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Full Length Research Paper

A class of exponentially - Fitted third derivative methods for solving stiff differential equations

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A class of exponentially fitted third derivative methods of order eight for solving stiff initial value problems in ordinary differential equations is derived. The derivation of the method which allows free parameters is cast into predictor-corrector form for efficient implementation. The analysis of the method shows that it is A-stable. The numerical implementation of the method to standard stiff problems shows that it is more efficient when compared with some existing methods which some the same set of problems ([1]2000 Mathematics subject classification: primary: 65L05; Secondary 65L06, 65L20).

Key words: Exponential fitting, stiff IVPs, A-stable.

INTRODUCTION

We shall consider initial value problems of ordinary differential equations of the form.

$$y' = f(x, y),\ y(a) = \eta \tag{1.1}$$

Where $y = (y_1,\ y_2, ..., y_n)^T$ and $\eta = (\eta_1, \eta_2 ..., \eta_n)$.

It is assumed that the function *f*(*x*,*y*) is defined and continuous in the region ψ defined as,

$\psi = R \times R^n$, where $R = [a, b]$ is a finite closed interval on the real line and $y \in R^n$. In addition, the function *f*(*x*,*y*) also satisfies a Lipschitz condition of order one with respect to *y*.

Definition 1.1: A numerical integration formula is said to be exponentially fitted at a complex value $\lambda = \lambda_0$ if when this applied to test function.

$$y' = \lambda y, \quad y(x_0) = y_0 . \tag{1.2}$$

with exact initial condition, the characteristic equation ρ satisfies the relation $\rho(\lambda_0 h) = e^{\lambda_0 h}$

Definition 1.2: A differential system of the form (1.1) is said to be stiff over a finite interval $[a, b]$ if for every $x \in [a, b]$, the eigen values $\lambda_i(x),\ i = ,2, ..., n$ of the Jacobian matrix arising form (1.1) satisfies the following equations:

1. $\operatorname{Re} \lambda_i(x) < 0, \quad i = 1,2, ..., n$ and
2. The stiffness ratio

$$= \frac{\max|\operatorname{Re}\lambda_i(x)|}{\min|\operatorname{Re}\lambda_i(x)|} >> 1, \quad i = 1,2, ..., n .$$

These classes of problems has a lot of applications in many areas such as control theory, chemical kinetics, electrical circuit theory, mechanical, biological and economics systems. However, most conventional numerical integration solvers can not cope effectively with stiff problems as they lack adequate stability characteristics. So, A-stability, a concept introduced by the author in Dahlquist (1963) is widely used in connection with stiff systems.

Definition 1.3: A numerical integration scheme is said to be A-stable if the region of absolute stability contain the whole of the left-hand half of the complex plane.

Several authors including (Liniger and Willoughby, 1970; Abhulimen and Otunta, 2005; Makela et al., 1974; Cash, 1981; Okunuga, 1904; Vigo et al., 2005; Vigo and Martin 2007, 2006) have earlier proposed exponentially fitted methods to guarantee stability properties for stiff systems of ordinary differential equations (ODEs). We shall however, adopt the mechanism in (Otunta and Abhulimen, 2005; Cash 1981; Abhulimen and Otunta, 2007) to construct formula which possesses adequate stability characteristics to cope with stiff systems, for which exponential fitting is appropriate.

Development of the integration formula

The general multistep methods consider is given by

$$\sum_{i=0}^{k}\alpha_j y_{n+i}-\sum_{i=1}^{t}h^i\sum_{j=0}^{k}\phi_{i,j}f^{(i-1)}{}_{n+j}=0,\ n=0,1,2 \qquad (2.1)$$

Where, $f^{(i)}{}_{n+j}$ is the i^{th} derivative of $f(x,y)$ evaluated at $(x_{n+j},\ y_{n+j})$, α_j and $\phi_{i,j}$ are real constant with $\alpha_k \neq 0$ and y_{n+j} is the approximate numerical solution evaluated at the point x_{n+j}.

The aim of this paper is to derive a third derivative exponentially-fitted order eight. So for effective implementation of the method, we cast the method into predictor corrector from, hence we reduce equation (2.1) as follows;

$$\sum_{j=0}^{k}\alpha_j y_{n+j}-\left[h\sum_{j=0}^{k}\phi_{i,j}f_{n+j}+h^2\sum_{j=0}^{k}\phi_{2,j}g_{n+j}+h^3\sum_{j=0}^{k}\phi_{3j}z_{n+j}\right]=0 \qquad (2.2)$$

Assume, $\beta_j=\phi_{i,j},\ \gamma_j=\phi_{2,j}$ and $\omega_i=\phi_{3,j}$ so that equation (2.2) now becomes;

$$\sum_{j=0}^{k}\alpha_j y_{n+j}-\left[h\sum_{j=0}^{k}\beta_j f_{n+j}+h^2\sum_{j=0}^{k}\gamma_j g_{n+j}+h^3\sum_{j=0}^{k}\omega_j z_{n+j}\right]=0 \qquad (2.3)$$

Where,

$$f_{n+j}=f\left[\left(x_{n+j},\ y\left(x_{n+j}\right)\right]=y'_{n+j},\right.$$

$$g_{n+j}=f'\left[\left(x_{n+j},\ y\left(x_{n+j}\right)\right]=y''_{n+j},\right.$$

$$z_{n+j}=f''\left[\left(x_{n+j},\ y\left(x_{n+j}\right)\right]=y'''_{n+j},\right.$$

are respectively the first, second and third derivatives of y_{n+j}.

However, since the implementation of the method involves predictor-corrector form, equation (2.3) becomes the predictor and equation (2.4) below serves as the corrector.

$$\sum_{j=0}^{k}\alpha_j y_{n+j}-\left[h\sum_{j=0}^{k+1}\beta_j f_{n+j}+h^2\sum_{j=0}^{k}\gamma_j g_{n+j}+h^3\sum_{j=0}^{k}\omega_j z_{n+j}\right]=0 \qquad (2.4)$$

In order to remove the arbitrary constant in (2.3), and (2.4), we shall always assume that $\alpha_k=+1$, $\sum_{j=0}^{k}|a_j|>0$ and $\sum_{j=0}^{k}\left|\phi_{i,j}\right|>0,\quad i=1,2,..t$.

To obtain a two-step method of order eight predictor formula, we expand (2.3), using Taylor series to obtain eight set of simultaneous equation with eleven unknown parameters. In order to solve for unknown parameters, we assume $\omega_1=0,$ and $\beta_2=a$ as free parameter with $\alpha_2=1$ to obtain the following equations;

$$\begin{aligned}
&\alpha_0+1=0\\
&\beta_0+\beta_1+a=2\\
&\beta_1+2a+\gamma_0+\gamma_1+\gamma_2=2\\
&\tfrac{3}{2}\beta_1+6a+3\gamma_1+6\gamma_2+3(\omega_0+\omega_2)=4\\
&\tfrac{1}{2}\beta_1+4a+\tfrac{3}{2}\gamma_1+12\gamma_2+6\omega_2=2\\
&\tfrac{5}{8}\beta_1+10a+\tfrac{5}{2}\gamma_1+20\gamma_2+30\omega_2=4\\
&\tfrac{3}{8}\beta_1+12a+\tfrac{15}{8}\gamma_1+30\gamma_2+60\omega_2=4\\
&\tfrac{7}{16}\beta_1+28a+\tfrac{21}{8}\gamma_1+84\gamma_2+210\omega_2=8
\end{aligned}$$

Solving the above equations, we obtain the values of the unknown parameters as;

$$\begin{aligned}
&\alpha_0=-1,\\
&\beta_0=\tfrac{38}{35}-a,\ \beta_1=\tfrac{32}{35},\ \beta_2=a\\
&\gamma_0=\tfrac{193}{525}-\tfrac{7}{15}a,\ \gamma_1=\tfrac{294}{525}-\tfrac{16}{15}a,\ \gamma_2=\tfrac{73}{525}-\tfrac{7}{15}a\\
&\omega_0=\tfrac{8}{175}-\tfrac{1}{15}a,\ \omega_1=\tfrac{-2}{15}+\tfrac{1}{15}a
\end{aligned}$$

Substituting these values into (2.3), we then have the predictor integrator of order 8 as;

$$\begin{aligned}y_{n+2}-y_n=\ &h\left[\left(\tfrac{38}{35}-a\right)y'_n+\tfrac{32}{35}y'_{n+1}+ay'_{n+2}\right]+h^2\left[\left(\tfrac{193}{525}-\tfrac{7}{15}a\right)y''_n+\left(\tfrac{294}{525}-\tfrac{16}{15}a\right)y''_{n+1}+\left(\tfrac{73}{525}-\tfrac{7}{15}a\right)y''_{n+2}\right]\\&+h^3\left[\left(\tfrac{8}{175}-\tfrac{1}{15}a\right)y'''_n+\left(\tfrac{-2}{75}-\tfrac{1}{15}a\right)y'''_{n+2}\right]\end{aligned} \qquad (2.5)$$

For the purpose of exponential fitting conditions, we apply equation (2.5) to test function

$$y'=\lambda y,\ \ y(0)=1,\ \ \lambda h=q \qquad (2.6)$$

We obtain,

$$\frac{\overline{y}_{n+2}}{y_n}=\frac{1+\left(\tfrac{38}{35}-a\right)q+\tfrac{32}{35}qe^q+\left(\tfrac{193}{525}-\tfrac{7}{15}a\right)q^2+\left(\tfrac{8}{175}-\tfrac{7}{15}a\right)q^3+\left(\tfrac{98}{135}-\tfrac{16}{15}a\right)q}{1-aq-\left(\tfrac{73}{525}-\tfrac{7}{15}a\right)q^2-\left(\tfrac{2}{75}-\tfrac{1}{15}a\right)q^3} \qquad (2.7)$$

For the purpose of stability of the method, we need to obtain the free parameter '*a*' from (2.7) as

$$a=\frac{1+\tfrac{38}{35}q+\tfrac{32}{35}qe^q+\tfrac{193}{525}q^2+\tfrac{98}{175}q^2e^q+\tfrac{8}{175}q^3+\tfrac{73}{525}q^2e^{2q}-\tfrac{2}{15}q^3e^{2q}-e^{2q}}{\tfrac{7}{15}q\left(e^{2q}+1\right)+\tfrac{16}{15}q^2e^q-\tfrac{1}{15}q^3\left(e^{2q}-1\right)-q\left(e^{2q}-1\right)} \qquad (2.8)$$

Similarly to obtain the method of order eight corrector formula, we expand equation (2.4) by Taylor series to obtain nine set of simultaneous equations with twelve unknown parameters. To solve the set of equations, we let $\beta_3=b$ as the free parameter, and imposing similar condition as in the predictor, we then obtain nine simultaneous equations from equations follows;

$$
\begin{aligned}
&\alpha_0 + 1 = 0\\
&\beta_1 + \beta_2 + b = 2\\
&\beta_1 + 2\beta_2 + 3b + \gamma_0 + \gamma_1 + \gamma_2 = 2\\
&\tfrac{3}{2}\beta_1 + 6\beta_2 + \tfrac{27}{2}b + 3\gamma_1 + 6\gamma_2 + 3(\omega_0 + \omega_2) = 4\\
&\tfrac{1}{2}\beta_1 + 4\beta_2 + \tfrac{27}{2}b + \tfrac{3}{2}\gamma_1 + 12\gamma_2 + 6\omega_2 = 2\\
&\tfrac{5}{8}\beta_1 + 10\beta_2 + \tfrac{405}{2}b + \tfrac{15}{2}\gamma_1 + 20\gamma_2 + 30\omega_2 = 4\\
&\tfrac{3}{8}\beta_1 + 12\beta_2 + \tfrac{729}{8}b + \tfrac{15}{8}\gamma_1 + 30\gamma_2 + 60\omega_2 = 4\\
&\tfrac{7}{16}\beta_1 + 28\beta_2 + \tfrac{5103}{16}b + \tfrac{21}{8}\gamma_1 + 84\gamma_2 + 210\omega_2 = 8\\
&\tfrac{1}{16}\beta_1 + 8\beta_2 + \tfrac{729}{16}b + \tfrac{7}{6}\gamma_1 + 28\gamma_2 + 84\omega_2 = 2
\end{aligned}
$$

Solving the equation, the following parameters are obtained as;

$$
\begin{aligned}
&\alpha_0 = -1,\\
&\beta_0 = \tfrac{19}{35} + \tfrac{97}{2}b, \quad \beta_1 = \tfrac{32}{35} + \tfrac{27}{b}b, \quad \beta_2 = \tfrac{19}{35} - \tfrac{135}{2}b, \quad \beta_3 = b,\\
&\gamma_0 = \tfrac{1}{105} + \tfrac{69}{4}b, \quad \gamma_1 = 0, \quad \gamma_2 = \tfrac{-4}{35} + \tfrac{135}{4}b\\
&\omega_0 = \tfrac{1}{105} + \tfrac{9}{4}b, \quad \omega_2 = \tfrac{1}{105} - \tfrac{27}{4}b
\end{aligned}
$$

By substituting the values of the parameters in equation (2.4), we obtain the method of order eight corrector formula as,

$$
\begin{aligned}
y_{n+2} - y_n = {} & h\left[\left(\tfrac{19}{35} + \tfrac{79}{2}b\right)y'_n + \left(\tfrac{32}{35} + 27b\right)y'_{n+1} + bq\tfrac{y_{n+3}}{y_n} + \left(\tfrac{19}{35} - \tfrac{135}{2}b\right)y'_{n+2}\right] + h^2\left[\left(\tfrac{4}{35} + \tfrac{69}{4}b\right)y''_n + \left(\tfrac{-4}{35} + \tfrac{135}{4}b\right)y''_{n+2}\right]\\
& + h^3\left[\left(\tfrac{1}{105} + \tfrac{9}{4}b\right)y'''_n + \left(\tfrac{1}{105} - \tfrac{27}{4}b\right)y'''_{n+2}\right]
\end{aligned}
$$

As it was done in the predictor, we apply (2.9) to scalar test function (2.6), to obtain,

$$
\frac{y_{n+2}}{y_n} = \frac{1 + \left(\tfrac{19}{35} + \tfrac{79}{2}b\right)q + \left[\left(\tfrac{32}{35} + 27b\right)q\right]R_1(\bar{q}) + \left(\tfrac{4}{15} + \tfrac{69}{4}b\right)q^2 + \left(\tfrac{1}{105} + \tfrac{9}{4}b\right)q^3 + bqR_2(q)}{1 - \left(\tfrac{19}{35} - \tfrac{135}{2}b\right)q + \left(\tfrac{4}{35} - \tfrac{135}{4}b\right)q^2 - \left(\tfrac{1}{105} - \tfrac{27}{4}b\right)q^3} = R(q)\ say \tag{2.10}
$$

Where, $R_1(\bar{q}) = \dfrac{y_{n+1}}{y_n} = \left[\dfrac{\bar{y}_{n+2}}{y_n}\right]^{1/2}$ and $R_2(q) = \dfrac{y_{n+3}}{y_n} = \left[\dfrac{\bar{y}_{n+2}}{y_n}\right]^{3/2}$.

Equation (2.10) now becomes the predictor-corrector method of order eight. For the purpose of stability of this method, we obtain the free parameter b as;

$$
b(q) = \frac{1 + \tfrac{32}{35}qe^q + \tfrac{4}{35}q^2\left(1 + e^{2q}\right) + \tfrac{19}{35}q\left(e^{2q} + 1\right) + \tfrac{1}{105}q^3\left(e^{2q} + 1\right) - e^{2q}}{\tfrac{9}{4}q^3\left(3e^{2q} - 1\right) + \tfrac{135}{2}qe^{2q} - \tfrac{1}{4}q^2\left(135e^{2q} - 69\right) - \tfrac{79}{2}q - 27qe^q - qe^{3q}} \tag{2.11}
$$

Stability of the method

Now, in order to investigate the stability of the method, the determination of the values of the free parameters a and b in the open left-half plane $(-\infty, 0]$ become very important. Hence, it is necessary to find the criteria which a and b needs to satisfy, such that,

$$
\left|\frac{y_{n+2}}{y_n}\right| < 1, \text{ that is; } |R(q)| < 1 \text{ for all } q \tag{3.1}
$$

Table 1. Parameter values of *a(q)* and *b(q)* corresponding to values of *q*.

q	*a(q)*	*b(q)*
-5	0.66068	-0.00140
-30	0.66312	-0.00552
-40	0.66853	-0.00612
-50	0.67185	-0.00651
-100	0.67867	-0.00741
-200	0.68217	-0.00792
-500	0.68429	-0.00824
-1000	0.68500	-0.00835
-2000	0.68536	-0.00841

However, necessary and sufficient conditions for equation (3.1) to hold is given by the application of the maximum modulus theorem (Cash,1981).
That is,

$R(q)$ analytic for $\mathrm{Re}(q)<0$
$|R(q)| < 1$ on $\mathrm{Re}(q) < 0$

If condition (i) holds, it follows that $R(q)$ is analytic as $q \to \infty$ and thus (i) and (ii) will guarantee A-stability.

Now, if we consider $\left|\frac{y_{n+2}}{y_n}\right| < 1$, then for $-1 < \frac{y_{n+2}}{y_n}$ is true.

So, for $\frac{y_{n+2}}{yn} < 1$, we have from equations (2.8) and (2.11), that a<1 and b<0.

Furthermore, we show analytically that a and b have finite limits.

$$
\lim_{q\to-\infty} a(q) = \lim_{q\to-\infty}\left[\frac{1 + \tfrac{38}{35}q + \tfrac{32}{35}qe^q + \tfrac{193}{525}q^2 + \tfrac{98}{175}q^2e^q + \tfrac{8}{175}q^3 + \tfrac{73}{525}q^2e^{2q} - \tfrac{2}{15}q^3e^{2q} - e^{2q}}{\tfrac{7}{15}q\left(e^{2q} + 1\right) + \tfrac{16}{15}q^2e^q - \tfrac{1}{15}q^3\left(e^{2q} - 1\right) - q\left(e^{2q} - 1\right)}\right] = \frac{24}{35} \tag{2.9}
$$

To obtain the limiting value of a as $q \to 0$ from (2.8), we apply L' Hospital rule at eight stages involving tedious and careful differentiation to yield $\lim\limits_{q\to 0} a = \dfrac{8}{9}$ that is for $q \in (-\infty, 0]$, $a \in \left(\tfrac{8}{9}, \tfrac{24}{35}\right)$.

Similarly, for the finite limits of b as $q \to \infty$, we have from equation (2.11)

$$
\lim_{q\to-\infty} b(q) = \lim_{q\to-\infty}\left[\frac{1 + \tfrac{32}{35}qe^q + \tfrac{4}{35}q^2\left(1 + e^{2q}\right) + \tfrac{19}{35}q\left(e^{2q} + 1\right) + \tfrac{1}{105}q^3\left(e^{2q} + 1 - e^{2q}\right)}{\tfrac{9}{4}q^3\left(3e^{2q} - 1\right) + \tfrac{135}{2}qe^{2q} - \tfrac{1}{4}q^2\left(135e^{2q} - 69\right) - \tfrac{79}{2}q - 27qe^q - qe^{3q}}\right] = \frac{-4}{945}
$$

Now, to obtain the limiting value of b as $q \to 0$, we apply L'Hospital rule at nine stages to yield,

$$
\lim_{q\to 0} b(q) = \left[\frac{1 + \tfrac{32}{35}qe^q + \tfrac{4}{35}q^2\left(1 + e^{2q}\right) + \tfrac{19}{35}q\left(e^{2q} + 1\right) + \tfrac{1}{105}q^3\left(e^{2q} + 1 - e^{2q}\right)}{\tfrac{9}{4}q^3\left(3e^{2q} - 1\right) + \tfrac{135}{2}qe^{2q} - \tfrac{1}{4}q^2\left(135e^{2q} - 69\right) - \tfrac{79}{2}q - 27qe^q - qe^{3q}}\right] = \frac{-8}{135}
$$

That is for $q \in (-\infty, 0]$, $b \in \left(\tfrac{-8}{135}, \tfrac{-4}{945}\right)$. Thus, the stability intervals of this method are given by:

Table 2. Numerical result on Non-linear stiff problem 1.

Step length *h*	Method	y_1	Y_2	y_3
0.0625	AB8	0.5884667145	1.0090563343	-2.7919757498
0.1	AB8	0.5882826902	1.0092403584	-2.7914604809
Exact solution		0.5882826881	1.0092403605	-2.7914604750
Errors				
0.0625	AB8	-1.8×10^{-4}	1.8×10^{-4}	5.2×10^{-4}
0.1	AB8	-2.2×10^{-8}	2.2×10^{-9}	6.3×10^{-9}

Table 3. Efficiency of method of order eight on problem 2, for *h* = 0.5, $x \in [0, 5]$.

Step length *h*	*x*	Exact solutions $y_i(x), \ i = 1,2,\ldots,4$	Approximated solution $y_i(x_n), \ i = 1,\ldots,4$	Absolute errors $\lvert y_i(x) - y_i(x_n)\rvert, \ i = 1,\ldots,4$
0.5	1.000	0.90237405E+00	0.9023740E+00	0.1110E-04
		0.4584262E-04	0.458422E-04	0.3004E-05
		-0.8224315E+01	-0.8224315E+01	0.000E+00
		0.9056815E-02	-0.9056815E-02	0.1421E-02
	2.000	0.81873075E+00	0.81873075E+00	0.1100E-04
		0.20310425E+01	0.20510425E+01	0.3524E+05
		-0.7412006E+01	-0.7412006E+01	0.8661E+00
		-0.8065206E-02	-0.8065206E-02	0.1200E-02
	3.000	0.72061710E+00	0.76061710E+00	0.2010E-05
		0.9346411E-05	0.9346411E-05	0.23544E-06
		-0.6624610E+01	-0.6624610E+01	0.1664E+04
		-0.7206061E-02	-0.7206061E-02	0.2621E-03
	4.000	0.6603200E+00	0.6603200E+00	0.2110E-06
		0.426232E-05	0.4126232E-05	0.1276E-08
		-0.6072605E+01	-0.6072605E+01	0.1664E-06
		-0.6602100E-02	-0.6602100E-02	0.1310E-05
	5.000	0.6003204E+00	0.6003204E+00	0.2110E-06
		0.1916538E-07	0.1916538E-07	0.8302E-10
		-0.5312130E+01	-0.5312130E+01	0.1664E-06

$a \in \left(\frac{8}{9}, \frac{24}{35}\right)$ and $b \in \left(\frac{-8}{135} - \frac{4}{945}\right)$.

By analytic procedure, we can verify that *a* and *b* are contained in the stability interval for all values of $q \in (-\infty, 0]$, as shown in Table 1.

So, from Table 3 above, all the values of *a* and *b* are bounded within the ranges of $a \in \left(\frac{8}{9}, \frac{24}{35}\right)$ and $b \in \left(\frac{-8}{135}, \frac{-4}{945}\right)$ for all $q \in (-\infty, 0]$. And also, as *q* decreases both *a* and *b* increases monotonically. We have that within these ranges of values of *a* and *b*, the predictor-corrector formula will be A-stable for all choices of fitting parameters. Infact, the stability function of the predictor-corrector method of order eight is given in equation (2.10). when we compute for *R*(*q*) for all values of *a* and *b* within their specified intervals above, we have $|R(q)| < 1$ for all $q \in (-\infty, 0]$, satisfying the inequality (ii) in equation (3.2), hence, the exponentially fitted method of order eight is A-stable for all choices of values of free parameters.

NUMERICAL COMPUTATIONS AND RESULTS

To show the effectiveness and validity of our newly derived methods, we present some numerical examples below. All numerical examples are coded in FORTRAN 77 and implemented on digital computer.

Problem 1: Non-linear stiff problems: Chemical Kinetic problem

The authors (Enright and Pryce, 1983) discussed the application of integration formulae based on two FORTRAN packages for assessing IVPs using the following non-linear stiff problems to illustrate the method.

Table 4 Comparison of results on problem 3.

(ERRORS)					
Method	**Step**	**$y_1(1)$**	**$y_2(1)$**	**$y_3(1)$**	**$y_4(1)$**
ERA8 ERK7	0.05	-2.7×10^{-2} 1.3×10^{3}	-2.7×10^{-1} 1.3×10^{3}	2.5×10^{-3} -1.1×10^{1}	-2.7×10^{-5} 1.3×10^{-3}
ERA8 ERA7	0.1	-8.4×10^{-2} -2.7×10^{-2}	-8.6×10^{-1} 2.1×10^{-1}	7.3×10^{-3} -2.4×10^{-3}	8.6×10^{-5} 2.7×10^{-4}

The exact solutions of problem 3 are given as follows;
$Y_1(1)$ = -5911.90736573
$Y_2(1)$ = -596.61978376
$Y_3(1)$ = 5.36957981
$Y_4(1)$ = 0.05966201

$$y_1' = -0.013y_1 + 1000y_1y_3, \qquad y_1(0)=1$$
$$y_2' = -2500y_2y_3, \qquad y_2(0)=1$$
$$y_3' = 0.013y_1 - 1020y_1y_3\text{-}2500y_2y_3, \qquad y_3(0)=0$$

The eigenvalues of the system are given as $\lambda_1 = 0,\ \ \lambda_2 = -0.00928572,\ \ \lambda_3 = -3500.003714$. The stiffness ratio is 376923.14. the exact solution is given by $y_j = C_j + D_j e^{\lambda_2 x}\ \ j = 1,2,3.$ C_j and D_i are determined using the initial value conditions.

The numerical results of problem 1 the method at x=1 is given in Table 2. We denote our method of order eight by AB8.

We observed from our numerical results in Table 4 that the error tolerance could be raise to 10^{-9} as against 10^{-4} prescribed in Enright and Pryce (Enright and Pryce, 1983). The result obtained at x=1 for h=0.0625 involves eight steps. While for h=0.1, it requires only five steps. Thus the result for h=0.1 has a higher degree of accuracy.

Problem 2: Consider the stiff system of ODEs by enright

$$y^1 = \begin{bmatrix} -0.1 & 0 & 0 & 0 \\ 0 & -10 & 0 & 0 \\ 0 & 0 & -100 & 0 \\ 0 & 0 & 0 & -1000 \end{bmatrix}, \qquad y(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad x \in [0,10]$$

The error tolerance given by Enright is 10^{-6}.The eigen values of the system are the non-zero element in the leading diagonal of the Jacobian. The stiffness ratio is given as 10^4. The exact solution is given as $y_i = A_i e^{\lambda_1 x} + B_i e^{\lambda_2 x}$. While the values of A_i and B_i are determined from the initial condition imposed on the derivations of $y(x)$. The numerical results are given in Table 2.

From Table 3, we observed that our method has low error tolerance up to 10^{-11} which is far better than the error prescribed in Enright.

Problem 3: Test problem from Enright and Pryce [7]

$$y_1' = -10^4 y_1 + 100y_2 - 10y_3 + y_4 \qquad y_1(0)=1$$
$$y_2' = -1000y_2 + 10y_3 - 10y_4 \qquad y_2(0)=1$$
$$y_3' = -y_3 + 10y_4 \qquad y_3(0)=1$$
$$y_4' = -0.1y_4 \qquad y_4(0)=1$$

The eigenvalues arising from the Jacobian of the system are

$$\lambda_1 = -0.1,\ \ \lambda_2 = -1,\ \ \lambda_3 = -1000, \lambda_4 = -10000$$

The stiffness ratio of the system is 100000. The general exact solution is in the form of $y_i(x) = A_i e^{\lambda i} + B_i e^{\lambda ix}\ \ i = 1,2,3,4$, where, A_i, B_i are constants to be determined using the initial conditions. The numerical results of problem 3 are given in Table 4. For the purpose of comparison, we denote errors in (Okunuga, 1904), method of order 7 and (Abhulimen, 2006) method of order 7 as ERK7 and ERA7 respectively. While we represent errors in present method of order 8 as ERA8, as shown in the Table 4.

Note: Discretization error is denoted by E, so that

$$E = Y_i(x) - y_i(x);\ \ i = 1,2,3,4$$

Where $Y_i(x)$ is the exact solution of the problem, and $y_i(x)$ is the numerical solution. From the numerical result displayed on Table 4, our present method of order 8 is comparatively more efficient and accurate than methods (Okunuga, 1904) and (Abhulimen and Otunta, 2007).

Conclusion

From the above results in Tables 2, 3 and 4, it can beseen that our proposed method of order eight is A-stable, more efficient and accurate when compared with existing methods of Okunuga, (1904) and Abhulimen and Tunta 2007), which have solved the same sets of stiff IVPs problems.

Conclusively, our method of order 8 can effectively handle stiff problems whose stiffness ratios are very large and for which exponential fitting are appropriate.

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