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Review

A class of totally umbilical slant submanifolds of Lorentzian para-Sasakian manifolds

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In the present paper, we study slant submanifolds of a Lorentzian para (LP)-Sasakian manifold. We consider M as a totally umbilical proper slant submanifold of an LP-Sasakian manifold and show that every totally umbilical proper slant submanifold of an LP-Sasakian manifold $\bar{M}$ is either totally geodesic or if it is not totally geodesic in $\bar{M}$, then we derive a formula of its slant angle and give an example.

Key words: Slant submanifold, totally umbilical, totally geodesic, minimal submanifold, Lorentzian para (LP)-Sasakian manifold.

INTRODUCTION

Slant submanifolds of an almost Hermitian manifold were defined and studied by Chen (1990) as a natural generalization of both holomorphic and totally real submanifolds. Later, many research articles appeared on slant submanifolds in different known spaces. Lotta (1996) introduced the notion of slant immersion of a Riemannian manifold into an almost contact metric manifold and he has proved some geometirc properties of such immersions. He also studied the intrinsic geometry of 3-dimensional non-anti-invariant slant submanifolds of K-Contact manifolds (Lotta, 1998). Later, Cabrerizo et al. (2000) investigated slant submanifolds of a Sasakian manifold and obtained many interesting results. Recently, Khan et al. (2010) defined and studied slant submanifolds in Lorentzian almost paracontact manifolds.

Recently, Sahin (2009) studied totally umbilical slant submanifolds of Kaehler manifold. In this paper, we studied a special class of slant submanifolds which are totally umbilical. We proved that for a totally umbilical slant submanifold, the mean curvature vector H of M is in μ, where μ is an invariant normal subbundle under ϕ.

Finally, we proved that a totally umbilical proper slant submanifold M of an LP- Sasakian manifold $\bar{M}$ is either totally geodesic in $\bar{M}$ or if it is not totally geodesic, then, the slant angle is $\theta = tan^{-1}(\sqrt{\frac{g(X,Y)}{\eta(X)\eta(Y)}})$, for any $X, Y \in TM$.

PRELIMINARIES

Let $\bar{M}$ be a n dimensional differentiable manifold. An LP-Contact structure (ϕ, ξ, η, g) on $\bar{M}$ consists of a tensor field ϕ of type $(1,1)$, a vector field ξ, a 1-form η and a Lorentzian metric *g* with signature $(-,+,+,\cdots,+)$ on $\bar{M}$ satisfying (Matsumoto, 1989):

$$\phi^2 X = X + \eta(X)\xi, \eta(\xi) = -1, \phi(\xi) = 0, \eta \circ \phi = 0 \quad (1)$$

and

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \eta(X) = g(X, \xi) \quad (2)$$

for any $X, Y \in T\bar{M}$.

Moreover, if on $\bar{M}$ the following additional conditions hold:

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$$(\bar{\nabla}_X \phi)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi \quad (3)$$

$$\bar{\nabla}_X \xi = \phi X, \quad (4)$$

for any vector fields $X, Y \in T\bar{M}$, then $\bar{M}$ is said to be an LP-Sasakian manifold (Matsumoto, 1989).

Now, let M be a submanifold of $\bar{M}$ with Lorentzian almost paracontact structure (ϕ, ξ, η, g). Let the induced metric on M also be denoted by g and let TM and $T^{\perp}M$ be the Lie algebra of vector fields tangential to M and normal to M, respectively and ∇ be the induced Levi-Civita connections on M, then the Gauss and Weingarten formulae are given by:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \quad (5)$$

$$\bar{\nabla}_X V = -A_V X + \nabla_X^{\perp} V \quad (6)$$

For any $X, Y \in TM$ and $V \in T^{\perp}M$, where $\nabla^{\perp}$ is the connection on the normal bundle $T^{\perp}M$, h is the second fundamental form and A_V is the Weingarten map associated with $V \in T_x^{\perp}M$ as:

$$g(A_V X, Y) = g(h(X,Y), V) \quad (7)$$

For any $x \in M$ and $X \in T_x M$, we write:

$$\phi X = TX + FX \quad (8)$$

where $TX \in T_x M$ and $FX \in T_x^{\perp}M$. Similarly, for any $V \in T_x^{\perp}M$, we have:

$$\phi V = tV + nV \quad (9)$$

where tV (resp. nV) is the tangential component (resp. normal component) of ϕV.

From Equations 1 and 8, it is easy to observe that for each $x \in M$ and $X, Y \in T_x M$:

$$g(TX, Y) = g(X, TY). \quad (10)$$

The covariant derivative of the morphisms T and F are defined, respectively as:

$$(\bar{\nabla}_X T)Y = \nabla_X TY - T\nabla_X Y \quad (11)$$

$$(\bar{\nabla}_X F)Y = \nabla_X^{\perp} FY - F\nabla_X Y \quad (12)$$

for any $X, Y \in TM$.

Throughout, the structure vector field ξ is assumed to be tangential to M, otherwise M is simply anti-invariant. For any $X, Y \in TM$ on using Equations 4 and 5, we may obtain:

$$(a)\nabla_X \xi = TX, (b)h(X,\xi) = FX. \quad (13)$$

On using Equations 3, 5, 6, 7, 9, 11 and 12, we obtain:

$$(\bar{\nabla}_X T)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi + A_{FY}X + th(X,Y) \quad (14)$$

$$(\bar{\nabla}_X F)Y = -h(X,TY) + nh(X,Y). \quad (15)$$

A submanifold M of an LP-contact manifold $\bar{M}$ is said to be totally umbilical if:

$$h(X,Y) = g(X,Y)H, \quad (16)$$

for any $X, Y \in TM$, where H is the mean curvature vector. Furthermore, if $h(X,Y) = 0$ for all $X, Y \in TM$, then M is said to be totally geodesic and if $H = 0$, then M is minimal in $\bar{M}$.

SLANT SUBMANIFOLDS

Here, we consider M as a proper slant submanifold of a Lorentzian para-Sasakian manifold $\bar{M}$. We always consider such submanifolds tangent to the structure vector fields ξ.

A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be a slant submanifold if for any $x \in M$ and $X \in T_x M - \langle\xi\rangle$, the angle between ϕX and $T_x M$ is constant. The constant angle $\theta \in [0, \pi/2]$ is then called slant angle of M in $\bar{M}$. The tangent bundle TM of M is decomposed as:

$$TM = D \oplus \langle\xi\rangle$$

where the orthogonal complementary distribution D of $\langle\xi\rangle$ is known as the slant distribution on M. If μ is ϕ −invariant subspace of the normal bundle $T^{\perp}M$, then:

$$T^{\perp}M = FTM \oplus \mu. \quad (17)$$

For a proper slant submanifold M of an LP-contact manifold $\bar{M}$ with a slant angle θ, Khan et al. (2010) proved the following theorem.

Theorem 1

Let M be a submanifold of an LP-Contact manifold $\bar{M}$

such that $\xi \in TM$. Then, M is slant submanifold if and only if there exists a constant $\lambda \in [0,1]$ such that:

$$T^2 = \lambda(I + \eta \otimes \xi). \tag{18}$$

Furthermore, if θ is slant angle of M, then $\lambda = \cos^2\theta$.

The following relations are straight forward consequence of Equation (18)

$$g(TX,TX) = \cos^2\theta[g(X,Y) + \eta(X)\eta(Y)] \tag{19}$$

$$g(FX,FY) = \sin^2\theta[g(X,Y) + \eta(X)\eta(Y)] \tag{20}$$

for any $X, Y \in TM$.

In the following theorems we consider M as a totally umbilical proper slant submanifold of an LP-Sasakian manifold $\bar{M}$.

Theorem 2

Let M be a totally umbilical slant submanifold of an LP-Sasakian manifold $\bar{M}$, then the following statements are equivalent:

1. $H \in \mu$
2. Either M is trivial or invariant submanifold of $\bar{M}$.

Proof

For any $X, Y \in TM$, then from Equation 14, we have:

$$(\bar{\nabla}_X T)Y = A_{FY}X + th(X,Y) + g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi. \tag{21}$$

Taking the product in Equation 21 with ξ, we obtain:

$$g(\nabla_X TY,\xi) = g(h(X,\xi),FY) + g(th(X,Y),\xi) - g(X,Y)$$
$$+\eta(X)\eta(Y) - 2\eta(X)\eta(Y).$$

As M is a totally umbilical slant submanifold of $\bar{M}$, then from Equation 16, the above equation takes the form:

$$-g(TY,\nabla_X\xi) = g(H,FY)\eta(X) + g(X,Y)g(tH,\xi) - g(X,Y) - \eta(X)\eta(Y).$$

Using Equations 13 and 19, we get:

$$\cos^2\theta[g(X,Y) + \eta(X)\eta(Y)] = -g(H,FY)\eta(X) + g(X,Y) + \eta(X)\eta(Y).$$

The above equation can be written as:

$$\sin^2\theta[g(X,Y) + \eta(X)\eta(Y)] = g(H,FY)\eta(X). \tag{22}$$

If $H \in \mu$, then from Equation 17, the right hand side of Equation 22 is identically zero, hence (ii) holds. Conversely, if (ii) holds then from Equation 22, we get $H \in \mu$. This completes the proof of the theorem.

Theorem 3

Let M be a totally umbilical proper slant submanifold of an LP-Sasakian manifold $\bar{M}$ such that $\nabla_U^{\perp}H \in \mu$, for all $U \in TM$. Then,

1. either M is totally geodesic,
2. or, the slant angle $\theta = tan^{-1}(\sqrt{\frac{g(X,Y)}{\eta(X)\eta(Y)}})$ for any $X, Y \in TM$.

Proof

For any $X, Y \in TM$, we have:

$$\bar{\nabla}_X\phi Y - \phi\bar{\nabla}_X Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi.$$

From Equations 5 and 8, we obtain:

$$\bar{\nabla}_X TY + \bar{\nabla}_X FY - \phi(\nabla_X Y + h(X,Y)) = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi.$$

Again using Equations 5, 6 and 8, we get:

$$g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi = \nabla_X TY + h(X,TY) - A_{FY}X + \nabla_X^{\perp}FY$$
$$-T\nabla_X Y - F\nabla_X Y - \phi h(X,Y).$$

As M is totally umbilical proper slant, then:

$$g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi = \nabla_X TY + g(X,TY)H - A_{FY}X + \nabla_X^{\perp}FY$$
$$-T\nabla_X Y - F\nabla_X Y - g(X,Y)\phi H. \tag{23}$$

Taking the product in Equation 23 with ϕH, we obtain:

$$g(X,TY)g(H,\phi H) + g(\nabla_X^{\perp}FY,\phi H) = g(F\nabla_X Y,\phi H) + g(X,Y)g(\phi H,\phi H).$$

Using Equation 2 and the fact that $H \in \mu$ (Theorem 2), we get:

$$g(\nabla_X^{\perp}FY,\phi H) = g(X,Y) \parallel H \parallel^2.$$

Then, from Equation 6, we derive:

$$g(\overline{\nabla}_X FY, \phi H) = g(X,Y) \parallel H \parallel^2. \qquad (24)$$

Now, for any $X \in TM$, we have:
$(\overline{\nabla}_X \phi)H = \overline{\nabla}_X \phi H - \phi \overline{\nabla}_X H.$

Using Equation 3 and the fact that $H \in \mu$, we obtain:

$$0 = \overline{\nabla}_X \phi H - \phi \overline{\nabla}_X H.$$

Using Equations 5, 6, 8 and 9, we obtain:

$$-A_{\phi H}X + \nabla_X^{\perp} \phi H = -TA_H X - FA_H X + t\nabla_X^{\perp} H + n\nabla_X^{\perp} H. \quad (25)$$

Taking the product in Equation 25 with FY for any $Y \in TM$ and using the fact $n\nabla_X^{\perp} H \in \mu$, the above equation gives:

$$g(\nabla_X^{\perp} \phi H, FY) = -g(FA_H X, FY).$$

Using Equation (3.4), we obtain that:

$$g(\overline{\nabla}_X FY, \phi H) = \sin^2\theta [g(A_H X, Y) + \eta(A_H X)\eta(Y)],$$

that is,

$$g(\overline{\nabla}_X FY, \phi H) = \sin^2\theta [g(X,Y) + \eta(X)\eta(Y)] \parallel H \parallel^2. \qquad (26)$$

Thus, from Equations 24 and 26, we derive:

$$[\cos^2\theta g(X,Y) - \sin^2\theta \eta(X)\eta(Y)] \parallel H \parallel^2 = 0. \qquad (27)$$

Since M is a proper slant submanifold, then it follows from Equation 27 that either $H = 0$, that is, M is totally geodesic in $\overline{M}$ or as θ is acute angle, then $\theta = tan^{-1}(\sqrt{\frac{g(X,Y)}{\eta(X)\eta(Y)}})$. This proves the theorem completely.

Now, we give an example of slant submanifold of an LP-contact manifold.

Example 1

Consider a 3-dimensional submanifold of $\mathbb{R}^5$ with its usual structure defined as:

$$\phi(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_i}, (i = 1,2),$$

$$\phi(\frac{\partial}{\partial y_j}) = \frac{\partial}{\partial y_j}, (j = 1,2),$$

$$\eta = dt, \xi = -\frac{\partial}{\partial t}$$

$$\phi(\frac{\partial}{\partial t}) = 0 \; and \; g = dx_i^2 + dy_j^2 - \eta \otimes \eta.$$

Now, for any $\theta \in [0, \pi/2]$,

$$x(u,v,t) = 2(u\cos\theta, u\sin\theta, v, 0, t).$$

If we denote by M a slant submanifold, then its tangent space TM span by the vectors:

$$e_1 = \frac{\partial}{\partial u} + 2\cos\theta v \frac{\partial}{\partial t} = \cos\theta(2(\frac{\partial}{\partial x^1} + y^1 \frac{\partial}{\partial t})) + \sin\theta(2(\frac{\partial}{\partial x^2} + y^2 \frac{\partial}{\partial t})),$$

$$e_2 = \frac{\partial}{\partial v} = 2\frac{\partial}{\partial y^1}, e_3 = -\frac{\partial}{\partial t} = \xi.$$

Moreover, the vector fields:

$$e_1^{\star} = -\sin\theta(2(\frac{\partial}{\partial x^1} + y^1 \frac{\partial}{\partial t})) + \cos\theta(2(\frac{\partial}{\partial x^2} + y^2 \frac{\partial}{\partial t})),$$

$$e_2^{\star} = 2\frac{\partial}{\partial y^2}$$

form a basis of $T^{\perp}M$. Furthermore, using Gauss formula, we get $\overline{\nabla}_{e_i} e_i = 0$, for $i = 1,2$. Thus, we have:

$$h(e_1, e_1) = 0, h(e_2, e_2) = 0, h(e_1, e_2) = 0$$

and hence, we conclude that M is totally geodesic.

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