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Full Length Research Paper

Analysis and optimization of a repairable $M^{[x]}/G/1$ queue with an additional optional vacation

Jau-Chuan Ke[1] and Dong-Yuh Yang[2]*

[1]Department of Applied Statistics, National Taichung University of Science and Technology, Taichung 404, Taiwan.
[2]Institute of Information and Decision Sciences, National Taipei College of Business, Taipei 100, Taiwan.

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This paper investigates an $M^{[x]}/G/1$ queueing system with an unreliable server, where the server may take an additional vacation after the essential vacation. If the system becomes empty, the server leaves the system and takes the essential vacation. At the end of the essential vacation, the server may return to the system with probability *p* or take another vacation with probability 1-*p*. When the additional vacation is completed, the server returns from the vacation. If there are no customers waiting for service in the system, the server waits idly for the first arrival and starts working. It is assumed that the server is subject to break down according to a Poisson process and the repair time obeys a general distribution. For such a system, we derive the system size distribution at a random epoch, as well as various system characteristics. Finally, we develop an iterative procedure to find the optimal threshold values under a linear cost structure. Some numerical experiments are also presented.

Key words: Cost, optimization, server breakdowns, vacation queue.

INTRODUCTION

Queueing systems with vacations have been studied extensively in the past, mainly because of their wide applications in various fields such as production/inventory systems, communication networks, computer systems and so on (Doshi, 1986). A comprehensive and excellent discussion on the vacation models can be found in Levy and Yechiali (1975) and Takagi (1991). In this paper, we consider an $M^{[x]}/G/1$ queue, in which an unreliable server operates an additional optional vacation. Being different from classical vacation policies, the vacation policy of our model is that when no customers are found in the system, the server leaves the system and takes the essential vacation. After the essential vacation, the serve may either remain idly in the system or takes an additional vacation. At an optional vacation completion epoch, the server waits for the customers in the system. Specifically, the server may perform secondary tasks utilizing the idle period, such as machine maintenance work. Consequently, this queueing model has potential applications in real life phenomenon, for example, computer/telecommunication systems.

According to the related literature of $M^{[x]}/G/1$ queue with vacations, Baba (1986) first considered batch arrival queue with multiple vacations, where the server goes on vacations repeatedly till he finds at least one waiting customer at the end of a vacation. Choudhury (2002a) modeled a batch arrival $M^{[x]}/G/1$ queueing system with a single vacation, which extends the results of Levy and Yechiali (1975) and Takagi (1991). The variations and extensions of these models can be referred to Lee et al. (1995), Krishna Reddy et al. (1998), Choudhury (2002b) and many others. More recent studies are as follows: Ke and Chu (2006) proposed a new vacation policy for the $M^{[x]}/G/1$ queueing system where the server may leave for at most *J* vacations. The results in Ke and Chu (2006) generalized those of the multiple vacation policy and the single vacation policy $M^{[x]}/G/1$ queueing system. Based on the supplementary variable technique, Ke (2007) analyzed an $M^{[x]}/G/1$ queueing system with balking under a variant vacation. Choudhury et al. (2007) made an extensive analysis of a batch arrival queue with two phases of heterogeneous service and Bernoulli schedule vacation under multiple vacation policy. Choudhury (2007) considered a retrial $M^{x}/G/1$ queueing system with two phases of heterogeneous service and Bernoulli vacation schedule. Recently, Ke et al. (2010) examined

*Corresponding author. E-mail: yangdy@webmail.ntcb.edu.tw.

an $M^{[x]}/G/1$ queueing system with a randomized vacation policy and at most *J* vacations.

As for $M^{[x]}/G/1$ queueing models with server breakdowns, Ke (2003) studied the optimal strategy of the controllable $M^{[x]}/G/1$ queueing system with server breakdowns and multiple vacations. Ke and Lin (2006) applied the maximum entropy principle to study the system characteristics of the $M^{[x]}/G/1$ queueing system with an unreliable server and delaying vacations. Ke and Chang (2009) dealt with a batch arrival retrial queue with general retrial times was investigated, where the server is subject to starting failures and provides two phases of heterogeneous service to all customers under Bernoulli vacation schedules. Recent work about $M^{[x]}/G/1$ queues with server breakdowns and vacation was presented by Choudhury and Tadj (2011). They applied supplementary variables technique to deal with an $M^{X}/G/1$ queue with two phases of service and Bernoulli vacation under *N*-policy for unreliable server.

To the best of our knowledge, there is no work that combines batch arrival, server breakdown and an additional optional vacation. This motivates us to investigate an $M^{[x]}/G/1$ queueing system with an unreliable server under additional vacation policy. Conveniently, we represent this variant vacation system as $\mathrm{M}^{[X]}/\mathrm{G(G)}/1/\mathrm{V_A}$ queue, where the first G and the second G represent the service time and repair time, respectively. V_A represents the additional vacation. Moreover, this model can be applied to model many real world systems. For example, we consider a production system with a production machine. All arriving job orders arrive according to a compound Poisson process. After processing all jobs, the production machine undergoes maintenance. The production machine may be available to perform another optional job when maintenance is done. Upon the completion of each optional job, the production is ready for accomplishing the job orders. During a manufacturing process, the production may be interrupted when the production machine fails unpredictably. When the breakdown occurs, it is immediately sent for repair with a random time. This production system can be modeled as a $\mathrm{M}^{[X]}/\mathrm{G(G)}/1/\mathrm{V_A}$ queue, that is, the maintenance period can be interpreted as an essential vacation and the period to perform another optional job can be referred to the additional vacation.

The rest of the paper is organized as follows: we make assumptions and give a brief description of the $\mathrm{M}^{[X]}/\mathrm{G(G)}/1/\mathrm{V_A}$ queue. Next, we develop the differential-equations governing the system. Some important performance measures of the $\mathrm{M}^{[X]}/\mathrm{G(G)}/1/\mathrm{V_A}$ queue are derived thereafter. Subsequently, a long-run expected cost function per unit time for the $\mathrm{M}^{[X]}/\mathrm{G(G)}/1/\mathrm{V_A}$ queue with a fixed vacation time is constructed to determine the joint optimum threshold values. Then, we present some numerical examples for illustrative purposes. Based on various system parameters and cost elements, a sensitivity analysis on the optimal threshold value is also performed. Finally, some conclusions are drawn.

Model description and assumption

We consider a batch arrival queueing system, where customers arrive in batches according to a compound Poisson process with rate λ. Let X_k denote the number of customers belonging to the *k*th arrival batch, where X_k, k = 1, 2, 3 are independent and identically distributed (i. i. d.) random variables with a continuous distribution $\Pr[X = n] = X_n$, $n \geq 1$, the probability generating function (PGF) $X(z)$ and $k-\mathrm{th}$ factorial moment of X, $E[X(X-1)...(X-k+1)]$. Arriving customers form a single waiting line based on the order of their arrivals; that is, according to the first-come, first-served (FCFS) discipline. The service time provided by a single server is an independent and identically distributed random variable (S) with distribution function $S(t)$, Laplace-Stieltjes transform (LST) $S^*(\theta)$ and $k-\mathrm{th}$ finite moment $E[S^k]$; for $k = 1, 2$. The server can serve only one customer at a time. If the server is busy or on vacation or under repair, arrivals in the queue wait until the server is available. Subsequently, we employ the following scheme to discuss the additional vacation policy of our system.

Whenever the system becomes empty, the server deactivates and leaves for a vacation with a random length V. Upon termination of the vacation period, the server inspects the system and decides whether to take additional vacation, to stay dormant in the system or to resume serving the customers exhaustively. If there is at least one customer found waiting in the queue upon returning from a vacation, the server immediately activates for service. As soon as the system empties, the server leaves the system and takes the essential vacation. At the end of the essential vacation, the server returns to the system and waits idle for customers in the system with probability *p* or may take another vacation with probability *q* $(=1-p)$. When the additional vacation is completed, the server returns from the vacation. If there are no customers waiting for service in the system, the server waits idly for the first arrival and starts working. Alternatively, one or more customers arrive at the idle state, and the server immediately starts providing service for the arrived customers. The vacation time V has a distribution function $V(x)$, LST $V^*(\theta)$ and $k-\mathrm{th}$ finite moment $E[V^k]$; for $k = 1, 2$. When the server is working, it may meet unpredictable breakdowns at any time but is immediately repaired. The server is subject to breakdowns at any time with Poisson breakdown rate α when it is working. Whenever service interruptions occur (breakdowns), the server is immediately repaired at a repair facility. After the server is repaired, it returns and starts the remaining service to customers until there are no customers in the system. The repair time is an independent and identically distributed random variable R with a general distribution function $R(t)$, LST $R^*(\theta)$ and $k-\text{th}$ finite moment $E[R^K]$; for $k=1,2$. As soon as the broken server is repaired, the server immediately returns to the system and provides service. Although no service occurs during the repair period of a broken server, customers continue to arrive according to a compound Poisson process. A customer who arrives and finds the server busy or broken down must wait in the queue until the server is available. Immediately after the server is fixed, he starts to serve customers until the system becomes empty and the service time is cumulative. Naturally, various stochastic processes involved in the system are assumed to be independent of each other.

ANALYSIS OF THE QUEUEING MODEL

We first set up the steady-state difference-differential equations for the $M^{[X]}/G(G)/1/V_A$ queue by treating the elapsed service time and the elapsed vacation time as supplementary variables. Then we solve these system equations and derive the PGF at a random epoch. In steady-state, we assume that $S(0)=0$, $S(\infty)=1$, $R(0)=0$, $R(\infty)=1$, $V(0)=0$, $V(\infty)=1$, and that $S(x)$ and $V(x)$ are continuous at $x=0$, and $R(y)$ is continuous at $y=0$, so that

$$\mu(x)dx=\frac{dS(x)}{1-S(x)},\ \eta(y)dy=\frac{dR(y)}{1-R(y)} \text{ and } w(x)dx=\frac{V(x)}{1-V(x)}.$$

The state of the system at time t is described by the random variables (r.v.s), namely:

$N(t)\equiv$ the number of customers in the system,
$S^-(t)\equiv$ the elapsed service time,
$R^-(t)\equiv$ the elapsed repair time,
$V_1^-(t)\equiv$ the elapsed time of the essential vacation, and
$V_2^-(t)\equiv$ the elapsed time of the additional optional vacation.

For further development of this variant vacation model, let us define the r.v. $\Delta(t)$ as follows:

$$\Delta(t)=\begin{cases}0, & \text{if the server is idle in the system at time } t,\\ 1, & \text{if the server is busy at time } t,\\ 2, & \text{if the server is under repair at time } t,\\ 3, & \text{if the server is on the essential vacation at time } t,\\ 4, & \text{if the server is on the additonal optional vacation at time } t.\end{cases}$$

Thus, the supplementary variables $S^-(t)$, $R^-(t)$, $V_1^-(t)$ and $V_2^-(t)$ are introduced to obtain a higher Markov process $\{N(t),\Delta(t),\delta(t)\}$, where $\delta(t)=0$ if $\Delta(t)=0$, $\delta(t)=\delta^-(t)$ if $\Delta(t)=1$, $\delta(t)=R^-(t)$ if $\Delta(t)=2$ and $\delta(t)=V_j^-(t)$ if $\Delta(t)=j+2$ ($j=1,2$).

Now, we define the following probabilities:

$$P_0(t)=P_r\{N(t)=0,\delta(t)=0\},$$
$$P_n(x,t)dx=P_r\{N(t)=n,\delta(t)=S^-(t);x<S^-(t)\le x+dx\},$$
$x>0$, $n\ge 1$,
$$Q_n(x,y,t)dy=P_r\{N(t)=n,\delta(t)=R^-(t);y<R^-(t)\le y+dy \mid S^-(t)=x\},$$
$x>0,\ y>0,\ n\ge 1,$
$$\Omega_{j,n}(x,t)dx=P_r\{N(t)=n,\delta(t)=V^-(t);x<V_j^-(t)\le x+dx\}$$
$x>0$, $n\ge 0$, $j=1,2.$

As we shall discuss the model in steady-state, that is, when $t\to\infty$, the aforementioned probabilities will be denoted by P_0, $P_n(x)$, $Q_n(x,y)$, $Q_{j,n}(x)$, respectively. That is, $P_0=\lim_{t\to\infty}P_0(t)$, $P_n(x)=\lim_{t\to\infty}P_n(x,t)dx$, $Q_n(x,y)=\lim_{t\to\infty}Q_n(x,y,t)$ and $\Omega_{j,n}(x)=\lim_{t\to\infty}\Omega_{j,n}(x,t)$.

According to Cox (1955), the Kolmogorov forward equations, which govern the system under steady-state conditions, can be written as follows:

$$\lambda P_0=p\int_0^\infty\Omega_{1,0}(x)w(x)dx+\int_0^\infty\Omega_{2,0}(x)w(x)dx, \tag{1}$$

$$\frac{d}{dx}P_n(x)+[\lambda+\alpha+\mu(x)]P_n(x)=\lambda\sum_{k=1}^{n}\chi_kP_{n-k}(x)+\int_0^\infty Q_n(x,y)\eta(y)dy,$$
$$x>0,\ y>0,\ n\ge 1, \tag{2}$$

$$\frac{\partial}{\partial y}Q_n(x,y)+[\lambda+\eta(y)]Q_n(x,y)=\lambda\sum_{k=1}^{n}\chi_kQ_{n-k}(x,y),$$
$$x>0,\ y>0,\ n\ge 1, \tag{3}$$

$$\frac{d}{dx}\Omega_{j,0}(x)+[\lambda+w(x)]\Omega_{j,0}(x)=0,\quad x>0,\ n=0,\ j=1,\ 2, \tag{4}$$

$$\frac{d}{dx}\Omega_{j,n}(x)+[\lambda+w(x)]\Omega_{j,n}(x)=\lambda\sum_{k=1}^{n}\chi_k\Omega_{j,n-k}(x),\quad x>0,\quad n\ge 1,$$
$$j=1,2. \tag{5}$$

These sets of equations are to be solved under the following boundary conditions at $x=0$.

$$P_n(0)=\int_0^\infty\Omega_{1,0}(x)w(x)dx+\int_0^\infty\Omega_{2,0}(x)w(x)dx+\int_0^\infty P_{n+1}(x)\mu(x)dx+\lambda\chi_nP_0,$$
$$n\ge 1, \tag{6}$$

$$\Omega_{1,n}(0)=\begin{cases}\int_0^\infty P_1(x)\mu(x)dx, & n=0,\\ 0, & n\ge 1.\end{cases} \tag{7}$$

$$\Omega_{2,n}(0)=\begin{cases}q\int_0^\infty\Omega_{1,n}(x)w(x)dx, & n=0,\\ 0, & n\ge 1.\end{cases} \tag{8}$$

and at $y=0$ for fixed values of x

$$Q_n(x,0)=\alpha P_n(x),\quad x>0,\ n\ge 1, \tag{9}$$

with the normalization condition

$$P_0+\sum_{n=1}^{\infty}\int_0^{\infty}P_n(x)dx+\sum_{n=1}^{\infty}\int_0^{\infty}\int_0^{\infty}Q_n(x,y)dxdy+\sum_{n=0}^{\infty}\int_0^{\infty}\Omega_{1,n}(x)dx+\sum_{n=0}^{\infty}\int_0^{\infty}\Omega_{2,n}(x)dx=1. \tag{10}$$

Let us define the probability generating functions for $\{P_n(\cdot)\}$, $\{Q_n(\cdot)\}$ and $\{\Omega_{j,n}(\cdot)\}$ as follows:

$$P(x;z)=\sum_{n=1}^{\infty}z^nP_n(x),\quad |z|<1,$$

$$Q(x,y;z)=\sum_{n=1}^{\infty}z^nQ_n(x,y),\quad |z|<1,$$

$$\Omega_j(x;z)=\sum_{n=0}^{\infty}z^n\Omega_{j,n}(x),\quad |z|<1,\quad j=1,2.$$

Now multiplying Equation 2 by z^n ($n=1,2,3,...$) and then taking summation over all possible values of n, we get

$$\frac{\partial P(x;z)}{\partial x}+[a(z)+\mu(x)+\alpha]P(x;z)=\int_0^{\infty}\eta(y)Q(x,y;z)\mathrm{d}y, \tag{11}$$

where $a(z)=\lambda(1-X(z))$.

Proceeding in the usual manner with Equations 3 to 6, then it follows that

$$\frac{\partial Q(x,y;z)}{\partial y}+[a(z)+\eta]Q(x,y;z)=0, \tag{12}$$

$$\frac{\partial \Omega_j(x;z)}{\partial y}+[a(z)+w(x)]\Omega_j(x;z)=0,\quad j=1,\ 2, \tag{13}$$

and

$$P(0;z)=\int_0^{\infty}\Omega_1(x;z)w(x)dx+\int_0^{\infty}\Omega_2(x;z)w(x)dx+\frac{1}{z}\int_0^{\infty}P(x;z)\mu(x)dx$$
$$+\lambda X(z)P_0-\Omega_1(0;z)-\Omega_2(0;z)-\lambda P_0, \tag{14}$$

where $X>0$.

Solving the partial differential equations (Equations 11 to 13), we have

$$P(x;z)=P(0;z)[1-S(x)]e^{-A(z)x}, \tag{15}$$

$$Q(x,y;z)=Q(x,0;z)[1-R(y)]e^{-a(z)y}, \tag{16}$$

and

$$\Omega_j(x;z)=\Omega_j(0;z)[1-V(x)]e^{-a(z)x},\quad j=1,\ 2, \tag{17}$$

Where $A(z)=a(z)+\alpha(1-R^*(a(z)))$.

Utilizing Equation 4, we obtain

$$\Omega_{1,0}(x)=\Omega_{1,0}(0)[1-V(x)]e^{-\lambda x}, \tag{18}$$

$$\Omega_{2,0}(x)=\Omega_{2,0}(0)[1-V(x)]e^{-\lambda x}. \tag{19}$$

Multiplying Equation 19 by $w(x)$ on both sides and integrating with respect to x over 0 to ∞, we get :

$$\int_0^{\infty}\Omega_{2,n}(x)w(x)dx=\Omega_{2,n}(0)\gamma_0,\quad \text{where }\gamma_0=V^*(\lambda). \tag{20}$$

where $\gamma_0=V^*(\lambda)$.

Inserting Equation 20 into Equation 8 leads to:

$$\Omega_{1,0}(0)=\frac{\Omega_{2,0}(0)}{q\gamma_0}. \tag{21}$$

Substituting Equation 21 into Equation 1, and after algebraic manipulation, we get:

$$\Omega_2(0;z)=\Omega_{2,0}(0)=\frac{q\lambda P_0}{q\gamma_0+p}. \tag{22}$$

From Equations 21 and 22, $\Omega_{1,0}(0)$ can be written as:

$$\Omega_{1,0}(0)=\frac{\lambda P_0}{\gamma_0[q\gamma_0+p]}. \tag{23}$$

Integrating Equations 18 and 19 with respect to x from 0 to ∞, it yields the following result:

$$\Omega_{j,0}=\Omega_{j,0}(0)\int_0^{\infty}[1-V(x)]e^{-\lambda x}dx=\frac{1}{\lambda}\Omega_{j,0}(0)(1-\gamma_0),\ j=1,2. \tag{24}$$

From Equations 22 and 24, one obtains:

$$\Omega_{1,0}=\frac{P_0(1-\gamma_0)}{\gamma_0(q\gamma_0+p)} \tag{25}$$

and

$$\Omega_{2,0}=\frac{qP_0(1-\gamma_0)}{q\gamma_0+p}. \tag{26}$$

Let us define Ω as the probability that no customers appear in the system when the server is on vacation. Then Ω can be expressed as follows:

$$\Omega=\Omega_{1,0}+\Omega_{2,0}=\frac{P_0(1-\gamma_0)(1+q\gamma_0)}{\gamma_0(q\gamma_0+p)}. \tag{27}$$

Substituting Equations 15 and 17 into Equation 14, we can obtain after simplifying that:

$$P(0;z)=\frac{\lambda P_0(1+q\gamma_0)V^*(a(z))}{\gamma_0(q\gamma_0+p)}+\frac{P(0;z)S^*(A(z))}{z}+\lambda X(z)P_0$$
$$-\Omega_1(0;z)-\Omega_2(0;z)-\lambda P_0. \tag{28}$$

Applying Equations 22 and 23, and Equation 28 becomes:

$$P(0;z)=\frac{\lambda P_0 z\left(\dfrac{(1+q\gamma_0)(V^*(a(z))-1)}{\gamma_0(q\gamma_0+p)}-1+X(z)\right)}{z-S^*(A(z))}. \quad (29)$$

From Equation 29, Equation 15 can be rewritten as:

$$P(x;z)=\frac{\lambda P_0 z\left(\dfrac{(1+q\gamma_0)(V^*(a(z))-1)}{\gamma_0(q\gamma_0+p)}-1+X(z)\right)}{z-S^*(A(z))}[1-S(x)]e^{-A(z)x} \quad (30)$$

It follows that:

$$\begin{aligned}P(z)&=\int_0^\infty P(x;z)dx \\ &=\frac{\lambda P_0 z\left(\dfrac{(1+q\gamma_0)(V^*(a(z))-1)}{\gamma_0(q\gamma_0+p)}-1+X(z)\right)}{z-S^*(A(z))}\times\frac{1-S^*(A(z))}{A(z)}.\end{aligned} \quad (31)$$

From Equation 9, the relationship between $Q(x,0;z)$ and $P(x;z)$ is given by:

$$Q(x,0;z)=\alpha P(x;z). \quad (32)$$

Then the boundary condition (Equation 9) can be written as:

$$Q(x,y;z)=\alpha P(x;z)[1-R(y)]e^{-a(z)y}. \quad (33)$$

Inserting Equation 15 into Equation 33, we get:

$$Q(x,y;z)=\alpha P(0;z)[1-S(x)]e^{-A(z)x}[1-R(y)]e^{-a(z)y}. \quad (34)$$

Using Equation 30, Equation 34 becomes:

$$Q(x,y;z)=\frac{\lambda\alpha P_0 z\left(\dfrac{(1+q\gamma_0)(V^*(a(z))-1)}{\gamma_0(q\gamma_0+p)}-1+X(z)\right)}{z-S^*(A(z))}\times[1-S(x)]e^{-A(z)x}\times[1-R(y)]e^{-a(z)y}. \quad (35)$$

Now we derive $Q(z)$ by solving the double integral of Equation 35 with respect to x and y, that is $\int_0^\infty\int_0^\infty Q(x,y;z)dxdy$. It results that:

$$\begin{aligned}Q(z)&=\frac{\lambda P_0 z\left(\dfrac{(1+q\gamma_0)(V^*(a(z))-1)}{\gamma_0(q\gamma_0+p)}-1+X(z)\right)}{z-S^*(A(z))}\times\frac{1-S^*(A(z))}{A(z)}\times\frac{\alpha[1-R^*(a(z))]}{a(z)} \\ &=P(z)\times\frac{\alpha[1-R^*(a(z))]}{a(z)}.\end{aligned} \quad (36)$$

Integrating Equation 17 with respect to x over $(0,\infty)$, and using Equations 22 and 23, we have:

$$W_1(z)=\frac{P_0[V^*(a(z))-1]}{\gamma_0(X(z)-1)(q\gamma_0+p)} \quad (37)$$

and

$$W_2(z)=\frac{qP_0[V^*(a(z))-1]}{\gamma_0(X(z)-1)(q\gamma_0+p)}. \quad (38)$$

Thus, we can compute the unknown constant P_0 by the normalization condition (10). It follows that:

$$P_0=\frac{\gamma_0(1-\rho_H)(q\gamma_0+p)}{\gamma_0(q\gamma_0+p)+\lambda E(V)(1+q\gamma_0)}, \quad (39)$$

where $\rho_H=\rho(1+\alpha E[R])$ and $\rho=\lambda E[X]E[S]$. From Equation 39, we have $\rho_H<1$, which is the steady-state condition under which the steady-state solution exists.

Let $\Phi(z)$ be the PGF of the system size distribution at arbitrary time, and $\Phi(z)$ can be expressed by:

$$\begin{aligned}\Phi(z)&=P_0+P(z)+Q(z)+W_1(z)+W_2(z) \\ &=\frac{(1-\rho_H)S^*(A(z))(z-1)}{z-S^*(A(z))}\times\frac{(1+q\gamma_0)(V^*(a(z))-1)-\gamma_0[1-X(z)][q\gamma_0+p]}{(\lambda E[V](1+q\gamma_0)+\gamma_0[q\gamma_0+p])[X(z)-1]} \\ &=\xi(z)\times\eta(z),\end{aligned} \quad (40)$$

where $\xi(z)=\dfrac{(1-\rho_H)S^*(A(z))(z-1)}{z-S^*(A(z))}$ is the PGF of the number of customers in an $M^{[x]}/G/1$ queue with server breakdowns, and $\eta(z)=\dfrac{(1+q\gamma_0)(V^*(a(z))-1)-\gamma_0[1-X(z)][q\gamma_0+p]}{(\lambda E[V](1+q\gamma_0)+\gamma_0[q\gamma_0+p])[X(z)-1]}$ denotes the system size distribution due to residual generalized-vacation period.

Remark 1

It should be noted that the system size distribution at a random epoch of an $M^{[X]}/G(G)/1/V_A$ queue in Equation 40 can be decomposed into two independent random variables:

(i) the system size distribution of an $M^{[x]}/G/1$ queue with server breakdowns; and
(ii) the system size distribution due to residual generalized-vacation period.

This confirms the stochastic decomposition property of Fuhrmann and Cooper (1985).

Remark 2

Suppose that we let $p=1$ and $\alpha=0$, our model can be reduced to the $M^{[x]}/G/1$ queue with a reliable server and single vacation. $\Phi(z)$ can be written as: $\Phi(z)=\dfrac{(1-\rho)(z-1)S^*(a(z))}{S^*(a(z))-z}\times\dfrac{(1-V^*(a(z)))+\gamma_0[1-X(z)]}{[1-X(z)](\lambda E[V]+\gamma_0)}$, which confirms the result obtained by Choudhury (2002a).

System performance measures

We consider system performance measures for the $M^{[X]}/G(G)/1/V_A$ queue as follows: (i) the expected number of customers in the system; (ii) the expected waiting time in the queue; (iii) the expected length of the completion period; (iv) the expected length of the idle period; and (v) the expected length of the busy cycle. The results are summarized subsequently.

Expected number of customers in the system and expected waiting time in the queue

First, let L_S be the expected number of customers in the system and W_q be the expected waiting time in the queue. By differentiating Equation 40 with respect to z, then taking the limit as $z \to 1$ by using the L'hopital's rule, we obtain:

$$L_S = \frac{\lambda E[X(X-1)]E[S](1+\alpha E[R]) + \left(\lambda E[X](1+\alpha E[R])\right)^2 E[S^2] + \alpha(\lambda E[X])^2 (E[R^2]E[S])}{2(1-\rho_H)} + \frac{\lambda^2 E[X][1-(q\gamma_0)^2]E[V^2]}{2(1-q\gamma_0)\left(\lambda E[V](1+q\gamma_0)+\gamma_0(p+q\gamma_0)\right)} + \rho_H. \quad (41)$$

By using the Little's formula, W_q can be obtained and expressed as:

$$W_q = \alpha E[S]E[R] + \frac{E[X(X-1)]E[S](1+\alpha E[R]) + \lambda\left(E[X](1+\alpha E[R])\right)^2 E[S^2]}{2E[X](1-\rho_H)} + \frac{\alpha\lambda E[X](E[R^2]E[S])}{2(1-\rho_H)} + \frac{\lambda[1-(q\gamma_0)^2]E[V^2]}{2(1-q\gamma_0)\left(\lambda E[V](1+q\gamma_0)+\gamma_0(p+q\gamma_0)\right)}. \quad (42)$$

Expected length of the completion period, the idle period and the busy cycle

Let $H^*(\theta)$ and $I^*(\theta)$ be the LST of the completion period (including busy period and breakdown period) and idle period for the $M^{[X]}/G(G)/1/V_A$ queue. Applying the arguments by Takagi (1991) and Tang (1997), $H^*(\theta)$ and $I^*(\theta)$ can be written as:

$$H^*(\theta) = (1+q\gamma_0)\left(V^*[\lambda(1-X(H_0^*(\theta)))] - \gamma_0\right) + \gamma_0\left(p+q\gamma_0\right)X(H_0^*(\theta)), \quad (43)$$

$$I^*(\theta) = (1+q\gamma_0)\left(V^*(\theta) - \gamma_0\right) + \gamma_0\left(p+q\gamma_0\right)\left(\frac{\lambda}{\lambda+\theta}\right), \quad (44)$$

where $H^*(\theta) = G^*[\theta + \lambda - \lambda X(H_0^*(\theta))]$ is the LST of the completion period in the ordinary $M^{[X]}/G/1$ queueing model with server breakdowns.

Now, we further define that $E[H]$, $E[I]$ and $E[C]$ are the expected length of completion period, the expected length of idle period and the expected length of the busy cycle, respectively. From Equations 43 and 44, we have:

$$E[H] = \frac{\rho_H\left[(1+q\gamma_0)\lambda E[V] + \gamma_0(p+q\gamma_0)\right]}{\lambda(1-\rho_H)}, \quad (45)$$

$$E[I] = (1+q\gamma_0)E[V] + \frac{\gamma_0(p+q\gamma_0)}{\lambda}. \quad (46)$$

Then, it yields that

$$E[C] = E[H] + E[I] = \frac{(1+q\gamma_0)\lambda E[V] + \gamma_0(p+q\gamma_0)}{\lambda(1-\rho_H)}. \quad (47)$$

Reliability indices

Here, we develop two main reliability indices of the presented model, namely, the system availability and failure frequency under the steady-state conditions. We define $A_v(t)$ as the system availability at time *t*, which is the probability that the server is either working for a customer or remaining idle in the system. The steady-state availability of the server is given by $A_v = \lim_{t\to\infty} A_v(t)$. It follows that:

$$A_v = P_0 + \int_0^\infty P(x,1)dx = P_0 + \lim_{z\to 1} P(z).$$

From Equations 31 and 39, we obtain:

$$A_v = \rho + \frac{(1-\rho_H)\gamma_0(q\gamma_0+p)}{\gamma_0(q\gamma_0+p) + \lambda E[V](1+q\gamma_0)}.$$

Next, let the steady-state failure frequency of the server be M_f. Following the argument by Li et al. (1997), we have:

$$M_f = \alpha\int_0^\infty P(x,1)dx.$$

Applying Equation 31 again, it yields that $M_f = \alpha\rho$.

Optimization of the cost model

We are interested in situations where the server deactivates and leaves for the first essential vacation with a fixed length *T* whenever the system is empty. At the end of the essential vacation, the server returns to the system and waits idle for customers in the system with probability *p* or may take another vacation of the same length *T* with probability *q* ($=1-p$). Thus, the vacation time of length is fixed rather than variable. This queueing model can be regarded as a special case of the $M^{[X]}/G(G)/1/V_A$ queue. It follows that $E[V]=T$, $E[V^2]=T^2$ and $\gamma_0 = e^{-\lambda T}$. Determination of an optimal policy is an important issue, which has received considerable attention for a queueing system (Tadj and Choudhury, 2005). To this end, we develop a steady-state expected cost function per unit time for the $M^{[X]}/G(G)/1/V_A$ queue with a fixed vacation time, in which *p* and *T* are decision variables. Our objective is to determine the joint optimal thresholds (say (p^*, T^*)), so as to minimize this cost function. To do this, let us define the cost elements in the following:

$C_h \equiv$ holding cost per unit time for each customer present in the system;

$C_S \equiv$ setup cost for per busy cycle.

Employing the definition of each cost element and its corresponding system performance, the expected cost function per unit time is given by:

$$F_0(p,T)=C_hL_S+\frac{C_S}{E[C]}$$
$$=C_hL_h+C_h\frac{\lambda^2T^2E[X][1-(q\gamma_0)^2]}{2(1-q\gamma_0)\left(\lambda T(1+q\gamma_0)+\gamma_0(p+q\gamma_0)\right)}+C_S\frac{\lambda(1-\rho_H)}{\lambda T(1+q\gamma_0)+\gamma_0(p+q\gamma_0)}, \tag{48}$$

where

$$L_h=\rho_H+\frac{\lambda E[X(X-1)]E[S](1+\alpha E[R])+\left(\lambda E[X](1+\alpha E[R])\right)^2E[S^2]+\alpha(\lambda E[X])^2(E[R^2]E[S])}{2(1-\rho_H)}$$
.

Since L_h is independent of *p* and *T*, we omit this term. Now, we are interested in obtaining the joint optimal thresholds (p,T) , say (p^*,T^*), which minimizes $F_0(p,T)$ is equivalent to minimize the following equation:

$$F(p,T)=\frac{C_h\lambda^2T^2E[X](1+q\gamma_0)+2C_S\lambda(1-\rho_H)}{2\left(\lambda T(1+q\gamma_0)+\gamma_0(p+q\gamma_0)\right)}=\frac{A_1T^2(1+q\gamma_0)+A_2}{\lambda T+p\gamma_0+q\gamma_0(\lambda T+\gamma_0)}, \tag{49}$$

where $A_1=C_h\lambda^2E[X]/2$ and $A_2=C_s\lambda(1-\rho_H)$.
Differentiating $F(p,T)$ with respect to *p*, we have:

$$\frac{\partial F(p,T)}{\partial p}=\frac{\gamma_0(\lambda A_2T+A_2\gamma_0-A_1T^2-A_2)}{\left[\lambda T+p\gamma_0+q\gamma_0(\lambda T+\gamma_0)\right]^2}=\frac{\lambda A_2T+A_2e^{-\lambda T}-A_1T^2-A_2}{e^{\lambda T}\left[\lambda T+pe^{-\lambda T}+qe^{-\lambda T}(\lambda T+e^{-\lambda T})\right]^2}. \tag{50}$$

This implies that for any p in (0, 1),

$$\frac{\partial F(p,T)}{\partial p}\begin{cases}>0 & \text{if } A_2(\lambda T+e^{-\lambda T}-1)>A_1T^2,\\ =0 & \text{if } A_2(\lambda T+e^{-\lambda T}-1)=A_1T^2,\\ <0 & \text{if } A_2(\lambda T+e^{-\lambda T}-1)<A_1T^2.\end{cases}$$

Apparently, the following results can be obtained:

(i) When $A_2(\lambda T+e^{-\lambda T}-1)>A_1T^2$, $F(p,T)$ is an increasing function in $p\in[0,1]$. Based upon the "first derivative test", it implies that:

$$h(T)=\min_{0\le p\le 1}F(p,T)=F(0,T)=\frac{A_1T^2+A_1T^2e^{-\lambda T}+A_2}{\lambda T+\lambda Te^{-\lambda T}+e^{-2\lambda T}} \tag{51}$$

As shown in Equation 51, $h(T)$ is non-linear and complex. It is rather difficult to derive the closed-form expression of T^* to minimize $h(T)$. The bisection algorithm proposed to calculate the optimal value T^* is described as follows:

Input: endpoints $T_1=0$ and $T_2(>0)$; tolerance ε ; maximum number of iterations I .
Output: approximate solution T^* and the minimum cost $h(T^*)$
Step 1: Set $i=1$.
Step 2: While $i\le I$ do Steps 3-6
Step 3: Compute $T^*=(T_2+T_1)/2$
Step 4: If $\left.\frac{dh(T)}{dT}\right|_{T=T^*}=0$ then T^* is the optimum point. STOP.
Else if $\left.\frac{dh(T)}{dT}\right|_{T=T^*}<0$ then $T_1=T^*$
Else $\left.\frac{dh(T)}{dT}\right|_{T=T^*}>0$ then $T_2=T^*$
Step 5: Set $i=i+1$.
Step 6: Until $T_2-T_1<\varepsilon$, where $\varepsilon=10^{-6}$. STOP.

(ii) When $A_2(\lambda T+e^{-\lambda T}-1)=A_1T^2$, which is equivalent to $T=0$ and $F(p,T)$ is independent of p . It yields that $\min_{0\le p\le 1}F(p,0)=C_S\lambda(1-\rho_H)$ for $p\in[0,1]$.

(iii) When $A_2(\lambda T+e^{-\lambda T}-1)<A_1T^2$, $F(p,T)$ is a decreasing function in $p\in[0,1]$. From the "first derivative test", we have:

$$g(T)=\min_{0\le p\le 1}F(p,T)=F(1,T)=\frac{A_1T^2+A_2}{\lambda T+e^{-\lambda T}}. \tag{52}$$

Differentiating $g(T)$ with respect to T , it can be seen that

$$\frac{dg(T)}{dT}=\frac{A_1\lambda T^2+A_2\lambda(e^{-\lambda T}-1)+\lambda e^{-\lambda T}+2A_1Te^{-\lambda T}}{(\lambda T+e^{-\lambda T})^2}. \tag{53}$$

Since $A_2(\lambda T+e^{-\lambda T}-1)<A_1T^2$, it follows that $A_1T^2\lambda>A_2T^2\lambda+A_2T^2\lambda(e^{-\lambda T}-1)>A_2T^2\lambda(e^{-\lambda T}-1)$. Using the "first derivative test" again, one sees that $dg(T)/dT>0$ from Equation 53. Thus, the minimum of $g(T)$ is achieved when T approaches to zero. That is, $\min g=\lim_{T\to 0}g(T)=C_S\lambda(1-\rho_H)$.

Summarizing the preceding discussed results, we find that the minimum of $F(p,T)$, $F(p^*,T^*)$, occurs at (i) $p^*=0$ and $T^*\neq 0$ or (ii) p^* is an arbitrary number ($p^*\in[0,1]$) and $T^*=0$. Subsequently, a numerical illustration is provided to capture the effect of system parameters and cost elements on the T^*.

NUMERICAL COMPUTATIONS

Here, some numerical examples are provided to illustrate the optimal threshold policy based on changes in the values of the system parameters and cost elements. First, we perform an extensive computation with the following parameters:

i. The batch arrival rate is $\lambda=1.2$;
ii. Geometric batch size with mean $E[X]=2.5$;
iii. The mean service time per batch $E[S]=0.3$;
iv. The breakdown rate $\alpha=0.05$;
v. The mean repair time is $E[R]=0.2$;
vi. The holding cost $C_h=10$ and the set-up cost $C_S=600$.

A computer program using MATLAB software was implemented. The expected cost of the $M^{[X]}/G(G)/1/V_A$ queue with a fixed vacation time under different values of

Table 1. The long-run expected cost for different values of p and T.

p \ T	0	1	2	3	4	5	6	7	8
0.0	65.520	53.834	54.857	62.703	73.538	85.893	99.093	112.798	126.825
0.1	65.520	53.999	54.891	62.705	73.536	85.892	99.093	112.798	126.825
0.2	65.520	54.167	54.925	62.707	73.535	85.891	99.093	112.798	126.825
0.3	65.520	54.339	54.960	62.709	73.534	85.890	99.092	112.798	126.824
0.4	65.520	54.514	54.995	62.711	73.532	85.890	99.092	112.798	126.824
0.5	65.520	54.692	55.030	62.714	73.531	85.889	99.092	112.798	126.824
0.6	65.520	54.873	55.066	62.716	73.529	85.888	99.091	112.798	126.824
0.7	65.520	55.058	55.102	62.718	73.528	85.887	99.091	112.797	126.824
0.8	65.520	55.247	55.139	62.720	73.527	85.886	99.090	112.797	126.824
0.9	65.520	55.439	55.176	62.722	73.525	85.885	99.090	112.797	126.824
1.0	65.520	55.636	55.213	62.724	73.524	85.885	99.090	112.797	126.824

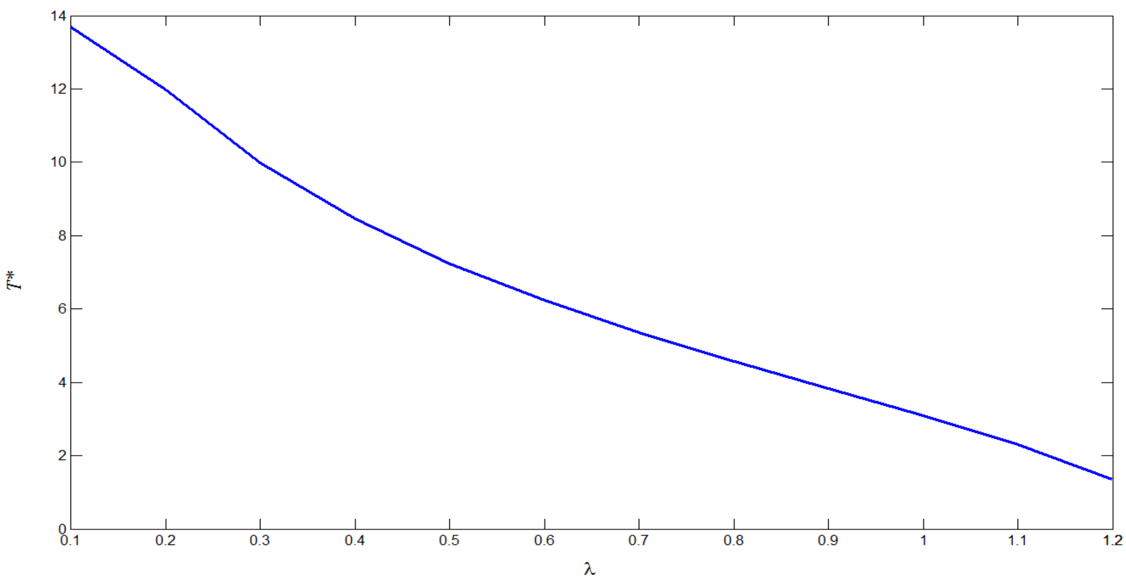

Figure 1. The optimal value of T versus λ.

p and T was shown in Table 1. One can easily see from Table 1 that (i) the minimum cost occurs at $p=0$ or 1 when T is fixed; (ii) the cost is a constant for any p when $T=0$; and (iii) the cost function is concave up when p is fixed. Next, we perform a sensitivity analysis on the T^* under various system parameters and cost elements. The following six cases are considered as follows:

Case 1: $E[X]=2.5$, $E[S]=0.3$, $\alpha=0.05$, $E[R]=0.2$, $C_h=10$ and $C_S=600$ for different values of λ.
Case 2: $\lambda=1.2$, $E[X]=2.5$, $\alpha=0.05$, $E[R]=0.2$, $C_h=10$ and $C_S=600$ for different service rates $(1/E[S])$.
Case 3: $\lambda=1.2$, $E[X]=2.5$, $E[S]=0.3$, $E[R]=0.2$, $C_h=10$ and $C_S=600$ for different values of α.
Case 4: $\lambda=1.2$, $E[X]=2.5$, $E[S]=0.3$, $\alpha=0.05$, $C_h=10$ and $C_S=600$ for different repair rates $(1/E[R])$.
Case 5: $\lambda=1.2$, $E[X]=2.5$, $E[S]=0.3$, $\alpha=0.05$, $E[R]=0.2$ and $C_S=600$ for different values of C_h.
Case 6: $\lambda=1.2$, $E[X]=2.5$, $E[S]=0.3$, $\alpha=0.05$, $E[R]=0.2$ and $C_h=10$ for different values of C_S.

The numerical illustration is graphically presented in Figures 1 to 6. We observe from Figures 1 to 6 that (i) T^* decreases as one of λ, α and C_h increases; and (ii) T^*

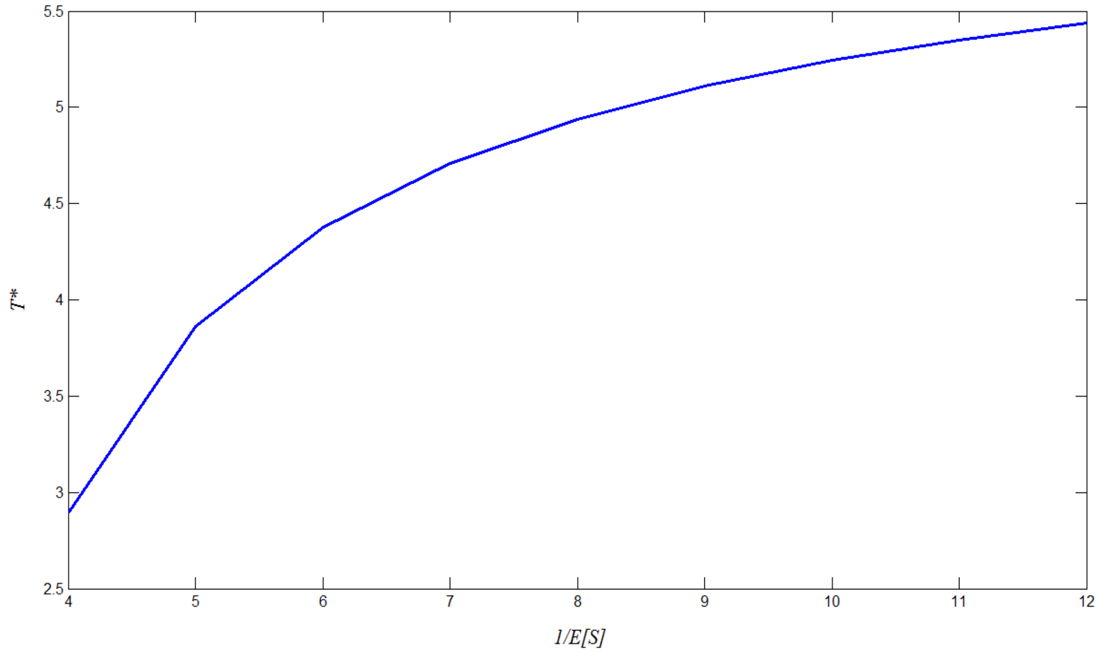

Figure 2. The optimal value of T versus service rate $1/E[S]$.

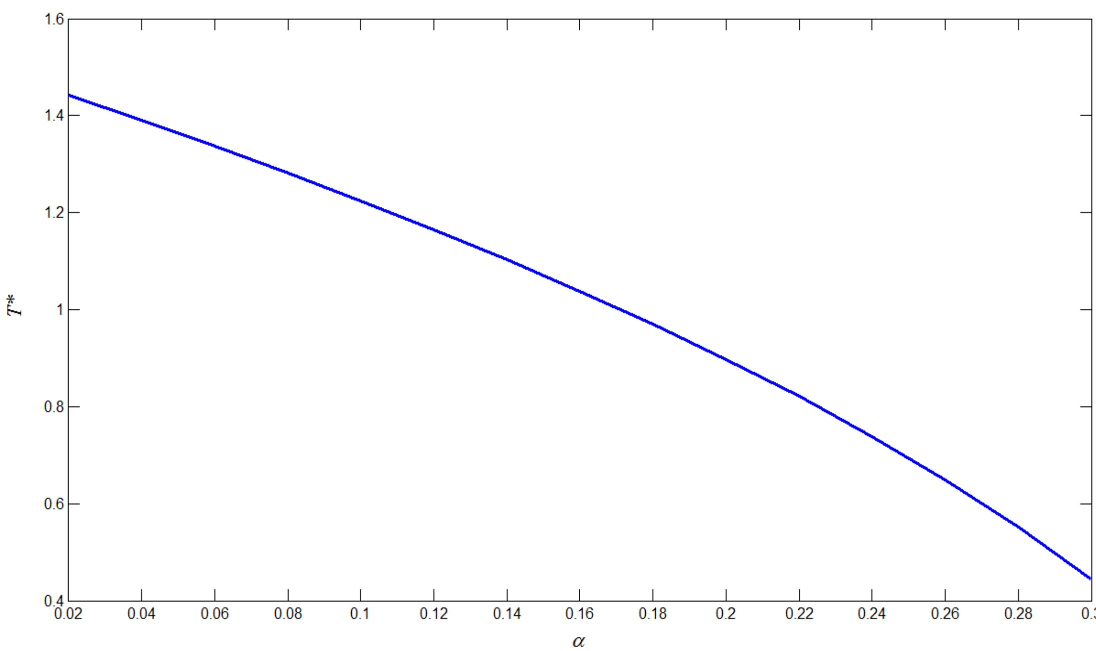

Figure 3. The optimal value of T versus α.

increases as one of service rate $1/E[S]$, repair rate $1/E[R]$ and C_S increases. Moreover, it is interesting to mention that the repair rate $1/E[R]$ rarely affects T^* when $1/E[R]$ is sufficiently large.

Conclusions

In this paper, we investigated an $M^{[X]}/G(G)/1/V_A$ queue, in which the server may take an additional vacation after the essential vacation. Some important system

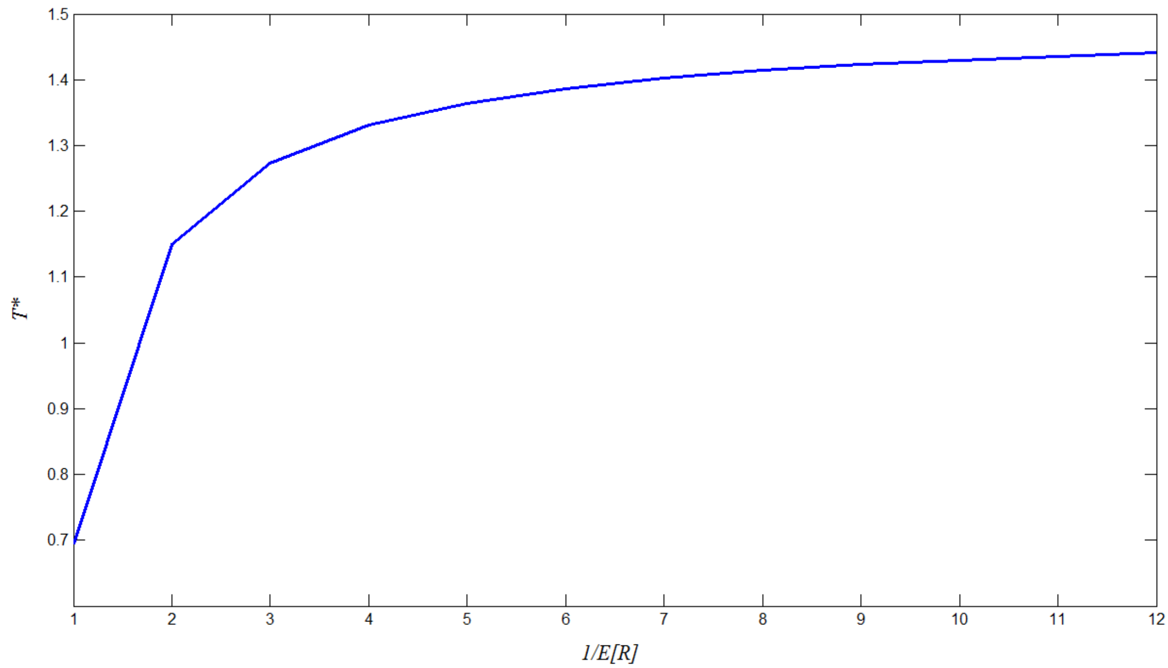

Figure 4. The optimal value of T versus repair rate $1/E[R]$.

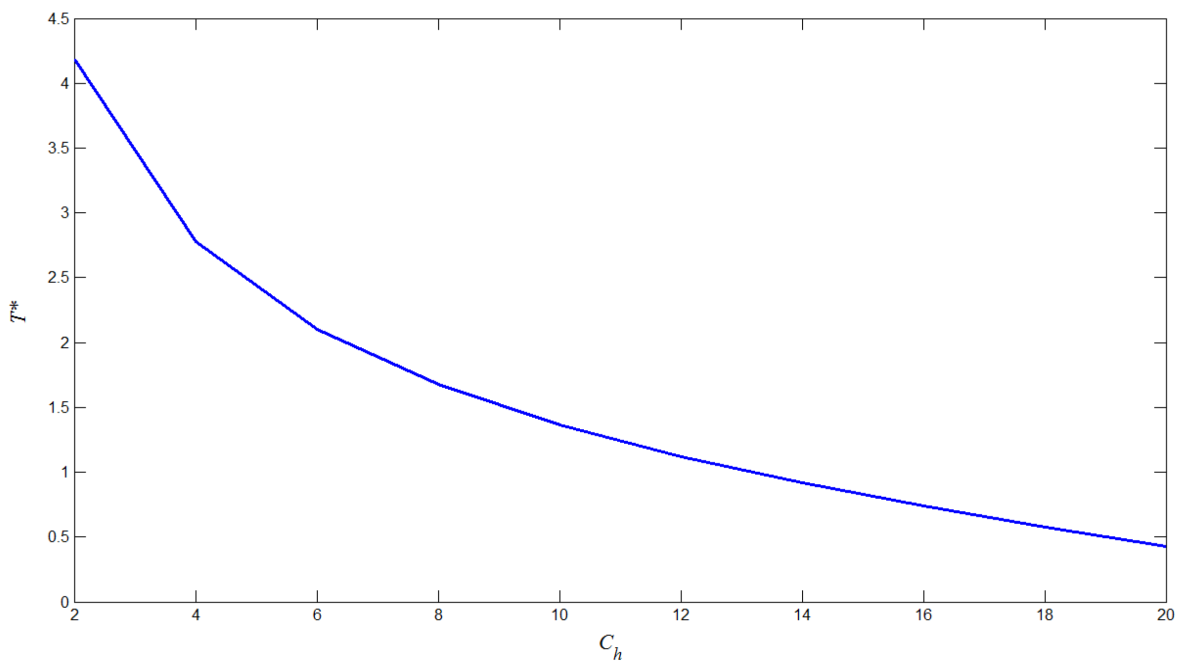

Figure 5. The optimal value of T versus C_h.

characteristics were also performed. Then, the expected cost function per unit time for the $M^{[X]}/G(G)/1/V_A$ queue with a fixed length T was constructed. We optimized the joint threshold values of (p,T) to minimize the expected cost per unit time. More importantly, an efficient iterative procedure was developed to determine the joint optimum thresholds (p^*,T^*). We finally presented some numerical results to illustrate the effect of various system

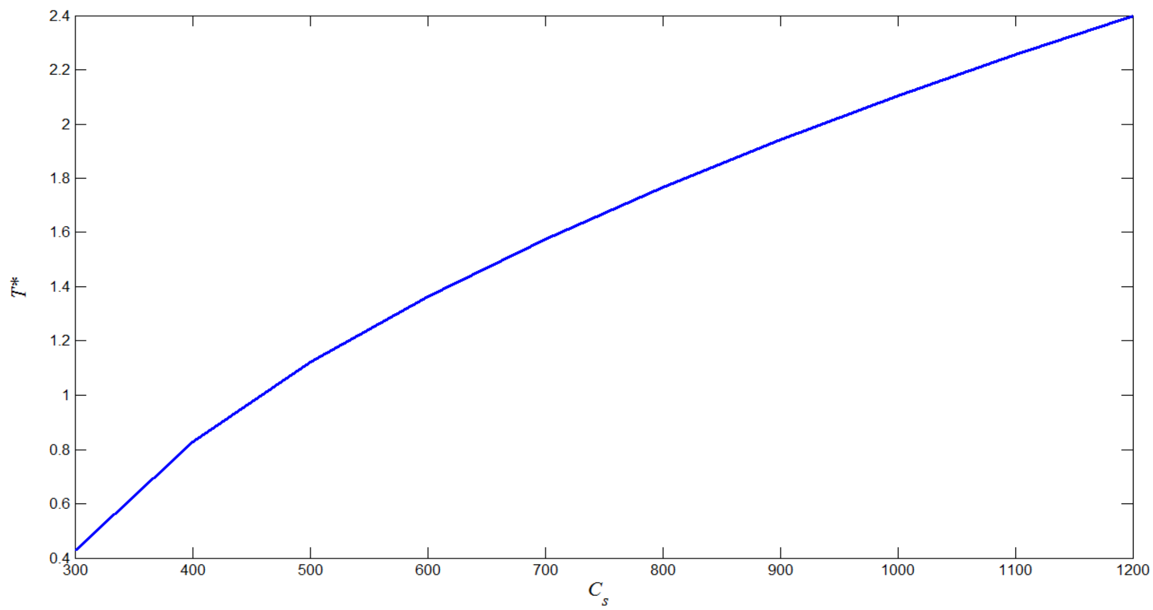

Figure 6. The optimal value of T versus C_S .

parameters and cost elements on the T^* . The analysis of this model would be helpful for further performance evaluation in many real systems such as flexible manufacturing systems, production systems, inventory systems, and many other related systems.

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