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Full Length Research Paper

On boundry value problems for random nonlinear differential equations

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This paper deals with the nonlinear boundary value problem including random effects. An approximate probability distribution function is constructed by numerical integrated of a set of related deterministic problems. Convergence of the approximate distribution function in the actual distribution function is established.

Key words: Random boundary value problem, distribution function, numerical integrated differential equation.

INTRODUCTION

Mathematical modeling of dynamic process lead us to a system of differential equation. In formulating the mathematical models, one can ignore the random uncertainties in the system and derive a deterministic model. Such a deterministic model of a dynamic process can be described by a system of deterministic differential equations.

The dynamic of the process will be described by a system of stochastic differential equations with random parameters. Therefore, one is interested in approximating a model by means of a deterministic model. Such an approximate in will lead us to study the estimations of the error response between the solutions and the solutions of the mean of a nonlinear boundary value problems with random parameters.

Very recently problems of this nature with regard to roots of random polynomials (Ladde and Sambandham, 1982), random initial value problem (1.4) have been investigated.

In this work, many areas of applications there have recently increasing interest mathematical models that include random effects, for example initial or bounded value problems for random differential equations. While there are powerful fairly general methods available for the treatment of certain types of random differential equations, these methods sometimes are difficult to apply to special problems.

An alternative approach involves a direct numerical construction of the used information about the solution of a random differential equation.

There are have been a many papers dealing with direct numerical method, for examples (Ladde et al., 2003; Bernfeld and Lakshmikantham, 1974; Deimling et al., 1985) with initial and boundary value problems for the n^{th} order linear equations.

In this paper, we will be and primarily concerned with boundary value problems for nonlinear second order equations.

Any direct numerical method involves the discretization of the random input problem, for example, by assuming that this input is described by number of random variables with known properties.

In other cases, a mathematical approximation is involved such as replacement of a stochastic process by a random polynomial or the truncation an appropriate series expansion.

In any case, we will be primarily concerned with the nonlinear two-point value problem.

$$\left.\begin{aligned} y'' &= f\left(t, y, y', w_1, w_2, \ldots\ldots\ldots, w_n\right) \\ \mu_1 y\left(t_o\right) &+ \mu_2 y'\left(t_o\right) = \mu_3 \\ \mu_4 y\left(t_1\right) &+ \mu_5 y'\left(t_1\right) = \mu_6 \end{aligned}\right\} \quad (1.1)$$

in the internal $\left[t_o, t_1\right]$ where $w_1, w_2, \ldots\ldots\ldots, w_n, \mu_1, \mu_2, \ldots\ldots\ldots, \mu_6$ are bounded continuous random variables.

Our main object is to provide a feasible algorithm for the computation of the marginal distribution function

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$$f_t(z)=P\{y(t)\le z\} \tag{1.2}$$

A numerical integration of a set of related deterministic problems we construct approximate distribution function $f_t^*(z)$. The case in which the problem involves two random variables in discussed in detail two methods are given the first with μ_2 and μ_5 as the only random variables and the second with μ_1 and μ_2 random.

In each case convergence of $f_t^*(z)$ and $f_1(z)$ is established in the sense that for any $\varepsilon>0$ one can insure that

$$\left|f_t(z)-f_t^*(z)\right|<\varepsilon \tag{1.3}$$

Eq. (1.1) are those necessary to insure that the solution $y(t)$ is a continuously differentiable function of those parameters.

Theorem 1-1:

Consider the boundary value problem

$$\begin{aligned} &y''=f(t,y,y'),\ t_o<t<t_1\\ &y(t_o)+Ry'(t_o)=\alpha\\ &y(t_1)+By'(t_1)=\beta \qquad \alpha,\beta,B \text{ are constants}\end{aligned} \tag{1.4}$$

and R is a bounded random variable taking on values in closed interval $I_R=[R_1,R_2]$ with probability distribution function $f_R(a), P(R\le r)$ for $r\in I_R$.

Assume that the problem (1.4) has a unique solution for each value in I_R. For a fixed $t\in[t_o,t_1]$ we want to determine a numerical approximation $f_t^*(z)$ to the probability distribution function, $f_t(z)=P(y(t)\le z)$. We want to determine $f_t''(z)$ so that given $\varepsilon>0$ we can insure that $\left|f_t^*(z)-f_t(z)\right|<\varepsilon$.

Let the set $\{r_o,\ldots\ldots,r_m\}$ be a partition of I_R with $r_i<r_{i+1},\ i=0,\ldots\ldots,m$ and $\delta r=\sup_i(r_{i+1},\ldots\ldots,r_m)$ using a method of order of accuracy at least P_i numerically that $m+1$ boundary value problems:

$$\begin{aligned} &y_i''=f(r,y_i,y_i') \qquad i=0,\ldots\ldots,m\\ &y_i(t_o)+r_iy_i'(t_o)=\alpha\\ &y_i(t_1)+B\,y_i'(t_1)=\beta\end{aligned} \tag{1.5}$$

On a set $\{t_j\}$ where $t_j=t_o+j\,h,\ i=1,\ldots\ldots,J$ with $h=(t_1-t_o)/J$. If the value calculated for $y_i(t_1)$, there exist constant C_i such that

$$\left|y_{ij}-y_i(t_1)\right|=C_ih^p \qquad i=0,1,\ldots\ldots,m \quad j=0,1,\ldots\ldots,J \tag{1.6}$$

Using linear interpolation we can approximate $y_i(t)$ between mesh points by

$$y_i^*(t)=y_{ij}+\frac{(t-t_j)}{(t_{j+1}-t_j)}\left(y_{i,j+1}-y_{ij}\right) \tag{1.7}$$
$$where\ \ t_j\le t\le t_{j+1}\ \&\ j=0,1,\ldots\ldots,J-1$$

It follows
$$\left|y_i^*(t)-y_i(t)\right|\le Ch^P+\frac{1}{2}h^2y \tag{1.8}$$

$$C=\max C_i\ ,\ y=\max y_i\ \ \&\ \ y_i=\sup_{t\in[a,b]}\left|y_i''(t)\right| \tag{1.9}$$

Hence A is the only random variable in (1.4), the probability attached to a solution $y_i(t)$ is the same as the probability corresponding to r_i. If for $[t_o,t_1], y(t,r)$ is r monotonically increasing function of r, then if for yi we have

$$f_i(z_i)=P(y(t)\le z_i)=f_R(r_i)=P(R\le r_i)$$

We will assume that $\frac{\partial y}{\partial r}>0$ and $\frac{\partial 2y}{\partial r^2}$ are continuous for $t\in[t_o,t_1]$ and $r\in I_R$. There are constants M_1 and M_2 such that:

$$0<m<\left|\frac{\partial y}{\partial r}(t,r)\right|\le M_1 \qquad \left|\frac{\partial^2 y}{\partial r^2}(1,r)\right|\le M_2 \qquad t\in[t_o,t_1],\ r\in I_R \tag{1.10}$$

Then, we define $f_t^*(z)$ to be

$$\begin{aligned} f_t^*(z)&=f_t(z_i)+\frac{(z-z_1^*)}{(z_{i+1}^*-z_i^*)}\left[f_R(z_{i+1})-f_R(z_i)\right]\\ &=f_A(a_i)+\frac{(z-z_1^*)}{(z_{i+1}^*-z_i^*)}\left[f_R(r_{i+1})-f_R(r_i)\right]\end{aligned} \tag{1.11}$$

Since z_i^* is a numerical approximation to $y_i(t)$ rather than the exact value. It actually corresponds not to a, but to some nearly value of r2 Using this value obtain the approximate distribution function $f_t^*(z)$ given by :

$$f_t^*(z)=f_t(z_i^*)+\frac{(z-z_i^*)}{(z_{i+1}^*-z_i^*)}\left[f_t(z_{i+1}^*)-f_t(z_i^*)\right] \quad (1.12)$$
$$=f_R(r_i^*)+\frac{(z-z_i^*)}{(z_{i+1}^*-z_i^*)}\left[f_R(r_{i+1}^*)-f_R(r_i^*)\right]$$

Then by the triangle inequality the error can be expressed as

$$\left|f_t(z)-f_t^*(z)\right|\le\left|f_t(z)-\bar f_t(z)\right|+\left|\bar f_t(z)-f_t^*(z)\right| \quad (1.13)$$

We will now construct an upper bound for each of the terms on the right side in (1.13). The first term is the error due to numerical integration and the second term is the error due to replacing *R* by a discrete random variable. For the first term, we have Newton's Interpolation formula that

$$f_r(z)=\bar f_t(z)+\frac{1}{2}(z-z_i^*)(z-z_{i+1}^*)f_t^*(\eta) \quad (1.14)$$

For some $\eta\in\left[z_i^*,z_{i+1}^*\right]$. Thus

$$\left|f_t(z)-\bar f_t(2)\right|\le\frac{1}{8}\left(z_{i+1}^*-z_i^*\right)^2\sup\left|f''(2)\right| \quad (1.15)$$

Using (1.8) we have that

$$\left|z_{i+1}^*-z_i^*\right|\le\left|z_{i+1}^*-z_{i+1}\right|+\left|z_{i+1}-z_i\right|+\left|z_i-z_i^*\right| \quad (1.16)$$
$$\le\left|z_{i+1}-z_i\right|+2\left(Ch^P+\frac{1}{2}h^2y\right)$$

Since $y\in C'(I_R)$, then

$$z_{i+1}-z_i=y(t,r_{i+1})-y(t,r_i)=\frac{\partial y}{\partial r}(t,r^*)(r_{i+1}-r_i) \quad (1.17)$$
$$r^*\in\left[r_i,r_{i+1}\right]$$

And it follows that

$$\left|f_t(z)-\bar f_t(2)\right|\le\frac{1}{8}\left[M_1\delta r+2\left(Ch^P+\frac{1}{2}h^2y\right)\right]^2\sup\left|f_t'(t)\right| \quad (1.18)$$

Hence, if sup $\left|f_t''(z)\right|$ is finite, and if δr and *h* are sufficiently small, then

$$\left|f_t(z)-\bar f_t(z)\right|<\frac{1}{2}\varepsilon \quad (1.19)$$

$$\left|f_t(z)-\bar f_t(z)\right|<\frac{1}{2}\varepsilon \quad (1.20)$$

For the second term on the right side of (1.13), we have that :

$$\left|f_t(z_i^*)-f_t(z_i)\right|=\left|f_t'(\bar z)(z_1^*-z_i)\right|$$
$$\le\sup_z\left|f_t'(z)\right|\left(Ch^P+\frac{1}{2}h^2y\right)$$
$$\bar z\in(z_1,z_i^*)$$

and thus

$$\left|\bar f_t(z)-f_t^*(z)\right|=\left[f_t(z_i^*)-f_t(z_i)\right]\frac{z_{i+1}^*-z}{z_{i+1}^*-z_i^*}$$
$$+\left[f_t(z_{i+1}^*)-f_t(z_{i+1})\right]\frac{z-z_i^*}{z_{i+1}^*-z_i^*}$$
$$\le\sup_z\left|f_t'(z)\right|\left(Ch^P+\frac{1}{2}h^2y\right)$$

it follows that if $\sup_z\left|f_t'(z)\right|$ is finite and if *h* is sufficiently small, then

$$\left|\bar f_t(z)-f_t^*(z)\right|<\frac{1}{2}\varepsilon \quad (1.21)$$

obtaining (1.21) and (1.18) with (1.12) gives derived the result.

It remains to show that $f_t'(z) and f_t''(z)$ are bounded. Since $\partial y(t,r)/\partial r$ f 0. This exists by the inverse function theorem, a well- defined differentiable inverse function 1/y.

Further, if $\eta=y(r)$, then $r=y^{-1}(\eta)$ and

$$\frac{dr}{d\eta}=\frac{dy^{-1}(\eta)}{d\eta}=\frac{1}{\partial y(t,r)/\partial r} \quad (1.22)$$

if $\bar r=y^{-1}(z)$, then

$$f_t(z)=P(r/y(t,r)\le z)$$
$$=P(r/r\le y^{-1}(z))$$
$$=f_R(y^{-1}(z))=f_R(r)$$

By thus by chain rule

$$f_t'(z)=f_R'(r)\frac{dr}{d\eta}(z)=\frac{f_R'(\bar r)}{\partial y(t,\bar r)/\partial r}$$

Consequently $\left|f_t'(z)\right|\le\sup_{I_R}f_R'(r)/m$

Similarly

$$f_t''(z)=\left[f_R''(\bar r)\frac{\partial y(t,\bar r)}{\partial r}-f_R'(\bar r)\frac{\partial^2 y(t,\bar a)}{\partial a^2}\right]\Big/\left(\frac{\partial y(t,\bar r)}{\partial r}\right)^2$$

and hence

$$\left|f_t''(z)\right|\le\left[\sup_{I_R}\left|f_R''(r)\right|M_1+\sup_{I_R}\left|f_R'(r)\right|\Big/M_2\right]\Big/m^3$$

This complete the proof of the following theorem.

Theorem 1: Let y(t) be the solutions of

$$y'' = f(x, y, y') \,\&\, t_0 \text{ p } t \text{ p } t_j$$
$$y(t_0) + Ry'(t_0) = \alpha$$
$$y(t_i) + By'(t_0) = B$$

Where α, B are constants and R is a bounded random variable taken on values in closed interval $I_R = [R_1, R_2]$ with distribution function $f_R(a) \in C^2(R_A)$. Assume that for each value of R, the deterministic problem corresponding to (1,4) has unique solution. Assume that $\partial y / \partial r \,\&\, \partial^2 y / \partial r^2$ are continuous for $t \in [t_0, t_1]$ and $r \in I_n$ and that $\partial y / \partial r \text{ f } 0$ there. Let $F_t(z), F^*_t(z), \Delta r, and \quad h$ as defined as above. Then for any $\in \text{f } 0$, it is choose Δr and h so small that $\left| f_t\ (z) - f_t^{\ *}(z)\ \right| \text{p} \in$

§ 2. For the case of two random boundary value problem: We will now extend in a more formal to the case of are two random boundary conditions.

We consider the following boundary value problem:

$$y'' = f(t, y, y'), \qquad y(t_o) + Ry'(t_o) = \alpha \qquad (2.1)$$
$$y(t_1) + \bar{R}y'(t_1) = \beta$$

Where $R, \bar{R}$ are independent random variables taking on values in the real internal $I_R = [R_1, R_2]$ and $I_{\bar{R}} = [\bar{R}_1, \bar{R}_2]$ respectively.

Suppose that $R, \bar{R}$ are obtained by the respective distribution function $g_R(r)$ and $g_{\bar{R}}(\bar{r})$ for $r \in I_R$, $\bar{r} \in I_{\bar{R}}$. We assume that the density function $f_R(r) \,\&\, f_{\bar{R}}(\bar{r})$ exist and (2.1) has a unique solution for each of the possible values of R and $\bar{R}$.

Using the conditional probability distribution of $y(t) = z$ given $\bar{R} = \bar{r}$, we have:

$$g_t(z) = \int_{I_{\bar{R}}} g_t(z \mid \bar{R} = \bar{r}) f_{\bar{R}}(\bar{r}) d\,\bar{r} \qquad (2.2)$$

Which insures that $\left| \bar{g}_t(z) - \underline{g}_t(z) \right| < \eta$

Proof :

Let $S = [R_1, R_2] \times [\bar{R}_1 \times \bar{R}_2]$. Define a partition of S by the sets

$$S_{ij} = \left\{ (r, \bar{r}) / r_i \le r < r_{i+1}, \bar{r}_j \le \bar{r} < \bar{r}_{j+1} \right\}$$

i=0, 1, …., M-1 & j=0, 1, …., J-1 for some integers *M* and *J-1* .

$R_1 = r_o$, $R_2 = r_M$, $\bar{R}_1 = \bar{r}_o$ and $\bar{R}_2 = \bar{r}_1$. Let $\delta a = \max_{0 \le i \le M-1} \left| r_{i+1} - r_i \right|$, $\delta \bar{r} = \max_{0 \le j \le J-1} \left| \bar{r}_{j+1} - r_j \right|$.

Let $y_{ij}(t)$ denote the solution of

$$y''_{ij} = f\ (t, y, y') \quad t_o < t < t_1$$
$$\mu_1 y\ (t_o) + \mu_2 y'(t_o) = \mu_2$$
$$\mu_4 y\ (t_1) + \mu_5 y'(t_1) = \mu_6$$

Where $f_{ij}(t, y, y') = f\ (t, y, y', r_i, \bar{r}_i)$ for each *i* and *j*. Use their solutions and $g_{R\bar{R}}(r, \bar{r})$ to construct upper and lower bounds, $\bar{g}_t(z)$ for $g_t(z)$. We will show that for any $\eta > 0$, we can find δr and $\delta \bar{r}$ sufficiently small such that $\left| \bar{g}_t(z) - \underline{g}(z) \right| < \eta$.

Suppose that $y(t, r, \bar{r})$ is strictly increasing as a function of r, and the partition points of $I_{\bar{R}}$ have been chosen so that for any given interval $[\bar{r}_i, \bar{r}_{i+1}]$ we know that either $\frac{\partial y}{\partial F} \ge 0$ or $\frac{\partial y}{\partial \bar{r}} \le 0$ for all $(r, \bar{r}) \in S_{ij}$.

And suppose that $y_{i+1,j}(t) \le z$, $y_{i+1,j+1}(t) \le z$

Then $y(t, r, \bar{r}) \le z$ for all $(r, \bar{r}) \in S_{ij}$

Thus we can define upper and lower bounds for $g_t(z)$ as follows:

$$\bar{g}_t(z) = 1 - \sum_{i,j} P(S_{ij})$$

Such that $y_{ij}(t) > z$ and $y_{i,j+1}(t) > z$

$$\underline{g}_t(z) = \sum_{i,j} P(S_{ij}) \qquad (2.2)$$

Then $\underline{g}_t(z) \le \underline{g}_t(z) \le \bar{f}_t(z) \qquad (2.3)$

For each $\bar{r} \in [\bar{R}_1, \bar{R}_2]$, let $\bar{r}(r)$ be defined as the number $r \in [R_1, R_2]$ such that

$$\bar{g}_t(z)=\int_{\bar{R}_1}^{\bar{R}_2}\int_{R_1}^{r(\bar{r})}\frac{\partial^2 g_{R\bar{R}}(r,\bar{r})}{\partial r\,\partial \bar{r}}dr\,d\,\bar{r} \qquad (2.4)$$

Then for any $j\in(0,1,.....,J)$ there exists $m\in(0,1,.....,M)$ such that $P(S_{ij})$ in $\bar{g}_t(z)$ for all $i\geq j$.

For $\bar{r}\in(\bar{r}_j,r_{j+1})$ define $\bar{r}(r)=r_{ij}$. Thus $\bar{r}(r)$ is defined for all in $[\bar{R}_1,\bar{R}_2]$.

If we define $\underline{r}(r)=r_{ij}$, we can write

$$\underline{g}_t(z)=\int_{y_1}^{y_2}\int_{R_1}^{\underline{r}(r)}\frac{\partial^2 g_{R\bar{R}}(r,\bar{r})}{\partial r\,\partial \bar{r}}dr\,d\,\bar{r}$$

If we assume $\left|\frac{\partial^2 g_{R\bar{R}}(r,\bar{r})}{\partial r\,\partial \bar{r}}\right|\leq 0<\infty$, then we have that $\left|\bar{g}_t(z)-\underline{g}_t(z)\right|\leq(\bar{R}_2-\bar{R}_1)CK\,\delta r$ with K is an integer.
It is possible to show that for a given δr, the partition can be defined so that δr to zero.
We note for $k'=2$ it follows that
$\left|\bar{g}_t(z)-\underline{g}_t(z)\right|\leq 2(\bar{R}_2-\bar{R}_1)C\,\delta r$ Which completes the proof of the following theorem

Theorem 2: Let y(t) be the solution of (2.1) and $\partial y/\partial r\,\&\,\partial^2 f(r,\bar{r})/\partial r\partial\bar{r}\leq C$ for $r\in I_R\,\&\,\bar{r}\in I_{\bar{R}}$, Then there exist a partition which insures

$$\left|\overline{g_t}(z)-g_t(z)\text{p}\right|\eta$$

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