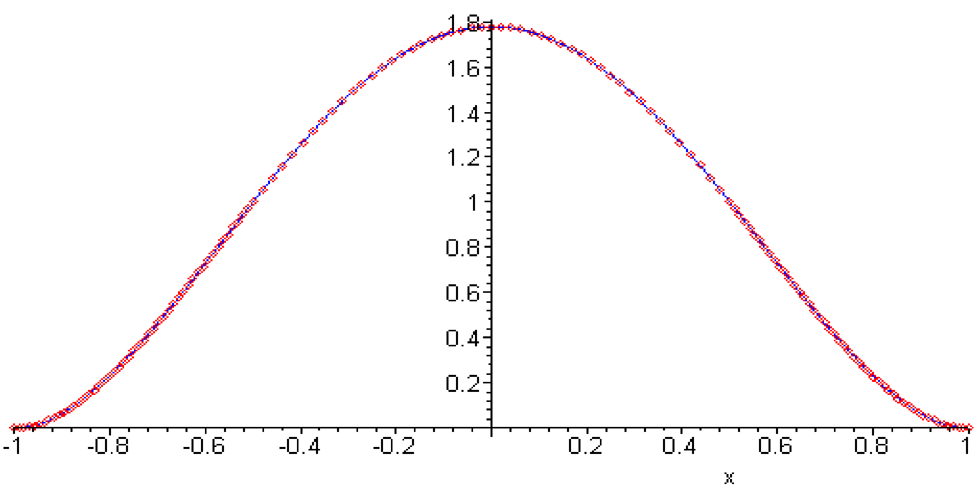

Figure 1. Represents the graphical representation of analytical solution of system of fifth-order boundary value problem by using variation of parameters method.

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