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# The modified (G'/G)-expansion method for the (1+1) Hirota-Ramani and (2+1) breaking soliton equation

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**In this article, we apply the modified (G'/G)-expansion method to construct hyperbolic, trigonometric and rational function solutions of nonlinear evolution equations. This method can be thought of as the generalization of the (G'/G)-expansion method given recently by Wang et al. (2008). To illustrate the validity and advantages of this method, the (1+1)-dimensional Hirota-Ramani equation and the (2+1)-dimensional breaking soliton equation are considered and more general traveling wave solutions are obtained. It is shown that the proposed method provides a more general powerful mathematical tool for solving nonlinear evolution equations in mathematical physics.**

**Key words:** Nonlinear evolution equations, modified (G'/G)-expansion method, hyperbolic Function solutions, trigonometric function solutions, rational function solutions.

## INTRODUCTION

Nonlinear evolution equations are often presented to describe the motion of isolated waves, localized in a small part of space, in many fields such as hydrodynamics, plasma physics, and nonlinear optics. Seeking exact solutions of these equations plays an important role in the study of these nonlinear physical phenomena. In the past several decades, many effective methods for obtaining exact solutions of these equations have been presented, such as the inverse scattering method (Ablowitz and Clarkson, 1991), Hirota bilinear method (Hirota, 1980), Backlund transformation (Hirota, 1980; Miura, 1978), Painleve expansion (Hearn et al., 2012; Kudryashov, 1988; Kudryashov, 1990; Kudryashov, 1991; Russo et al., 2012; Weiss et al., 1983), Sine-Cosine method (Wazwaz, 2004; Zayed and Abdelaziz, 2011), Jacobi elliptic function method (Lu, 2005; Liu et al., 2001), Tanh-function method (Fan, 2000; Yusufoglu and Bekir, 2008; Zayed and Abdel Rahman, 2010; Zhang and Xia, 2008), F-expansion method (Zhang and Xia, 2006), Exp-function method (Bekir, 2010; Bekir,

2009; He and Wu, 2006), (G'/G)-expansion method (Bekir, 2008; Islan, 2010; Kudryashov, 2010; Kudryashov, 2009; Ma et al., 2011; Peng, 2008, 2009; Reza and Rasoul, 2011; Wang et al., 2008; Zayed, 2009; Zayed et al., 2011; Zayed and Al-Joudi, 2010; Zayed and Al-Joudi, 2009; Zhang et al., 2011), the modified (G'/G)-expansion method (Ma et al., 2011; Reza and Rasoul, 2011; Wang et al., 2008) and so on. Wang et al. (2008) introduced the (G'/G)-expansion method to look for traveling wave solutions of nonlinear evolution equations. This method is based on the assumption that these solutions can be expressed by a polynomial in (G'/G), and that  $G = G(\xi)$  satisfies a second order linear ordinary differential equation (ODE).

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \quad (1)$$

Where  $\lambda$ ,  $\mu$  are constants and  $' = d/d\xi$ , while  $\xi = kx + \omega t$ , and  $k$ ,  $\omega$  are constants. The degree of this polynomial can be determined by considering the homogeneous balance between the highest-order derivatives and nonlinear terms appearing in the given nonlinear evolution equations. The coefficients of this polynomial can be obtained by solving a set of algebraic

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equations resulted from the process of using the method. The present paper is motivated by the desire to propose a modified (G'/G)-expansion method for constructing more general exact solutions of nonlinear evolution equation. To illustrate the validity and advantages of the proposed method, we would like to employ it to solve the (1+1)-dimensional Hirota-Ramani equation (Hirota and Ramani, 1980; Reza and Rasoul, 2011) and the (2+1)-dimensional breaking soliton equation (Zayed et al., 2011; Zayed and Al-Joudi, 2009).

**DESCRIPTION OF THE MODIFIED (G'/G)-EXPANSION METHOD**

A given nonlinear evolution equation is in the form

$$P(u, u_x, u_y, u_t, u_{xx}, u_{xt}, \dots), \tag{2}$$

Where  $u(x, y, t)$ , we use the wave transformation

$$u(x, y, t) = u(\xi), \quad \xi = k_1x + k_2y + \omega t, \tag{3}$$

Where  $k_1, k_2, \omega$  are constants, then Equation 1 is reduced into the ODE

$$Q(u^{(r)}, u^{(r+1)}, \dots) = 0, \tag{4}$$

Where  $u^{(r)} = \frac{d^r u}{d\xi^r}, r \geq 0$  and  $r$  is the least order of

derivatives in the equation. Setting  $u^{(r)} = V(\xi)$ , Where  $V(\xi)$  is a new function of  $\xi$ , we further introduce the following anstaz:

$$u^{(r)}(\xi) = V(\xi) = \sum_{i=0}^m \alpha_i \left(\frac{G'}{G}\right)^i, \quad \alpha_m \neq 0, \tag{5}$$

Where  $G = G(\xi)$  satisfies Equation 1, while  $\alpha_i (i = 0, 1, \dots, m)$  are constants to be determined later.

To determine  $u(\xi)$  explicitly, we take the following four steps (Zayed and Al-Joudi, 2010; Zayed and Al-Joudi, 2009; Zhang et al., 2011):

Step 1: Determine the positive integer  $m$  in Equation 5 by balancing the highest-order nonlinear terms and the highest-order derivatives in Equation 4.

Step 2: Substitute Equation 5 along with Equation 1 into Equation 4 and collect all terms with the same powers of

$$\left(\frac{G'}{G}\right)^i, \quad i = (0, 1, \dots, m) \text{ together; thus, the left-hand side}$$

of Equation 4 is converted into a polynomial in  $\left(\frac{G'}{G}\right)^i$ .

Then set each coefficient of this polynomial to zero, to derive a set of algebraic for  $\alpha_i, k_1, k_2, \omega, i = (0, 1, \dots, m)$ .

Step 3: Solve these algebraic equations by the use of Mathematica to find the values of  $\alpha_i, k_1, k_2, \omega,$

$$i = (0, 1, \dots, m).$$

Step 4: Use the results obtained in above steps to derive a series of fundamental solutions  $V(\xi)$  of Equation 4

depending on  $\left(\frac{G'}{G}\right)$ , since the solutions of Equation 1

have been well known for us as follows:

(i) If  $\lambda^2 - 4\mu > 0$ , then

$$\left(\frac{G'}{G}\right) = \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \left[ \frac{c_1 \sinh\left(\frac{\xi}{2} \sqrt{\lambda^2 - 4\mu}\right) + c_2 \cosh\left(\frac{\xi}{2} \sqrt{\lambda^2 - 4\mu}\right)}{c_1 \cosh\left(\frac{\xi}{2} \sqrt{\lambda^2 - 4\mu}\right) + c_2 \sinh\left(\frac{\xi}{2} \sqrt{\lambda^2 - 4\mu}\right)} \right] - \frac{\lambda}{2}. \tag{6}$$

(ii) If  $\lambda^2 - 4\mu < 0$ , then

$$\left(\frac{G'}{G}\right) = \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \left[ \frac{-c_1 \sin\left(\frac{\xi}{2} \sqrt{\lambda^2 - 4\mu}\right) + c_2 \cos\left(\frac{\xi}{2} \sqrt{\lambda^2 - 4\mu}\right)}{c_1 \cos\left(\frac{\xi}{2} \sqrt{\lambda^2 - 4\mu}\right) + c_2 \sin\left(\frac{\xi}{2} \sqrt{\lambda^2 - 4\mu}\right)} \right] - \frac{\lambda}{2}. \tag{7}$$

(iii) If  $\lambda^2 - 4\mu = 0$ , then

$$\left(\frac{G'}{G}\right) = \frac{c_2}{c_1 + c_2 \xi} - \frac{\lambda}{2}. \tag{8}$$

Where  $c_1$  and  $c_2$  are constants, we can obtain exact solutions of Equation 2 by integrating each of the obtained fundamental solutions  $V(\xi)$  with respect to  $\xi$  and  $r$  times as follows:

$$u(\xi) = \int \int \dots \int V(\xi) d\xi_1 \dots d\xi_{r-1} d\xi_r + \sum_{j=1}^r d_j \xi^{r-j}, \tag{9}$$

Where  $d_j (j = 1, 2, \dots, r)$  are constants

**Remark 1**

It can easily be found that when  $r = 0$ ,  $u(\xi) = V(\xi)$  then Equation 5 becomes the anstaz solutions obtained

in Wang et al. (2008). When  $r \geq 1$ , the solution of Equation 9 can be found in Zhang et al. (2011) and cannot be obtained by the methods in Wang et al. (2008).

**APPLICATIONS**

Here, we used the modified (G'/G)-expansion method to find the exact solutions of the following nonlinear partial differential equations (PDEs):

**Example 1: Nonlinear Hirota-Ramani equation**

Here, we used the proposed method previously used in the work, to find the solutions to Hirota-Ramani equation (Hirota and Ramani, 1980; Reza and Rasoul, 2011):

$$u_t - u_{xxt} + \alpha u_x (1 - u_t) = 0, \tag{10}$$

Where  $\alpha \neq 0$  is a constant. To this end, we use the wave transformation

$$u(x, t) = u(\xi), \quad \xi = kx + \alpha t, \tag{11}$$

Where  $k, \omega$  are constants, to reduce Equation 10 to the following ODE:

$$(\omega + \alpha k)u' - k^2 \alpha u''' - \alpha k \alpha u'^2 = 0. \tag{12}$$

Setting  $r = 1$  and  $u' = V$ , we have  $u(\xi) = \int V(\xi) d\xi + d_1$ , where  $V(\xi)$  satisfies the equation

$$(\omega + \alpha k)V - k^2 \alpha V'' - \alpha k \alpha V^2 = 0. \tag{13}$$

According to Step 1, we get  $m + 2 = 2m$ , and hence  $m = 2$ . We then suppose that Equation 13 has the formal solution

$$V(\xi) = \alpha_2 \left(\frac{G'}{G}\right)^2 + \alpha_1 \left(\frac{G'}{G}\right) + \alpha_0, \tag{14}$$

It is easy to see that

$$V''(\xi) = 6\alpha_2 \left(\frac{G'}{G}\right)^4 + (10\lambda\alpha_2 + 2\alpha_1) \left(\frac{G'}{G}\right)^3 + (8\mu\alpha_2 + 4\lambda^2\alpha_2 + 3\lambda\alpha_1) \left(\frac{G'}{G}\right)^2 + (6\lambda\mu\alpha_2 + 2\mu\alpha_1 + \lambda^2\alpha_1) \left(\frac{G'}{G}\right) + 2\mu^2\alpha_2 + \lambda\mu\alpha_1. \tag{15}$$

$$V^2(\xi) = \alpha_2^2 \left(\frac{G'}{G}\right)^4 + 2\alpha_1\alpha_2 \left(\frac{G'}{G}\right)^3 + (\alpha_1^2 + 2\alpha_0\alpha_2) \left(\frac{G'}{G}\right)^2 + 2\alpha_0\alpha_1 \left(\frac{G'}{G}\right) + \alpha_0^2. \tag{16}$$

Substituting Equation 14 to 16 into Equation 13 and collecting all terms with the same powers of  $\left(\frac{G'}{G}\right)$  together, the left-hand side of Equation 13 is converted into a polynomial in  $\left(\frac{G'}{G}\right)$ . Setting each coefficient of this polynomial to zero, we get the following algebraic equations:

$$0: \alpha_0(\omega + \alpha k) - k^2 \omega(2\mu^2\alpha_2 + 2\lambda\mu\alpha_1) - \alpha k \omega \alpha_0^2 = 0, \tag{17}$$

$$1: \alpha_1(\omega + \alpha k) - k^2 \omega(6\lambda\mu\alpha_2 + 2\mu\alpha_1 + \lambda^2\alpha_1) - 2\alpha_0\alpha_1\alpha k \omega = 0, \tag{18}$$

$$2: \alpha_2(\omega + \alpha k) - k^2 \omega(8\mu\alpha_2 + 4\lambda^2\alpha_2 + 3\lambda\alpha_1) - \alpha k \omega(\alpha_1^2 + 2\alpha_0\alpha_2) = 0, \tag{19}$$

$$3: -k^2 \omega(10\lambda\alpha_2 + 2\alpha_1) - 2\alpha_1\alpha_2\alpha k \omega = 0, \tag{20}$$

$$4: -6\alpha_2 k^2 \omega - \alpha_2^2 \alpha k \omega = 0. \tag{21}$$

On solving the algebraic Equations 17 to 21, we have the results:

$$\alpha_2 = \frac{-6k}{\alpha}, \quad \alpha_1 = \frac{-6k\lambda}{\alpha}, \quad \alpha_0 = \frac{-6k\mu}{\alpha}, \quad \omega = \frac{-\alpha k}{1 - k^2(\lambda^2 - 4\mu)}, \tag{22}$$

Where  $k^2(\lambda^2 - 4\mu) \neq 1$ . Consequently, we deduce the following exact solutions of Equation 10:

(i) If  $\lambda^2 - 4\mu > 0$  (Hyperbolic solutions)

When  $\lambda^2 - 4\mu > 0$ , we set  $\phi = \frac{\xi}{2} \sqrt{\lambda^2 - 4\mu}$ . Then we get

$$u(\xi) = \frac{-3k(\lambda^2 - 4\mu)}{2\alpha} \int \left[ \frac{c_1 \sinh \phi + c_2 \cosh \phi}{c_1 \cosh \phi + c_2 \sinh \phi} \right]^2 d\xi + \frac{3k(\lambda^2 - 4\mu)}{2\alpha} \xi + d_1. \tag{23}$$

Substituting Equations 8, 10, 12 and 14 obtained in Peng (2008, 2009) into Equation 23, we have respectively the following kink-type traveling wave solutions:

(1) If  $|c_1| > |c_2|$ , then

$$u(\xi) = \frac{-3k(\lambda^2 - 4\mu)}{2\alpha} \int \tanh^2[\phi + \text{sgn}(c_1 c_2) \psi_1] d\xi + \frac{3k(\lambda^2 - 4\mu)}{2\alpha} \xi + d_1 = \frac{3k\sqrt{\lambda^2 - 4\mu}}{\alpha} \tanh[\phi + \text{sgn}(c_1 c_2) \psi_1] + d_1. \tag{24}$$

(2) If  $|c_1| > |c_2| \neq 0$ , then

$$u(\xi) = \frac{-3k(\lambda^2 - 4\mu)}{2\alpha} \int \coth^2[\phi + \operatorname{sgn}(c_1 c_2) \psi_2] d\xi + \frac{3k(\lambda^2 - 4\mu)}{2\alpha} \xi + d_1$$

$$= \frac{3k\sqrt{\lambda^2 - 4\mu}}{\alpha} \coth[\phi + \operatorname{sgn}(c_1 c_2) \psi_2] + d_1. \tag{25}$$

(3) If  $|c_1| > |c_2| = 0$ , then

$$u(\xi) = \frac{-3k(\lambda^2 - 4\mu)}{2\alpha} \int \coth^2 \phi d\xi + \frac{3k(\lambda^2 - 4\mu)}{2\alpha} \xi + d_1$$

$$= \frac{3k\sqrt{\lambda^2 - 4\mu}}{\alpha} \coth \phi + d_1. \tag{26}$$

(4) If  $|c_1| = |c_2|$ , then

$$u(\xi) = d_1, \tag{27}$$

Where  $\psi_1 = \tanh^{-1}(|c_2|/|c_1|)$ ,  $\psi_2 = \tanh^{-1}(|c_1|/|c_2|)$  and  $\operatorname{sgn}(c_1 c_2)$  is the sign function, while  $\xi$  is given by:

$$\xi = k \left[ x - \frac{\alpha t}{1 - k^2(\lambda^2 - 4\mu)} \right].$$

(ii) If  $\lambda^2 - 4\mu < 0$  (Trigonometric solutions)

In this case, we have

$$u(\xi) = \frac{-3k(\lambda^2 - 4\mu)}{2\alpha} \int \frac{\left[ -c_1 \sin\left(\frac{\xi}{2} \sqrt{4\mu - \lambda^2}\right) + c_2 \cos\left(\frac{\xi}{2} \sqrt{4\mu - \lambda^2}\right) \right]^2}{\left[ c_1 \cos\left(\frac{\xi}{2} \sqrt{4\mu - \lambda^2}\right) + c_2 \sin\left(\frac{\xi}{2} \sqrt{4\mu - \lambda^2}\right) \right]^2} d\xi$$

$$- \frac{3k(4\mu - \lambda^2)}{2\alpha} \xi + d_1. \tag{28}$$

We now simplify Equation 28 to get the following periodic solutions:

$$(1) u(\xi) = \frac{6k\sqrt{4\mu - \lambda^2}}{2\alpha} \tan \left[ \xi_1 - \frac{1}{2} \xi \sqrt{4\mu - \lambda^2} \right] + d_1, \tag{29}$$

Where  $\xi_1 = \tan^{-1}\left(\frac{c_2}{c_1}\right)$  and  $c_1^2 + c_2^2 \neq 0$ .

$$(2) u(\xi) = \frac{6k\sqrt{4\mu - \lambda^2}}{2\alpha} \cot \left[ \xi_2 + \frac{1}{2} \xi \sqrt{4\mu - \lambda^2} \right] + d_1, \tag{30}$$

Where  $\xi_2 = \cot^{-1}\left(\frac{c_2}{c_1}\right)$  and  $c_1^2 + c_2^2 \neq 0$ .

(iii) If  $\lambda^2 - 4\mu = 0$  (Rational function solutions)

In this case, we have

$$u(\xi) = \frac{-6k}{\alpha} \int \left[ \frac{c_2}{c_1 + c_2 \xi} \right]^2 d\xi + d_1 = \frac{6kc_2}{\alpha[c_1 + c_2 k(x - \alpha)]} + d_1. \tag{31}$$

**Remark 2**

If we multiply Equation 13 by  $V'(\xi)$  and integrate with zero constant of integration, we deduce that

$$V'^2(\xi) = a_1 V^2(\xi) + b_1 V^3(\xi), \tag{32}$$

Where  $a_1 = \frac{\omega + \alpha k}{\omega k^2}$ ,  $b_1 = \frac{-2\alpha}{3k}$ .

On solving Equation 32 we have the two cases:

(i) If  $\frac{\omega + \alpha k}{\omega} > 0$ ,

$$V(\xi) = \frac{3(\omega + \alpha k)}{2\alpha k \omega} \operatorname{sech}^2 \left( \frac{\xi}{2k} \sqrt{\frac{\omega + \alpha k}{\omega}} \right), \tag{33}$$

$$V(\xi) = \frac{-3(\omega + \alpha k)}{2\alpha k \omega} \operatorname{csc}^2 \left( \frac{\xi}{2k} \sqrt{\frac{\omega + \alpha k}{\omega}} \right). \tag{34}$$

Integrating Equations 33 and 34, we have the solutions of Equation 10 in the forms:

$$u(\xi) = \frac{3}{\alpha} \sqrt{\frac{\omega + \alpha k}{\omega}} \tanh \left( \frac{\xi}{2k} \sqrt{\frac{\omega + \alpha k}{\omega}} \right) + d_1, \tag{35}$$

$$u(\xi) = \frac{3}{\alpha} \sqrt{\frac{\omega + \alpha k}{\omega}} \coth \left( \frac{\xi}{2k} \sqrt{\frac{\omega + \alpha k}{\omega}} \right) + d_1. \tag{36}$$

Substituting  $\omega = \frac{-\alpha k}{1 - k^2(\lambda^2 - 4\mu)}$ , into Equations 35

and 36, we arrive at the same solutions of Equations 24 and 25 or 26, respectively.

(ii)  $\frac{\omega + \alpha k}{\omega} < 0$ ,

$$V(\xi) = \frac{3(\omega + \alpha k)}{2\alpha k \omega} \sec^2 \left( \frac{\xi}{2k} \sqrt{-\left(\frac{\omega + \alpha k}{\omega}\right)} \right), \quad (37)$$

$$V(\xi) = \frac{3(\omega + \alpha k)}{2\alpha k \omega} \csc^2 \left( \frac{\xi}{2k} \sqrt{-\left(\frac{\omega + \alpha k}{\omega}\right)} \right). \quad (38)$$

Integrating Equations 37 and 38, we have the solutions of Equation 10 in the forms:

$$u(\xi) = \frac{-3}{\alpha} \sqrt{-\left(\frac{\omega + \alpha k}{\omega}\right)} \tan \left( \frac{\xi}{2k} \sqrt{-\left(\frac{\omega + \alpha k}{\omega}\right)} \right) + d_1, \quad (39)$$

$$u(\xi) = \frac{3}{\alpha} \sqrt{-\left(\frac{\omega + \alpha k}{\omega}\right)} \cot \left( \frac{\xi}{2k} \sqrt{-\left(\frac{\omega + \alpha k}{\omega}\right)} \right) + d_1, \quad (40)$$

Substituting  $\omega = \frac{-\alpha k}{1 - k^2(\lambda^2 - 4\mu)}$ , into Equations 39

and 40, we arrive at the same solutions of Equation 29 and 30, respectively.

### Example 2: Nonlinear breaking soliton equation

Here, we used the proposed method previously used in the work, to find the solutions of the breaking soliton equation (Zayed et al., 2011; Zayed and Al-Joudi, 2010; Zayed and Al-Joudi, 2009):

$$u_{xt} - 4u_x u_{xy} - 2u_{xx} u_y + u_{xxx} = 0. \quad (41)$$

To this end, we use the wave transformation

$$u(x, y, t) = u(\xi), \quad \xi = k_1 x + k_2 y + \omega t, \quad (42)$$

Where  $k_1, k_2$  and  $\omega$  are constants, to reduce Equation 41 to the following ODE:

$$\omega u'' - 6k_1 k_2 u' u'' + k_1^2 k_2 u''' = 0. \quad (43)$$

Integrating Equation 43 once with respect to  $\xi$ , with zero constant of integration, we get

$$\omega u' - 3k_1 k_2 u'^2 + k_1^2 k_2 u''' = 0. \quad (44)$$

Setting  $r = 1$ , and  $u' = V$ , we deduce that  $V(\xi)$  satisfies the equation:

$$\omega V - 3k_1 k_2 V^2 + k_1^2 k_2 V''' = 0. \quad (45)$$

According to Step 1, we get  $m = 2$ . Thus, the formal solution of Equation 45 has the same form of Equation 14.

Substituting Equations 14 to 16 into Equation 45 and collecting the coefficients of  $\left(\frac{G'}{G}\right)^j$   $j = 0, 1, 2, 3, 4$ .

Setting each coefficient to zero, we get the following algebraic equations:

$$0: \omega \alpha_0 - 3k_1 k_2 \alpha_0^2 + k_1^2 k_2 (2\mu^2 \alpha_2 + \lambda \mu \alpha_1) = 0, \quad (46)$$

$$1: \omega \alpha_1 - 6k_1 k_2 \alpha_0 \alpha_1 - k_1^2 k_2 (6\lambda \mu \alpha_2 + 2\mu \alpha_1 + \lambda^2 \alpha_1) = 0, \quad (47)$$

$$2: \omega \alpha_2 - 3k_1 k_2 (\alpha_1^2 + 2\alpha_0 \alpha_2) + k_1^2 k_2 (8\mu \alpha_2 + 4\lambda^2 \alpha_2 + 3\lambda \alpha_1) = 0, \quad (48)$$

$$3: -6k_1 k_2 \alpha_1 \alpha_2 + k_1^2 k_2 (10\lambda \alpha_2 + 2\alpha_1) = 0, \quad (49)$$

$$4: -3k_1 k_2 \alpha_2^2 + 6\alpha_2 k_1^2 k_2 = 0. \quad (50)$$

On solving the algebraic Equations 46 to 50, we have the following results:

$$\alpha_2 = 2k_1, \quad \alpha_1 = 2k_1 \lambda, \quad \alpha_0 = 2k_1 \mu, \quad \omega = -k_1^2 k_2 (\lambda^2 - 4\mu). \quad (51)$$

Consequently, we deduce the following exact solutions of Equation 41:

(i) If  $\lambda^2 - 4\mu > 0$  (Hyperbolic solutions)

Setting  $\phi = \frac{\xi}{2} \sqrt{\lambda^2 - 4\mu}$ . Then we get

$$u(\xi) = \frac{k_1}{2} (\lambda^2 - 4\mu) \int \left[ \frac{c_1 \sinh \phi + c_2 \cosh \phi}{c_1 \cosh \phi + c_2 \sinh \phi} \right]^2 d\xi - \frac{k_1}{2} (\lambda^2 - 4\mu) \xi + d_1. \quad (52)$$

Substituting the results of Equations 8, 10, 12 and 14 of Peng (2008, 2009) into Equation 52, we have respectively, the following Kink-type traveling wave solutions:

(1) If  $|c_1| > |c_2|$ , then

$$u(\xi) = -k_1 \sqrt{\lambda^2 - 4\mu} \tanh[\phi + \text{sgn}(c_1 c_2) \psi_1] + d_1 \quad (53)$$

(2) If  $|c_1| > |c_2| \neq 0$ , then

$$u(\xi) = -k_1 \sqrt{\lambda^2 - 4\mu} \coth[\phi + \text{sgn}(c_1 c_2) \psi_2] + d_1. \quad (54)$$

(3) If  $|c_1| > |c_2| = 0$ , then

$$u(\xi) = -k\sqrt{\lambda^2 - 4\mu} \coth \phi + d_1. \tag{55}$$

(4) If  $|c_1| = |c_2|$ , then

$$u(\xi) = d_1, \tag{56}$$

Where  $\xi = k_1x + k_2y - k_1^2k_2(\lambda^2 - 4\mu)t$ .

(ii) If  $\lambda^2 - 4\mu < 0$  (Trigonometric solutions)

In this case, we have

$$u(\xi) = \frac{k_1(4\mu - \lambda^2)}{2} \int \left[ \frac{-c_1 \sin\left(\frac{\xi}{2}\sqrt{4\mu - \lambda^2}\right) + c_2 \cos\left(\frac{\xi}{2}\sqrt{4\mu - \lambda^2}\right)}{c_1 \cos\left(\frac{\xi}{2}\sqrt{4\mu - \lambda^2}\right) + c_2 \sin\left(\frac{\xi}{2}\sqrt{4\mu - \lambda^2}\right)} \right] d\xi - \frac{k_1}{2}(4\mu - \lambda^2)\xi + d_1. \tag{57}$$

We now simplify Equation 57 to get the following periodic solutions:

$$(1) u(\xi) = -k_1\sqrt{4\mu - \lambda^2} \tan\left[\xi_1 - \frac{1}{2}\xi\sqrt{4\mu - \lambda^2}\right] + d_1, \tag{58}$$

Where  $\xi_1 = \tan^{-1}\left(\frac{c_2}{c_1}\right)$  and  $c_1^2 + c_2^2 \neq 0$ .

$$(2) u(\xi) = -k_1\sqrt{4\mu - \lambda^2} \cot\left[\xi_2 + \frac{1}{2}\xi\sqrt{4\mu - \lambda^2}\right] + d_1, \tag{59}$$

Where  $\xi_2 = \cot^{-1}\left(\frac{c_2}{c_1}\right)$  and  $c_1^2 + c_2^2 \neq 0$ .

(iii) If  $\lambda^2 - 4\mu = 0$  (Rational function solutions)

In this case, we have

$$u(\xi) = 2k_1 \int \left[ \frac{c_2}{c_1 + c_2\xi} \right]^2 d\xi + d_1 = \frac{2k_1c_2}{c_1 + c_2(k_1x + k_2y)} + d_1. \tag{60}$$

**Remark 3**

If we multiply Equation 45 by  $V'(\xi)$  and integrate with zero constant of integration, we deduce that

$$V'^2(\xi) = a_2V^2(\xi) + b_2V^3(\xi), \tag{61}$$

Where  $a_2 = \frac{-\omega}{k_1^2k_2}$ ,  $b_2 = \frac{2}{k_1}$ .

On solving Equation 61 we have the two cases:

(i) If  $\frac{\omega}{k_2} < 0$ ,

$$V(\xi) = \frac{\omega}{2k_1k_2} \sec h^2\left(\frac{\xi}{2k_1}\sqrt{\frac{-\omega}{k_2}}\right), \tag{62}$$

$$V(\xi) = \frac{-\omega}{2k_1k_2} \csc h^2\left(\frac{\xi}{2k_1}\sqrt{\frac{-\omega}{k_2}}\right). \tag{63}$$

Integrating Equations 62 and 63, we have the solutions of Equation 41 in the forms:

$$u(\xi) = -\sqrt{\frac{-\omega}{k_2}} \tanh\left(\frac{\xi}{2k_1}\sqrt{\frac{-\omega}{k_2}}\right) + d_1, \tag{64}$$

$$u(\xi) = -\sqrt{\frac{-\omega}{k_2}} \coth\left(\frac{\xi}{2k_1}\sqrt{\frac{-\omega}{k_2}}\right) + d_1. \tag{65}$$

Substituting  $\omega = -k_1^2k_2(\lambda^2 - 4\mu)$ .into Equations 64 and 65, we arrive at the same solutions of Equations 53 and 54 or 55, respectively

(ii)  $\frac{\omega}{k_2} > 0$ ,

$$V(\xi) = \frac{\omega}{2k_1k_2} \sec^2\left(\frac{\xi}{2k_1}\sqrt{\frac{\omega}{k_2}}\right), \tag{66}$$

$$V(\xi) = \frac{\omega}{2k_1k_2} \csc^2\left(\frac{\xi}{2k_1}\sqrt{\frac{\omega}{k_2}}\right). \tag{67}$$

Integrating Equations 66 and 67, we have the solutions of Equation 41 in the forms:

$$u(\xi) = -\sqrt{\frac{\omega}{k_2}} \tan\left(\frac{\xi}{2k_1}\sqrt{\frac{\omega}{k_2}}\right) + d_1, \tag{68}$$

$$u(\xi) = \sqrt{\frac{\omega}{k_2}} \cot\left(\frac{\xi}{2k_1}\sqrt{\frac{\omega}{k_2}}\right) + d_1, \tag{69}$$

Substituting  $\omega = -k_1^2 k_2 (\lambda^2 - 4\mu)$ . into Equations 68 and 69, we arrive at the same solutions of Equations 58 and 59, respectively.

## Conclusions

This study shows that the modified (G'/G)-expansion method is quite efficient and practically well suited for finding exact solutions to the Hirota-Ramani equation and the breaking soliton equation. Our solutions are in more general forms, and many known solutions to these equations are special cases of them. In Remarks 2 and 3, we have solved Equations 10 and 41 using a direct method, and we have arrived at the same solutions obtained by the modified (G'/G)-expansion method. With the aid of Mathematica, we have assured the correctness of the obtained solutions by putting them back into the original equations.

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