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Full Length Research Paper

A fixed point approach to the stability of a general mixed additive-cubic functional equation in quasi fuzzy normed spaces

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In this paper, we use the fixed point alternative theorem to establish Hyers-Ulam-Rassias stability of the general mixed additive-cubic functional equation where functions map a linear space into a complete quasi fuzzy p-normed space. In addition, some applications of our results in the stability of general mixed additive-cubic mappings from a linear space into a quasi p-normed space will be exhibited. Finally, we establish some results of continuous approximately general mixed additive-cubic mappings in quasi fuzzy p-normed spaces.

Key words: Fuzzy normed space, fuzzy stability, Hyers-Ulam-Rassias stability, additive function, cubic function, fixed point alternative.

INTRODUCTION

One of the most interesting questions in the theory of functional analysis concerning the Ulam stability problem of functional equations is as follows: when is it true that a mapping satisfying a functional equation approximately must be close to an exact solution of the given functional equation? In 1940, Ulam (1960) posed the famous Ulam stability problem. In 1941, Hyers solved the well-known Ulam stability problem for additive mappings subject to the Hyers condition on approximately additive mappings. He gave rise to the stability theory for functional equations.

In 1950, Aoki generalized Hyers' theorem for approximately additive functions. In 1978, Rassias provided a generalized version of Hyers for approximately linear mappings. In addition, Rassias et al. (2009, 2001, 1989, 2005, 2008, 2010, 2011) generalized the Hyers stability result by introducing two weaker conditions controlled by a product of different powers of norms and a mixed product-sum of powers of norms, respectively. In 2001, Rassias introduced the pioneering cubic functional equation:

$$f(x+2y)-3f(x+y)+3f(x)-f(x-y)=6f(y)$$

and established the solution of the Ulam stability problem for this cubic functional equation. Sadeghi and Moslehian (2008) proved the generalized stability of the cubic type functional equation,

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x)$$

and another functional equation,

$$f(kx+y)+f(x+ky)=(k+1)(k-1)^2[f(x)+f(y)]+k(k+1)f(x+y),$$

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where k is an integer with $k \neq 0, \pm 1$ in the framework of non-Archimedean normed spaces.

The idea of fuzzy norm was initiated by Katsaras (1984). Later, several notions of fuzzy norm have been introduced and discussed from different points of view (Cheng and Mordeson, 1994; Krishna and Sarma, 1994). In particular, Bag and Samanta (2005) gave a definition of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type (1975). They also studied some properties of the fuzzy norm (2005). The notion of fuzzy stability of the functional equations was given by Mirmostafaee and Moslehian (2008). Later, several various fuzzy versions of stability concerning Jensen, cubic, quadratic and mixed additive-cubic functional equations were investigated (Mihect, 2009; Mirmostafaee, 2009; Mirmostafaee et al., 2008; Xu et al., 2010.

In 2003, Radu noticed that a fixed point alternative method is very important for the solution of the Ulam problem. Mihet (2009) obtained a fuzzy version of a generalized Hyers-Ulam stability for the Jensen functional equation employing this method. Mirmostafaee (2009) established Hyers-Ulam-Rassias stability of the quartic functional equation in the setting of quasi fuzzy p-normed spaces using the fixed point method. In this paper, we consider the functional equation for fixed natural number k with $k \geq 2$:

$$f(kx+y)+f(kx-y)=kf(x+y)+kf(x-y)+2f(kx)-2kf(x) \quad (1)$$

with $f(0)=0$. It is easy to see that the function $f(x)=ax^3+bx$ is a solution of the functional Equation (1), which is called a general mixed additive-cubic functional equation. We observe that in case $k=2$ Equation (1) yields mixed additive-cubic equation,

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+2f(2x)-4f(x)$$

and there are many interesting results concerning the stability problems of the mixed additive-cubic equation (Najati and Eskandani, 2008). Therefore, Equation (1) is a generalized form of the mixed additive-cubic equation. The main purpose of this paper, is to establish the generalized Hyers-Ulam-Rassias stability of the functional Equation (1) where functions map a linear space into a complete quasi fuzzy p-normed spacea by using the fixed point alternative. In addition, some applications of our results in the stability of general mixed additive-cubic mappings from a linear space into a quasi p-norm space will be exhibited. Finally, we establish some interesting results of continuous approximately general mixed additive-cubic mappings in quasi fuzzy p-normed spaces.

Preliminaries

For the sake of convenience, here, we recall some notations and basic definitions used in this paper.

Definition 1 (Bag and Samanta, 2003): Let X be a real linear space. A function $N: X \times \mathrm{R} \to [0,1]$ (the so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathrm{R}$:

(N1) $N(x,c)=0$ for all $c \leq 0$;
(N2) $x=0$ if and only if $N(x,c)=1$ for all $c>0$;
(N3) $N(cx,t)=N(x,t/|c|)$ if $c \neq 0$;
(N4) $N(x+y,s+t) \geq \min\{N(x,s),N(y,t)\}$;
(N5) $\lim_{t\to\infty} N(x,t)=1$.

A fuzzy normed space is a pair (X,N), where X is a real linear space and N is a fuzzy norm on X.

Definition 2 (Mirmostafaee, 2009; Rolewicz, 1984): Let X be a real linear space. A quasi-norm is a real-valued function on X satisfying the following:

(1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\|=0$ if and only if $x=0$;
(2) $\|tx\|=|t|\|x\|$ for all $t \in \mathrm{R}$ and all $x \in X$;
(3) there is a constant $M \geq 1$ such that,
$\|x+y\| \leq M(\|x\|+\|y\|)$ for all $x, y \in X$.

A quasi-normed space is a pair $(X, \|\cdot\|)$, where $\|\cdot\|$ is a quasi-norm on X. The smallest possible M is called the modulus of concavity of $\|\cdot\|$. A quasi-Banach space is a complete quasi-normed space. A quasi-norm $\|\cdot\|$ is called a p-norm ($0<p\leq 1$) if $\|x+y\|^p \leq \|x\|^p+\|y\|^p$ for all $x, y \in X$.

In this case, a quasi-Banach space is called a p-Banach space. By the Aoki-Rolewicz theorem (1984), each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norm than quasi-norms; henceforth, we restrict our attention mainly to p-norms.

Definition 3 (Mirmostafaee, 2009): Let X be a real linear space. A function $N: X \times \mathrm{R} \to [0,1]$ is said to be a quasi fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathrm{R}$:

(a) $N(x,c)=0$ for all $c \leq 0$;
(b) $x=0$ if and only if $N(x,c)=1$ for all $c>0$;
(c) $N(cx,t)=N(x,t/|c|)$ if $c \neq 0$;
(d) there is a constant $M \geq 1$ such that,
$N(x+y, M(s+t)) \geq \min\{N(x,s),N(y,t)\}$ for all $x, y \in X$;
(e) $\lim_{t\to\infty} N(x,t)=1$.

A quasi fuzzy normed linear space is a pair (X,N), where X is a real linear space and N is a quasi fuzzy norm on X. A quasi fuzzy norm N is called a quasi fuzzy p-norm if

$$N(x+y,\sqrt[p]{s+t})\geq \min\{N(x,\sqrt[p]{s}),N(y,\sqrt[p]{t})\}$$

for all $x,y\in X$ and all $s,t>0$.

Lemma 1: Let (X,N) is a fuzzy normed space (quasi fuzzy p-normed space). Then $N(x,t)$ is non-decreasing with respect to $t\in(0,\infty)$ for each $x\in X$.

Proof. Let $s,t\in(0,\infty)$ such that $s>t$. Then $h=s-t$; for each $x\in X$, we have:

$$N(x,s)=N(x+0,t+h)\geq \min\{N(x,t),N(0,h)\}=N(x,t)$$

or

$$N(x,s)=N(x+0,\sqrt[p]{t^p+(s^p-t^p)})\geq \min\{N(x,t),N(y,\sqrt[p]{s^p-t^p})\}=N(x,t)$$

Example 1 (Mirmostafaee, 2009): Let $(X,\|\cdot\|)$ be a p-normed space. For all $x\in X$, consider

$$N(x,t)=\begin{cases}\dfrac{t}{t+\|x\|}, & t>0,\\[2ex] 0, & t\leq 0.\end{cases}$$

Then (X,N) is a quasi fuzzy p-normed space.

Definition 3: Let (X,N) be a quasi fuzzy normed space. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be convergent if there exists $x\in X$ such that $\lim_{n\to\infty} N(x_n-x,t)=1$ for all $t>0$. In that case, x is called the fuzzy limit of the $\{x_n\}$ and we denote it by $N-\lim_{n\to\infty} x_n=x$. A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if $\lim_{n\to\infty} N(x_{n+p}-x_n,t)=1$ for all $t>0$ and $p=1,2,3,\ldots$. It is known that every convergent sequence in a quasi fuzzy normed space is a Cauchy sequence. If every Cauchy sequence is convergent, then the quasi fuzzy norm is said to be quasi fuzzy complete and the quasi fuzzy normed space is called a quasi fuzzy Banach space. A complete quasi fuzzy p-normed space is called a quasi fuzzy p-Banach space.

Stability of functional equation in the fuzzy setting

From now on, unless otherwise stated, we will assume that $0<p\leq 1$ and $q=1/p$ are a real vector space, (Y,N) is a quasi fuzzy p-Banach space and (Z,N') is a fuzzy normed space. Utilizing the fixed point alternative, we will establish the following new stability for the generalized mixed additive-cubic functional equation in quasi fuzzy p-Banach space. For convenience, we use the following abbreviation for a given function $f:X\to Y$:

$$Df(x,y)=f(kx+y)+f(kx-y)-kf(x+y)-kf(x-y)-2f(kx)+2kf(x)$$

for all $x,y\in X$.

We start with the following lemma which plays a crucial role here.

Lemma 3 (Xu et al., 2011): Let $f:X\to Y$ be a function with $f(0)=0$ and satisfying Equation (1), then the function $G(x):=f(2x)-8f(x)$ is additive and $H(x):=f(2x)-2f(x)$ is cubic.

For explicitly later use, we recall the following result by Diaz and Margolis.

Lemma 4 (The fixed point alternative theorem, (Diaz and Margolis, 1968): Let (Ω,d) be a complete generalized metric space (that is one for which d may assume infinite values) and $J:\Omega\to\Omega$ be a strictly contractive mapping with Lipschitz constant $0<L<1$, that is

$d(Jx,Jy)\leq Ld(x,y)$ for all $x,y\in X$.

Then, for each given $x\in\Omega$, either $d(J^n x,J^{n+1}x)=\infty$ for all $n\geq 0$ or $d(J^n x,J^{n+1}x)<\infty$ for all $n\geq n_0$ for some natural number n_0. Actually, if the second alternative holds, then the sequence $\{J^n x\}$ is convergent to a fixed point x^* of J and

(1) x^* is the unique fixed point of J in the set $\Delta=\{y\in\Omega: d(J^{n_0}x,y)<\infty\}$;
(2) $d(y,y^*)\leq \frac{1}{1-L}d(y,Jy)$ for all $y\in\Delta$.

Lemma 2: Let $f:X\to Y$ be a function with $f(0)=0$ for which there is a function $\varphi:X\times X\to Z$ such that:

$$N(Df(x,y),t)\geq N'(\varphi(x,y),t) \qquad (1)$$

for all $x,y\in X$ and $t>0$. Then,

$$N(f(4x)-10f(2x)+16f(x),t) \geq N_1(x,(k^3-k)t) \qquad (2)$$

for all $x, y \in X$ and $t > 0$, where,

$$\begin{aligned}N_1(x,t)=\min\{&N'(\varphi(x,(2k+1)x),\frac{t}{384^q k}),\\
&N'(\varphi(x,(2k-1)x),\frac{t}{384^q k}),N'(\varphi(x,3kx),\frac{t}{384^q}),\\
&N'(\varphi(0,(3k-1)x),\frac{(k-1)t}{384^q k}),N'(\varphi(2x,2x),\frac{t}{96^q k^2}),\\
&N'(\varphi(0,(k+1)x),\frac{(k-1)t}{96^q k}),N'(\varphi(x,kx),\frac{t}{96^q}),\\
&N'(\varphi(0,2(k-1)x),\frac{(k-1)t}{96^q k^2}),N'(\varphi(0,2kx),\frac{t}{96^q (k+1)}),\\
&N'(\varphi(2x,2kx),\frac{t}{16^q}),N'(\varphi(3x,x),\frac{t}{56^q}),N'(\varphi(2x,x),\frac{t}{32^q}),\\
&N'(\varphi(x,(k+1)x),\frac{t}{128^q}),N'(\varphi(x,(k-1)x),\frac{t}{128^q}),\\
&N'(\varphi(0,x),\frac{(k-1)t}{128^q}),N'(\varphi(2x,kx),\frac{t}{16^q}),N'(\varphi(\frac{x}{2},\frac{(2k+1)x}{2}),\\
&\frac{t}{384^q k}),N'(\varphi(\frac{x}{2},\frac{(2k-1)x}{2}),\frac{t}{384^q k}),N'(\varphi(\frac{x}{2},\frac{3kx}{2}),\frac{t}{384^q}),\\
&N'(\varphi(0,\frac{(3k-1)x}{2}),\frac{(k-1)t}{384^q k}),N'(\varphi(x,x),\frac{t}{96^q k^2}),\\
&N'(\varphi(0,\frac{(k+1)x}{2}),\frac{(k-1)t}{96^q k}),N'(\varphi(\frac{x}{2},\frac{kx}{2}),\frac{t}{96^q}),\\
&N'(\varphi(0,(k-1)x),\frac{(k-1)t}{96^q k^2}),N'(\varphi(0,kx),\frac{t}{96^q (k+1)})\}.\end{aligned}$$

Proof: Letting $x=0$ in (1), we get

$$N(f(y)+f(-y),t) \geq N'(\varphi(0,y),(k-1)t) \qquad (3)$$

for all $y \in X$ and $t > 0$. Putting $y = x$ in (1), we have

$$N(f((k+1)x)+f((k-1)x)-kf(2x)-2f(kx)+2kf(x),t) \geq N'(\varphi(x,x),t) \qquad (4)$$

for all $x \in X$ and $t > 0$. Hence,

$$N(f(2(k+1)x)+f(2(k-1)x)-kf(4x)-2f(2kx)+2kf(2x),t) \geq N'(\varphi(2x,2x),t) \qquad (5)$$

for all $x \in X$ and $t > 0$. Letting $y = kx$ in (1), we get

$$N(f(2kx)-kf((k+1)x)-kf(-(k-1)x)-2f(kx)+2kf(x),t) \geq N'(\varphi(x,kx),t) \qquad (6)$$

for all $x \in X$ and $t > 0$. Letting $y = (k+1)x$ in (1), we have

$$N(f((2k+1)x)+f(-x)-kf((k+2)x)-kf(-kx)-2f(kx)+2kf(x),t) \geq N'(\varphi(x,(k+1)x),t) \qquad (7)$$

for all $x \in X$ and $t > 0$. Letting $y = (k-1)x$ in (1), we have

$$N(f((2k-1)x)-(k+2)f(kx)-kf(-(k-2)x)+(2k+1)f(x),t) \geq N'(\varphi(x,(k-1)x),t) \qquad (8)$$

for all $x \in X$ and $t > 0$. Replacing x and y by $2x$ and x in (1), respectively, we get

$$N(f((2k+1)x)+f((2k-1)x)-2f(2kx)-kf(3x)+2kf(2x)-kf(x),t) \geq N'(\varphi(2x,x),t) \qquad (9)$$

for all $x \in X$ and $t > 0$. Replacing x and y by $3x$ and x in (1), respectively, we get

$$N(f((3k+1)x)+f((3k-1)x)-2f(3kx)-kf(4x)-kf(2x)+2kf(3x),t) \geq N'(\varphi(3x,x),t) \qquad (10)$$

for all $x \in X$ and $t > 0$. Replacing x and y by $2x$ and kx in (1), respectively, we have:

$$N(f(3kx)+f(kx)-kf((k+2)x)-kf(-(k-2)x)-2f(2kx)+2kf(2x),t) \geq N'(\varphi(2x,kx),t) \qquad (11)$$

for all $x \in X$ and $t > 0$. Setting $y = (2k+1)x$ in (1), we have

$$N(f((3k+1)x)+f(-(k+1)x)-kf(2(k+1)x)-kf(-2kx)-2f(kx)+2kf(x),t) \geq N'(\varphi(x,(2k+1)x),t) \qquad (12)$$

for all $x \in X$ and $t > 0$. Letting $y = (2k-1)x$ in (1), we have

$$N(f((3k-1)x)+f(-(k-1)x)-kf(-2(k-1)x)-kf(2kx)-2f(kx)+2kf(x),t) \geq N'(\varphi(x,(2k-1)x),t) \qquad (13)$$

for all $x \in X$ and $t > 0$. Letting $y = 3kx$ in (1), we have,

$$N(f(4kx)+f(-2kx)-kf((3k+1)x)-kf(-(3k-1)x)-2f(kx)+2kf(x),t) \geq N'(\varphi(x,3kx),t) \qquad (14)$$

for all $x \in X$ and $t>0$. By (3), (4), (10), (12), and (13), we get

$$N(kf(2(k+1)x)+kf(-2(k-1)x)+6f(kx)-2f(3kx)-kf(4x)+2kf(3x)-6kf(x),t)$$
$$\geq \min\{N'(\varphi(x,(2k-1)x),\frac{t}{7^q}),N'(\varphi(x,(2k+1)x),\frac{t}{7^q}),N'(\varphi(3x,x),\frac{t}{7^q}),$$
$$N'(\varphi(x,x),\frac{t}{7^q}),\ N'(\varphi(0,(k+1)x),\frac{(k-1)t}{7^q}),N'(\varphi(0,(k-1)x),\frac{(k-1)t}{7^q}),$$
$$N'(\varphi(0,2kx),\frac{(k-1)t}{7^q k})\} \tag{15}$$

for all $x \in X$ and $t>0$. By (3), (7) and (8), we have

$$N\big(f((2k+1)x)+f((2k-1)x)-kf((k+2)x)-kf(-(k-2)x)-4f(kx)+4kf(x),t\big)$$
$$\geq \min\{N'(\varphi(x,(k+1)x),\frac{t}{4^q}),N'(\varphi(0,x),\frac{(k-1)t}{4^q}),$$
$$N'(\varphi(x,(k-1)x),\frac{t}{4^q}),N'(\varphi(0,kx),\frac{(k-1)t}{4^q k})\} \tag{16}$$

for all $x \in X$ and $t>0$. It follows from (9) and (16) that:

$$N\big(kf((k+2)x)+kf(-(k-2)x)-2f(2kx)+4f(kx)-kf(3x)+2kf(2x)-5kf(x),t\big)$$
$$\geq \min\{N'(\varphi(2x,x),\frac{t}{2^q}),N'(\varphi(x,(k+1)x),\frac{t}{8^q}),N'(\varphi(x,(k-1)x),\frac{t}{8^q}),$$
$$N'(\varphi(0,x),\frac{(k-1)t}{8^q}),N'(\varphi(0,kx),\frac{(k-1)t}{8^q k})\} \tag{17}$$

for all $x \in X$ and $t>0$. By (11) and (17), we have

$$N\big(f(3kx)-4f(2kx)+5f(kx)-kf(3x)+4kf(2x)-5kf(x),t\big)$$
$$\geq \min\{N'(\varphi(2x,x),\frac{t}{4^q}),N'(\varphi(x,(k+1)x),\frac{t}{16^q}),N'(\varphi(x,(k-1)x),\frac{t}{16^q}),$$
$$N'(\varphi(0,x),\frac{(k-1)t}{16^q}),N'(\varphi(0,kx),\frac{(k-1)t}{16^q k}),N'(\varphi(2x,kx),\frac{t}{2^q})\} \tag{18}$$

for all $x \in X$ and $t>0$. By (3), (12), (13) and (14), we have

$$N(kf(-(k+1)x)-kf(-(k-1)x)-k^2 f(2(k+1)x)+k^2 f(-2(k-1)x)$$
$$+k^2 f(2kx)-(k^2-1)f(-2kx)+f(4kx)-2f(kx)+2kf(x),t)$$
$$\geq \min\{N'(\varphi(x,(2k+1)x),\frac{t}{4^q k}),N'(\varphi(x,(2k-1)x),\frac{t}{4^q k}),$$
$$N'(\varphi(x,3kx),\frac{t}{4^q}),\ N'(\varphi(0,(3k-1)x),\frac{(k-1)t}{4^q k})\} \tag{19}$$

for all $x \in X$ and $t>0$. It follows from (3), (5), (6) and (19) that

$$N(f(4kx)-2f(2kx)-k^3 f(4x)+2k^3 f(2x),t)$$
$$\geq \min\{N'(\varphi(x,(2k+1)x),\frac{t}{24^q k}),N'(\varphi(x,(2k-1)x),\frac{t}{24^q k}),N'(\varphi(x,3kx),\frac{t}{24^q}),$$
$$N'(\varphi(0,(3k-1)x),\frac{(k-1)t}{24^q k}),N'(\varphi(2x,2x),\frac{t}{6^q k^2}),N'(\varphi(0,(k+1)x),\frac{(k-1)t}{6^q k}),$$
$$N'(\varphi(x,kx),\frac{t}{6^q}),\ N'(\varphi(0,2(k-1)x),\frac{(k-1)t}{6^q k^2}),N'(\varphi(0,2kx),\frac{t}{6^q(k+1)})\} \tag{20}$$

for all $x \in X$ and $t>0$. Hence,

$$N(f(2kx)-2f(kx)-k^3 f(2x)+2k^3 f(x),t)$$
$$\geq \min\{N'(\varphi(\frac{x}{2},\frac{(2k+1)x}{2}),\frac{t}{24^q k}),N'(\varphi(\frac{x}{2},\frac{(2k-1)x}{2}),\frac{t}{24^q k}),N'(\varphi(\frac{x}{2},\frac{3kx}{2}),\frac{t}{24^q}),$$
$$N'(\varphi(0,\frac{(3k-1)x}{2}),\frac{(k-1)t}{24^q k}),N'(\varphi(x,x),\frac{t}{6^q k^2}),N'(\varphi(0,\frac{(k+1)x}{2}),\frac{(k-1)t}{6^q k}),$$
$$N'(\varphi(\frac{x}{2},\frac{kx}{2}),\frac{t}{6^q}),\ N'(\varphi(0,(k-1)x),\frac{(k-1)t}{6^q k^2}),N'(\varphi(0,kx),\frac{t}{6^q(k+1)})\} \tag{21}$$

for all $x \in X$ and $t>0$. By (6), we have,

$$N\big(f(4kx)-kf(2(k+1)x)-kf(-2(k-1)x)-2f(2kx)+2kf(2x),t\big)$$
$$\geq N'\big(\varphi(2x,2kx),t\big) \tag{22}$$

for all $x \in X$ and $t>0$. From (20) and (22), we have:

$$N(kf(2(k+1)x)+kf(-2(k-1)x)-k^3 f(4x)+(2k^3-2k)f(2x),t)$$
$$\geq \min\{N'(\varphi(x,(2k+1)x),\frac{t}{48^q k}),N'(\varphi(x,(2k-1)x),\frac{t}{48^q k}),N'(\varphi(x,3kx),\frac{t}{48^q}),$$
$$N'(\varphi(0,(3k-1)x),\frac{(k-1)t}{48^q k}),N'(\varphi(2x,2x),\frac{t}{12^q k^2}),N'(\varphi(0,(k+1)x),\frac{(k-1)t}{12^q k}),$$
$$N'(\varphi(x,kx),\frac{t}{12^q}),N'(\varphi(0,2(k-1)x),\frac{(k-1)t}{12^q k^2}),N'(\varphi(0,2kx),\frac{t}{12^q(k+1)}),$$
$$N'(\varphi(2x,2kx),\frac{t}{2^q})\} \tag{23}$$

for all $x \in X$ and $t>0$. Also, from (15) and (23), we get

$$N(2f(3kx)-6f(kx)+(k-k^3)f(4x)-2kf(3x)+(2k^3-2k)f(2x)+6kf(x),t)$$
$$\geq \min\{N'(\varphi(x,(2k+1)x),\frac{t}{96^q k}),N'(\varphi(x,(2k-1)x),\frac{t}{96^q k}),N'(\varphi(x,3kx),\frac{t}{96^q}),$$
$$N'(\varphi(0,(3k-1)x),\frac{(k-1)t}{96^q k}),N'(\varphi(2x,2x),\frac{t}{24^q k^2}),N'(\varphi(0,(k+1)x),\frac{(k-1)t}{24^q k}),$$
$$N'(\varphi(x,kx),\frac{t}{24^q}),N'(\varphi(0,2(k-1)x),\frac{(k-1)t}{24^q k^2}),N'(\varphi(0,2kx),\frac{t}{24^q(k+1)}),$$
$$N'(\varphi(2x,2kx),\frac{t}{4^q}),N'(\varphi(3x,x),\frac{t}{14^q}),N'(\varphi(0,(k-1)x),\frac{(k-1)t}{14^q}),N'(\varphi(x,x),\frac{t}{14^q})\} \tag{24}$$

for all $x \in X$ and $t>0$. On the other hand, it follows from (18) and (24) that,

$$N(8f(2kx)-16f(kx)+(k-k^3)f(4x)+(2k^3-10k)f(2x)+16kf(x),t)$$
$$\geq \min\{N'(\varphi(x,(2k+1)x),\frac{t}{192^q k}),N'(\varphi(x,(2k-1)x),\frac{t}{192^q k}),N'(\varphi(x,3kx),\frac{t}{192^q}),$$
$$N'(\varphi(0,(3k-1)x),\frac{(k-1)t}{192^q k}),N'(\varphi(2x,2x),\frac{t}{48^q k^2}),N'(\varphi(0,(k+1)x),\frac{(k-1)t}{48^q k}),$$
$$N'(\varphi(x,kx),\frac{t}{48^q}),N'(\varphi(0,2(k-1)x),\frac{(k-1)t}{48^q k^2}),N'(\varphi(0,2kx),\frac{t}{48^q(k+1)}),$$
$$N'(\varphi(2x,2kx),\frac{t}{8^q}),N'(\varphi(3x,x),\frac{t}{28^q}),N'(\varphi(0,(k-1)x),\frac{(k-1)t}{28^q}),N'(\varphi(x,x),\frac{t}{28^q}),$$
$$N'(\varphi(2x,x),\frac{t}{16^q}),N'(\varphi(x,(k+1)x),\frac{t}{64^q}),N'(\varphi(x,(k-1)x),\frac{t}{64^q}),$$
$$N'(\varphi(0,x),\frac{(k-1)t}{64^q}),N'(\varphi(0,kx),\frac{(k-1)t}{64^q k}),N'(\varphi(2x,kx),\frac{t}{8^q})\} \tag{25}$$

for all $x \in X$ and $t>0$. Finally, by using (21) and (25), we get (2).

Lemma 6: Let $\varphi: X \times X \to Z$ be a function and $\Omega = \{g: X \to Y \mid g(0)=0\}$. For all $g,h \in \Omega$, define

$$d(g,h)=\inf\{K>0 \mid N(g(x)-h(x),K^q t)\geq N_1(x,(k^3-k)t), x\in X, t>0\}$$

where N_1 is defined as in Lemma 5. Then d is a generalized complete metric on Ω.

Proof: The proof is similar to the proof of Lemma 3 by Mirmostafaee (2009).

Theorem 1: Let $j\in\{-1,1\}$ be fixed and $\alpha_j>0$ such that $(\alpha_j/2)^j<1$. Let $\varphi: X\times X\to Z$ be a function with the following property:

$$N'(\varphi(2^j x,2^j y),t)\geq N'(\alpha_j^j \varphi(x,y),t) \quad (26)$$

and

$$\lim_{n\to\infty} N'(\varphi(2^{nj}x,2^{nj}y),2^{nj}t)=1 \quad (27)$$

for all $x,y\in X$ and $t>0$. Let $f: X\to Y$ be a function with $f(0)=0$ and satisfies the inequality

$$N(Df(x,y),t)\geq N'(\varphi(x,y),t) \quad (28)$$

for all $x,y\in X$ and $t>0$. Then there exists a unique additive function $A: X\to Y$ such that:

$$N(f(2x)-8f(x)-A(x),t)\geq N_1(x,(k^3-k)\,|\,\alpha_j^p-2^p\,|^q\, t) \quad (29)$$

for all $x\in X$, where N_1 is defined as in Lemma 5.

Proof: Consider the set $\Omega=\{g: X\to Y \mid g(0)=0\}$ and introduce the generalized metric on Ω,

$$d(g,h)=\inf\{K>0 \mid N(g(x)-h(x),K^q t)\geq N_1(x,(k^3-k)t), x\in X, t>0\}$$

Then d is a generalized complete metric on Ω by Lemma 6. Define a function $J:\Omega\to\Omega$ by $Jg(x)=\frac{1}{2^j}g(2^j x)$ for all $x\in X$. Let $d(g,k)<K$, by definition, $N(g(x)-h(x),K^q t)\geq N_1(x,(k^3-k)t)$, for all $x\in X, t>0$. By (26) and the definition, we get

$$N(2^{-j}g(2^j x)-2^{-j}h(2^j x),2^{-j}K^q t)\geq N_1(2^j x,(k^3-k)t)\geq N_1(x,(k^3-k)\alpha_j^{-j}t)$$

for all $x\in X$ and $t>0$. Hence,

$$N(2^{-j}g(2^j x)-2^{-j}h(2^j x),K^q(\alpha_j/2)^j t)\geq N_1(x,(k^3-k)t),$$

for all $x\in X, t>0$. By definition, $d(Jg,Jh)\leq(\alpha_j/2)^{jp}K$. Therefore, $d(Jg,Jh)\leq(\alpha_j/2)^{jp}d(g,h)$, for all $g,h\in\Omega$. This means that J is a strictly contractive self-mapping of Ω with Lipschitz constant $(\alpha_j/2)^{jp}$.

Now, let $\tilde f: X\to Y$ be the function defined by $\tilde f(x):=f(2x)-8f(x)$ for each $x\in X$. By (2) of Lemma 5, $N(\tilde f(2x)-2\tilde f(x),t)\geq N_1(x,(k^3-k))$, then we see that

$$N(\tilde f(x)-\frac{1}{2}\tilde f(2x),\frac{1}{2}t)\geq N_1(x,(k^3-k)t)$$

and

$$N(\tilde f(x)-2\tilde f(\frac{x}{2}),t)\geq N_1(\frac{x}{2},(k^3-k)t)\geq N_1(\frac{x}{2},(k^3-k)\alpha_{-1}t)$$

for all $x\in X$ and $t>0$. Hence,

$$d(\tilde f,J\tilde f)\leq\begin{cases}1/\alpha_{-1}, & j=-1;\\ 1/2^p, & j=1,\end{cases}$$

and therefore, by Lemma 4, J has a unique fixed point $A: X\to Y$ in the set $\Delta=\{g\in\Omega \mid d(\tilde f,g)<\infty\}$, where A is defined by

$$A(x):=N-\lim_{n\to\infty}J^n\tilde f(x)=N-\lim_{n\to\infty}\frac{1}{2^{nj}}\tilde f(2^{nj}x) \quad (30)$$

for all $x\in X$. Moreover,

$$d(\tilde f,A)\leq\frac{1}{1-L}d(\tilde f,J\tilde f)\leq\frac{1}{|\,\alpha_j^p-2^p\,|}.$$

This means that (29) holds. Now we show that A is additive. By (26) and (30), we have:

$$\begin{aligned}N(A(2x)-2A(x),t)&=N(A(2x)-\frac{1}{2^{nj}}\tilde f(2^{nj+1}x)+\frac{1}{2^{nj}}\tilde f(2^{nj+1}x)-2A(x),t)\\ &\geq\min\{N(A(2x)-\frac{1}{2^{nj}}\tilde f(2^{nj+1}x),\frac{t}{2^q}),N(\frac{1}{2^{nj+1}}\tilde f(2^{nj+1}x)-A(x),\frac{t}{2^{q+1}})\}\\ &\to 1\ (n\to\infty)\end{aligned}$$

for all $x\in X$ and $t>0$. So $A(2x)=2A(x)$ for all $x\in X$. Replacing x,y by $2^n x, 2^n y$, respectively in (28) we have:

$$N(\frac{1}{2^n}Df(2^n x,2^n y),t)\geq N'(\varphi(2^n x,2^n y),2^n t)$$

for all $x,y\in X$ and $t>0$. On the other hand, it can be easily verified that,

$$D\tilde f(x,y)=Df(2x,2y)-8Df(x,y)$$

for all $x, y \in X$. Hence,

$$
\begin{aligned}
&N(DA(x,y),t) \\
&= N(A(kx+y)+A(kx-y)-kA(x+y)-kA(x-y)-2A(kx))+2kA(x),t) \\
&\geq \min\{N(A(kx+y)-\frac{\tilde{f}(2^{nj}(kx+y))}{2^{nj}},\frac{t}{8^q}),N(A(kx-y)-\frac{\tilde{f}(2^{nj}(kx-y))}{2^{nj}},\frac{t}{8^q}), \\
&\quad N(A(x+y)-\frac{\tilde{f}(2^{nj}(x+y))}{2^{nj}},\frac{t}{8^q k}),N(A(x-y)-\frac{\tilde{f}(2^{nj}(x-y))}{2^{nj}},\frac{t}{8^q k}), \\
&\quad N(A(kx)-\frac{\tilde{f}(2^{nj}kx)}{2^{nj}},\frac{t}{2^{3q+1}}),N(A(x)-\frac{\tilde{f}(2^{nj}x)}{2^{nj}},\frac{t}{2^{3q+1}k}), \\
&\quad N'(\varphi(2^{nj+1}x,2^{nj+1}y),2^{nj-3q}t),N'(\varphi(2^{nj}x,2^{nj}y),2^{nj-3q-3}t)\}
\end{aligned}
$$

for all $x, y \in X$ and $t > 0$. The first six terms on the right hand side of the above inequality tend to 1 as $n \to \infty$ by (30) and the seventh and eighth terms tend to 1 as $n \to \infty$ by (27). Therefore $N(DA(x,y),t) \to 1$ for all $t > 0$. Then, A satisfies (1). By Lemma 3, the function $x \to A(2x) - 8A(x)$ is additive. Hence $A(2x) = 2A(x)$ implies that A is an additive function.

To prove the uniqueness assertion, let us assume that there exists an additive function $T: X \to Y$ which satisfies (29). Then T is a fixed point of J in Δ. However, by Lemma 4, J has only one fixed point in Δ, hence $A = T$. This completes the proof.

By a modification in the proof of Theorem 1, one can prove the following result:

Theorem 2: Let $j \in \{-1,1\}$ be fixed and $\beta_j > 0$ such that $(\beta_j / 8)^j < 1$. Let $\varphi: X \times X \to Z$ be a function with the following property:

$$N'(\varphi(2^j x, 2^j y), t) \geq N'(\beta_j^j \varphi(x,y), t)$$

and

$$\lim_{n\to\infty} N'(\varphi(2^{nj}x, 2^{nj}y), 8^{nj}t) = 1$$

for all $x, y \in X$ and $t > 0$. Let $f: X \to Y$ be a function with $f(0) = 0$ and satisfies the inequality $N(Df(x,y),t) \geq N'(\varphi(x,y),t)$ for all $x, y \in X$ and $t > 0$. Then there exists a unique cubic function $C: X \to Y$ such that

$$N(f(2x) - 2f(x) - C(x), t) \geq N_1(x, (k^3 - k) | \beta_j^p - 8^p |^q \, t)$$

for all $x \in X$, where N_1 is defined as in Lemma 5.

Now we establish the generalized Hyers-Ulam-Rassias stability of function Equation (1.1) as follows:

Theorem 3: Let $0 < \alpha < 2$ and $\varphi: X \times X \to Z$ be a function with the following property:

$$N'(\varphi(2x,2y),t) \geq N'(\alpha\varphi(x,y),t) \tag{31}$$

and

$$\lim_{n\to\infty} N'(\varphi(2^n x, 2^n y), 2^n t) = 1 \tag{32}$$

for all $x, y \in X$ and $t > 0$. Let $f: X \to Y$ be a function with $f(0) = 0$ and satisfies the inequality

$$N(Df(x,y),t) \geq N'(\varphi(x,y),t) \tag{33}$$

for all $x, y \in X$ and $t > 0$. Then there exist a unique additive function $A: X \to Y$ and a unique cubic function $C: X \to Y$ such that

$$N(f(x) - A(x) - C(x), t) \geq N_1(x, \frac{6(k^3-k)(2^p - \alpha^p)^q t}{2^q}) \tag{34}$$

for all $x \in X$ and $t > 0$, where N_1 is defined as in Lemma 5.

Proof: By theorems 1 and 2, there exists an additive mapping $A_0: X \to Y$ and a cubic mapping $C_0: X \to Y$ such that

$$N(f(2x) - 8f(x) - A_0(x), t) \geq N_1(x, (k^3 - k)(2^p - \alpha^p)^q t) \tag{35}$$

and

$$N(f(2x) - 2f(x) - C_0(x), t) \geq N_1(x, (k^3 - k)(|8^p - \alpha^p)^q t) \tag{36}$$

for all $x \in X$ and $t > 0$, where N_1 is defined as in Lemma 5. Therefore from (35) and (36), we get

$$N(f(x) + \frac{A_0(x)}{6} - \frac{C_0(x)}{6}, t) \geq N_1(x, \frac{6(k^3-k)(2^p - \alpha^p)^q t}{2^q}) \tag{37}$$

for all $x \in X$ and $t > 0$. Letting $A(x) = -A_0(x)/6$ and $C(x) = C_0(x)/6$ for all $x \in X$, it follows from (37) that (34) holds. To prove the uniqueness of A and C, let $A', C': X \to Y$ be different additive and cubic mapping satisfying (34). Let $A_1 = A - A'$ and $C_1 = C - C'$. So

$$
\begin{aligned}
N(A_1(x) + C_1(x), t) &= N(A(x) + C(x) - f(x) + f(x) - A'(x) - C'(x), t) \\
&\geq \min\{N(A(x) + C(x) - f(x), t/2^q), N(f(x) - A'(x) - C'(x), t/2^q)\} \\
&\geq N_1(x, 6(k^3 - k)(2^p - \alpha^p)^q t / 4^q)
\end{aligned}
$$

for all $x \in X$ and $t > 0$. Hence,

$$
\begin{aligned}
N(C_1(x), t) &= N(C_1(x) + A_1(x) - A_1(x), t) \\
&\geq \min\{N(C_1(x) + A_1(x), t/2^q), N(A_1(x), t/2^q)\} \\
&\geq \min\{N_1(x, 6(k^3 - k)(2^p - \alpha^p)^q t / 8^q), N(A_1(x), t/2^q)\}
\end{aligned} \tag{38}
$$

for all $x\in X$ and $t>0$. By $A(2x)=2A(x)$, $A'(2x)=2A'(x)$, $C(2x)=8C(x)$, $C'(2x)=8C'(x)$, (31) and (38), we get

$$\begin{aligned}&N(C_1(x),t)=N(C_1(2^n x),8^n t)\\&\geq \min\{N_1(2^n x,6(k^3-k)(2^p-\alpha^p)^q 8^{n-q}t),N(A_1(x),2^{2n-q}t)\}\\&\geq \min\{N_1(x,6(k^3-k)(2^p-\alpha^p)^q 8^{n-q}t/\alpha^n),N(A_1(x),2^{2n-q}t)\}\\&\to 1\,(n\to\infty)\end{aligned} \tag{39}$$

for all $x\in X$ and $t>0$. Therefore $C_1=0$ by (39) and then $A_1=0$. This completes the proof.

The proof of Theorems 4-5 is similar to the proof of Theorem 3, hence it is omitted.

Theorem 4. Let $\alpha>8$ and $\varphi:X\times X\to Z$ be a function with the following property:

$$N'(\varphi(x/2,y/2),t)\geq N'(\varphi(x,y),\alpha t)$$

and $\lim_{n\to\infty} N'(\varphi(2^{-n}x,2^{-n}y),8^{-n}t)=1$ for all $x,y\in X$ and $t>0$. Let $f:X\to Y$ be a function with $f(0)=0$ and satisfies the inequality, $N(Df(x,y),t)\geq N'(\varphi(x,y),t)$, for all $x,y\in X$ and $t>0$. Then there exists a unique additive function $A:X\to Y$ and a unique cubic function $C:X\to Y$ such that,

$$N(f(x)-A(x)-C(x),t)\geq N_1(x,\frac{6(k^3-k)(\alpha^p-8^p)^q t}{2^q})$$

for all $x\in X$ and $t>0$, where N_1 is defined as in Lemma 5.

Theorem 5: Let $2<\alpha,\beta<8$ and $\varphi:X\times X\to Z$ be a function with the following property:

$$N'(\varphi(x/2,y/2),t)\geq N'(\varphi(x,y),\alpha t),N'(\varphi(2x,2y),t)\geq N'(\beta\varphi(x,y),t)$$

and

$$\lim_{n\to\infty} N'(\varphi(2^{-n}x,2^{-n}y),2^{-n}t)=1,\lim_{n\to\infty} N'(\varphi(2^{n}x,2^{n}y),8^{n}t)=1$$

for all $x,y\in X$ and $t>0$. Let $f:X\to Y$ be a function with $f(0)=0$ and satisfies the inequality, $N(Df(x,y),t)\geq N'(\varphi(x,y),t)$, for all $x,y\in X$ and $t>0$. Then there exists a unique additive function $A:X\to Y$ and a unique cubic function $C:X\to Y$ such that:

$$\begin{aligned}&N(f(x)-A(x)-C(x),t)\\&\geq \min\{N_1(x,\frac{6(k^3-k)(\alpha^p-2^p)^q t}{2^q}),N_1(x,\frac{6(k^3-k)(8^p-\beta^p)^q t}{2^q})\}\end{aligned}$$

for all $x\in X$ and $t>0$, where N_1 is defined as in Lemma 5.

Applications of fuzzy stability

Here, we investigate the applications of fuzzy stability to the stability of general mixed additive-cubic functional equation in quasi p-normed spaces. Hereafter, we will assume that X is a linear space and Y is a p-Banach space with p-norm $\|\cdot\|_Y$. We will apply our results in Section 3 to obtain the stability of almost mixed additive-cubic mappings from X to Y.

Theorem 6: Let $\varphi:X\times X\to[0,\infty)$ be a function such that one of the following holds:

(i) for some $\alpha<2$, $\varphi(2x,2y)\leq\alpha\varphi(x,y)$ and $\lim_{n\to\infty} 2^{-n}\varphi(2^n x,2^n y)=0$ for all $x,y\in X$;

(ii) for some $\alpha>8$, $\alpha\varphi(x,y)\leq\varphi(2x,2y)$ and $\lim_{n\to\infty} 2^{3n}\varphi(2^{-n}x,2^{-n}y)=0$ for all $x,y\in X$;

(iii) for some $2<\alpha,\beta<8$, $\alpha\varphi(x,y)\leq\varphi(2x,2y)$, $\varphi(2x,2y)\leq\beta\varphi(x,y)$ and $\lim_{n\to\infty} 2^{n}\varphi(2^{-n}x,2^{-n}y)=0$ and $\lim_{n\to\infty} 2^{-3n}\varphi(2^{n}x,2^{n}y)=0$ for all $x,y\in X$.

Let $x,y\in X$ be a function with $f(0)=0$ and satisfies the inequality $\|Df(x,y)\|_Y\leq\varphi(x,y)$ for all $x,y\in X$. Then there exist a unique additive function $A:X\to Y$ and a unique cubic function $C:X\to Y$ such that:

$$\|f(x)-A(x)-C(x)\|_Y\leq\begin{cases}\dfrac{2^q\mu(x)}{6(k^3-k)(2^p-\alpha^p)^q}, & 0<\alpha<2;\\ \dfrac{2^q\mu(x)}{6(k^3-k)(\alpha^p-8^p)^q}, & \alpha>8;\\ \dfrac{2^q\mu(x)}{6(k^3-k)(\alpha^p-2^p)^q}, & 2<\alpha<8, 2<\beta<(8^p-2^p-\alpha^p)^q;\\ \dfrac{2^q\mu(x)}{6(k^3-k)(8^p-\beta^p)^q}, & 2<\alpha<8, \beta>(8^p-2^p-\alpha^p)^q,\end{cases}$$

for all $x\in X$, where,

$$\mu(x) = \max\{384^q k\varphi(x,(2k+1)x), 384^q k\varphi(x,(2k-1)x), 384^q \varphi(x,3kx), \frac{384^q k}{k-1}\varphi(0,(3k-1)x),$$
$$96^q k^2\varphi(2x,2x), \frac{96^q k}{k-1}\varphi(0,(k+1)x), 96^q\varphi(x,kx), \frac{96^q k^2}{k-1}\varphi(0,2(k-1)x), 96^q(k+1)\varphi(0,2kx),$$
$$16^q\varphi(2x,2kx), 56^q\varphi(3x,x), 32^q\varphi(2x,x), 128^q\varphi(x,(k+1)x), 128^q\varphi(x,(k-1)x), \frac{128^q}{k-1}\varphi(0,x),$$
$$16^q\varphi(2x,kx), 384^q k\varphi(\frac{x}{2},\frac{(2k+1)x}{2}), 384^q k\varphi(\frac{x}{2},\frac{(2k-1)x}{2}), 384^q\varphi(\frac{x}{2},\frac{3kx}{2}), \frac{384^q k}{k-1}\varphi(0,\frac{(3k-1)x}{2}),$$
$$96^q k^2\varphi(x,x), \frac{96^q k}{k-1}\varphi(0,\frac{(k+1)x}{2}), 96^q\varphi(\frac{x}{2},\frac{kx}{2}), \frac{96^q k^2}{k-1}\varphi(0,(k-1)x), 96^q(k+1)\varphi(0,kx)\}.$$

Proof: Consider the fuzzy quasi p-norm defined as in Example 1, and apply Theorems 3 to 5.

Corollary 1: Let X be a quasi-normed space with quasi-norm $\|\cdot\|_X$ and let ε, r be non-negative real numbers such that $r \in (0,1)\cup(1,3)\cup(3,\infty)$. Let $f: X \to Y$ be a function with $f(0)=0$ and satisfies the inequality $\| Df(x,y)\|_Y \le \varepsilon(\|x\|_X^r + \|y\|_X^r)$ for all $x, y \in X$. Then there exists a unique additive function $A: X \to Y$ and a unique cubic function $C: X \to Y$ such that:

$$\| f(x)-A(x)-C(x)\|_Y \le \begin{cases} \dfrac{768^q k^2 \varepsilon \|x\|_X^r}{(k^3-k)(2^p-2^{rp})^q}, & r\in(0,1); \\ \dfrac{768^q\cdot 3^{r-1}k^{r+1}\varepsilon\|x\|_X^r}{(k^3-k)(2^{rp}-8^p)^q}, & r\in(3,\infty); \\ \dfrac{768^q\cdot 3^{r-1}k^{r+1}\varepsilon\|x\|_X^r}{(k^3-k)(2^{rp}-2^p)^q}, & r\in(1,\dfrac{\ln(8^p-2^p)-\ln 2}{p\ln 2}); \\ \dfrac{768^q\cdot 3^{r-1}k^{r+1}\varepsilon\|x\|_X^r}{(k^3-k)(8^p-2^{rp})^q}, & r\in(\dfrac{\ln(8^p-2^p)-\ln 2}{p\ln 2},3), \end{cases}$$

for all $x \in X$.

Proof: Define $\varphi(x,y) = \varepsilon(\|x\|_X^r + \|y\|_X^r)$ for all $x, y \in X$, and apply theorem 6.

Corollary 2: Let X be a quasi-normed space with quasi-norm $\|\cdot\|_X$ and let ε, r, s be non-negative real numbers such that $\lambda := r+s \in (0,1)\cup(1,3)\cup(3,\infty)$. Let $f: X\to Y$ be a function with $f(0)=0$ and satisfies the inequality $\| Df(x,y)\|_Y \le \varepsilon\|x\|_X^r\|y\|_X^s$ for all $x, y\in X$. Then there exist a unique additive function $A: X\to Y$ and a unique cubic function $C: X\to Y$ such that:

$$\| f(x)-A(x)-C(x)\|_Y \le \begin{cases} \dfrac{768^q k^2 \varepsilon \|x\|_X^\lambda}{2(k^3-k)(2^p-2^{\lambda p})^q}, & \lambda\in(0,1); \\ \dfrac{768^q\cdot 3^{\lambda-1}k^{\lambda+1}\varepsilon\|x\|_X^\lambda}{2(k^3-k)(2^{\lambda p}-8^p)^q}, & \lambda\in(3,\infty); \\ \dfrac{768^q\cdot 3^{\lambda-1}k^{\lambda+1}\varepsilon\|x\|_X^\lambda}{2(k^3-k)(2^{\lambda p}-2^p)^q}, & \lambda\in(1,\dfrac{\ln(8^p-2^p)-\ln 2}{p\ln 2}); \\ \dfrac{768^q\cdot 3^{\lambda-1}k^{\lambda+1}\varepsilon\|x\|_X^\lambda}{2(k^3-k)(8^p-2^{\lambda p})^q}, & \lambda\in(\dfrac{\ln(8^p-2^p)-\ln 2}{p\ln 2},3), \end{cases}$$

for all $x \in X$.

Proof: Define $\varphi(x,y) = \varepsilon\|x\|_X^r\|y\|_X^s$ for all $x, y\in X$, and apply Theorem 6. This product stability function φ was introduced by Rassias (1989, 1982).

Corollary 3: Let X be a quasi-normed space with quasi-norm $\|\cdot\|_X$ and let ε, r, s be non-negative real numbers such that $\lambda := r+s\in(0,1)\cup(1,3)\cup(3,\infty)$. Let $f: X\to Y$ be a function with $f(0)=0$ and satisfies the inequality $\| Df(x,y)\|_Y \le \varepsilon[\|x\|_X^r\|y\|_X^s + (\|x\|_X^{r+s} + \|y\|_X^{r+s})]$ for all $x, y\in X$. Then there exists a unique additive function $A: X\to Y$ and a unique cubic function $C: X\to Y$ such that:

$$\left\| f(x)-A(x)-C(x)\right\|_Y \le \begin{cases} \dfrac{768^q\cdot 3k^2\varepsilon\left\|x\right\|_X^\lambda}{2(k^3-k)(2^p-2^{\lambda p})^q}, & \lambda\in(0,1); \\ \dfrac{768^q\cdot 3^\lambda k^{\lambda+1}\varepsilon\left\|x\right\|_X^\lambda}{2(k^3-k)(2^{\lambda p}-8^p)^q}, & \lambda\in(3,\infty); \\ \dfrac{768^q\cdot 3^\lambda k^{\lambda+1}\varepsilon\left\|x\right\|_X^r}{2(k^3-k)(2^{\lambda p}-2^p)^q}, & \lambda\in(1,\dfrac{\ln(8^p-2^p)-\ln 2}{p\ln 2}); \\ \dfrac{768^q\cdot 3^\lambda k^{\lambda+1}\varepsilon\left\|x\right\|_X^\lambda}{2(k^3-k)(8^p-2^{\lambda p})^q}, & \lambda\in(\dfrac{\ln(8^p-2^p)-\ln 2}{p\ln 2},3), \end{cases}$$

for all $x\in X$.

Proof: Define
$\varphi(x,y) = \varepsilon[\|x\|_X^r\|y\|_X^s + (\|x\|_X^{r+s} + \|y\|_X^{r+s})]$
for all $x, y\in X$, and apply Theorem 6. This mixed product-sum stability function φ was introduced by Rassias in 2008.

Fuzzy continuity

Here, we will establish some interesting results of continuous approximately general mixed additive-cubic mappings in quasi fuzzy p-normed spaces.

Definition 4 (Mirmostafaee, 2009): Let (Y,N) be a (quasi) fuzzy normed space, f: R $\to Y$ be a function and $0<\beta<1$. Then f is said to be β-fuzzy continuous, if for each $t>0$, there is some $\delta>0$ such that $N(f(\mu)-f(\mu_0),t)\ge\beta$ for each μ with $|\mu-\mu_0|<\delta$. f is called fuzzy continuous if it is β-fuzzy continuous for each $0<\beta<1$.

Hereafter, unless otherwise stated, we will assume that $0<p<1$ and $q=1/p$, (Y,N) is a quasi fuzzy p-Banach space and (Z,N') is a fuzzy normed space.

Theorem 7: Let X be a normed space with norm $\|\cdot\|_X$. Let $z_0 \in Z$ and r be a non-negative real number such that $r \in (0,1) \cup (1,3) \cup (3,\infty)$. Suppose that a function $f: X \to Y$ with $f(0)=0$ satisfies the inequality $N(Df(x,y),t) \geq N'((\|x\|_X^r + \|y\|_X^r)z_0, t)$ for all $x, y \in X$ and $t>0$. Then there exists a unique additive function $A: X \to Y$ and a unique cubic function $C: X \to Y$ such that:

$$N(f(x)-A(x)-C(x),t) \geq \begin{cases} N'(\|x\|_X^r z_0, \dfrac{(k^3-k)(2^p-2^{rp})^q}{768^q k^2} t, & r \in (0,1); \\ N'(\|x\|_X^r z_0, \dfrac{(k^3-k)(2^{rp}-8^p)^q}{768^q \cdot 3^{r-1} k^{r+1}} t, & r \in (3,\infty); \\ N'(\|x\|_X^r z_0, \dfrac{(k^3-k)(2^{rp}-2^p)^q}{768^q \cdot 3^{r-1} k^{r+1}} t), & r \in (1, \dfrac{\ln(8^p-2^p)-\ln 2}{p \ln 2}); \\ N'(\|x\|_X^r z_0, \dfrac{(k^3-k)(8^p-2^{rp})^q}{768^q \cdot 3^{r-1} k^{r+1}}), & r \in (\dfrac{\ln(8^p-2^p)-\ln 2}{p \ln 2}, 3), \end{cases} \tag{1}$$

for all $x \in X$ and $t>0$. Furthermore, if for each $x \in X$ and all $n \in$ N, the function $g:$ R $\to Y$ defined by $g(\mu) = f(2^n \mu x)$ is fuzzy continuous. Then the functions $\mu \to A(\mu x)$ and $\mu \to C(\mu x)$ are fuzzy continuous for each $x \in X$ and $A(\gamma x) = \gamma A(x)$ and $C(\gamma x) = \gamma^3 C(x)$ for each $x \in X$ and $\gamma \in$ R.

Proof: Let $r \in (0,1)$. If we define $\varphi(x,y) = (\|x\|_X^r + \|y\|_X^r) z_0$ for all $x, y \in X$. Existence and uniqueness of the additive function A and cubic function C satisfying (1) are deduced from Theorem 3. Note that for each $x \in X, t \in$ R, and $n \in$ N, by the proof of Theorem 3, we have

$$\begin{aligned} N(A(x) + \frac{f(2^{n+1}x) - 8f(2^n x)}{6 \cdot 2^n}, \frac{t}{6}) &= N(A(2^n x) + \frac{f(2^{n+1}x) - 8f(2^n x)}{6}, \frac{2^n t}{6}) \\ &\geq N_1(2^n x, (k^3-k)(2^p - 2^{rp})^q 2^n t) \\ &= N_1(x, \frac{(k^3-k)(2^p-2^{rp})^q 2^n t}{2^{nr}}) \end{aligned} \tag{2}$$

and

$$\begin{aligned} N(C(x) + \frac{f(2^{n+1}x) - 2f(2^n x)}{6 \cdot 8^n}, \frac{t}{6}) &= N(C(2^n x) + \frac{f(2^{n+1}x) - 2f(2^n x)}{6}, \frac{8^n t}{6}) \\ &\geq N_1(2^n x, (k^3-k)(8^p - 2^{rp})^q 8^n t) \\ &= N_1(x, \frac{(k^3-k)(8^p-2^{rp})^q 8^n t}{2^{nr}}) \end{aligned} \tag{3}$$

Fix $x \in X$ and $\mu_0 \in$ R. Given $\varepsilon > 0$ and $0 < \beta < 1$. From (2) and (3) it follows that:

$$\begin{aligned} N(A(\mu x) + \frac{f(2^{n+1}\mu x) - 8f(2^n \mu x)}{6 \cdot 2^n}, \frac{t}{6}) &\geq N_1(x, \frac{(k^3-k)(2^p-2^{rp})^q 2^n t}{|\mu|^r 2^{nr}}) \\ &\geq N_1(x, \frac{(k^3-k)(2^p-2^{rp})^q 2^n t}{(1+|\mu_0|)^r 2^{nr}}) \end{aligned}$$

and

$$\begin{aligned} N(C(\mu x) + \frac{f(2^{n+1}\mu x) - 2f(2^n \mu x)}{6 \cdot 8^n}, \frac{t}{6}) &\geq N_1(x, \frac{(k^3-k)(8^p-2^{rp})^q 8^n t}{|\mu|^r 2^{nr}}) \\ &\geq N_1(x, \frac{(k^3-k)(8^p-2^{rp})^q 8^n t}{(1+|\mu_0|)^r 2^{nr}}) \end{aligned}$$

for all $|\mu - \mu_0| < 1$ and $\mu \in$ R. Since,

$$\lim_{n\to\infty} \frac{(k^3-k)(2^p-2^{rp})^q 2^n t}{(1+|\mu_0|)^r 2^{nr}} = \lim_{n\to\infty} \frac{(k^3-k)(8^p-2^{rp})^q 8^n t}{(1+|\mu_0|)^r 2^{nr}} = \infty$$

there exists $n_0 \in$ N such that,

$$N(A(\mu x) + \frac{f(2^{n+1}\mu x) - 8f(2^n \mu x)}{6 \cdot 2^n}, \frac{\varepsilon}{3^q}) \geq \beta,$$

and

$$N(C(\mu x) + \frac{f(2^{n+1}\mu x) - 2f(2^n \mu x)}{6 \cdot 8^n}, \frac{\varepsilon}{3^q}) \geq \beta$$

for all $|\mu - \mu_0| < 1$ and $\mu \in$ R. By the fuzzy continuity of the mapping $\mu \to f(2^{n_0}\mu x)$, there exists $\delta < 1$ such that for each μ with $0 < |\mu - \mu_0| < 1$, we have,

$$N(\frac{f(2^{n_0+1}\mu x) - 8f(2^{n_0}\mu x)}{6 \cdot 2^{n_0}} - \frac{f(2^{n_0}\mu_0 x) - 8f(2^{n_0}\mu_0 x)}{6 \cdot 2^{n_0}}, \frac{\varepsilon}{3^q}) \geq \beta,$$

and

$$N(\frac{f(2^{n_0+1}\mu x) - 2f(2^{n_0}\mu x)}{6 \cdot 8^{n_0}} - \frac{f(2^{n_0}\mu_0 x) - 2f(2^{n_0}\mu_0 x)}{6 \cdot 8^{n_0}}, \frac{\varepsilon}{3^q}) \geq \beta,$$

It follows that,

$$\begin{aligned} N(A(\mu x) - A(\mu_0 x), \varepsilon) \geq \min\{ & N(A(\mu x) + \frac{f(2^{n_0+1}\mu x) - 8f(2^{n_0}\mu x)}{6 \cdot 2^{n_0}}, \frac{\varepsilon}{3^q}), \\ & N(\frac{f(2^{n_0+1}\mu x) - 8f(2^{n_0}\mu x)}{6 \cdot 2^{n_0}} - \frac{f(2^{n_0+1}\mu_0 x) - 8f(2^{n_0}\mu_0 x)}{6 \cdot 2^{n_0}}, \frac{\varepsilon}{3^q}), \\ & N(A(\mu_0 x) + \frac{f(2^{n_0+1}\mu_0 x) - 8f(2^{n_0}\mu_0 x)}{6 \cdot 2^{n_0}}, \frac{\varepsilon}{3^q})\} \geq \beta \end{aligned}$$

and $N(C(\mu x)-C(\mu_0 x),\varepsilon)\geq\beta$ for each μ with $0<|\mu-\mu_0|<\delta$. Hence, the functions $\mu\to A(\mu x)$ and $\mu\to C(\mu x)$ are fuzzy continuous.

Now, we use the fuzzy continuity of $\mu\to A(\mu x)$ and $\mu\to C(\mu x)$ to establish that $A(\gamma x)=\gamma A(x)$ and $C(\gamma x)=\gamma^3 C(x)$ for each $x\in X$ and $\gamma\in$ R. By induction on n, one can easily prove that $A(n x)=n A(x)$ and $C(n x)=n^3 C(x)$ for every natural number $n\in$ N. It follows that,

$$A(\frac{n}{m}x)=nA(\frac{1}{m}x)=\frac{n}{m}A(x), C(\frac{n}{m}x)=n^3C(\frac{1}{m}x)=(\frac{n}{m})^3C(x)$$

for all $m,n\in$ N and $x\in X$. Hence for every rational number $\gamma\in$ Q, we have $A(\gamma x)=\gamma A(x)$ and $C(\gamma x)=\gamma^3 C(x)$. Let γ be a real number, then there exists a sequence $\{\gamma_n\}$ of rational numbers such that $\gamma_n\to\gamma$. By the fuzzy continuity of $A(\bullet x)$ and $C(\bullet x)$, for every $x\in X$,

$$A(\gamma x)=\lim_{n\to\infty}A(\gamma_n x)=\lim_{n\to\infty}\gamma_n A(x)=\gamma A(x),\ C(\gamma x)=\lim_{n\to\infty}C(\gamma_n x)=\lim_{n\to\infty}\gamma_n^3 C(x)=\gamma^3 C(x).$$

For $r\in(1,3)$ or $r\in(3,\infty)$, we can prove the theorem by a similar technique.

Theorem 8: Let X be a normed space with norm $\|\cdot\|_X$. Let $z_0\in Z$ and r,s be non-negative real number such that $\lambda:=r+s\in(0,1)\cup(1,3)\cup(3,\infty)$. Suppose that a function $f:X\to Y$ with $f(0)=0$ satisfies the inequality

$$N(Df(x,y),t)\geq N'([\|x\|_X^r\ \|y\|_X^s+(\|x\|_X^{r+s}+\|y\|_X^{r+s})]z_0,t)$$

for all $x,y\in X$ and $t>0$. Then there exists a unique additive function $A:X\to Y$ and a unique cubic function $C:X\to Y$ such that,

$$N(f(x)-A(x)-C(x),t)\geq\begin{cases} N'(\|x\|_X^\lambda z_0,\dfrac{2(k^3-k)(2^p-2^{\lambda p})^q}{768^q\cdot 3k^2}t, & \lambda\in(0,1);\\ N'(\|x\|_X^\lambda z_0,\dfrac{2(k^3-k)(2^{\lambda p}-8^p)^q}{768^q\cdot 3^\lambda k^{\lambda+1}}t, & \lambda\in(3,\infty);\\ N'(\|x\|_X^\lambda z_0,\dfrac{2(k^3-k)(2^{\lambda p}-2^p)^q}{768^q\cdot 3^\lambda k^{\lambda+1}}t), & \lambda\in(1,\dfrac{\ln(8^p-2^p)-\ln 2}{p\ln 2});\\ N'(\|x\|_X^\lambda z_0,\dfrac{2(k^3-k)(8^p-2^{\lambda p})^q}{768^q\cdot 3^\lambda k^{\lambda+1}}), & \lambda\in(\dfrac{\ln(8^p-2^p)-\ln 2}{p\ln 2},3),\end{cases}$$

for all $x\in X$ and $t>0$. Furthermore, if for each $x\in X$ and all $n\in$ N, the function $g:$ R $\to Y$ defined by $g(\mu)=f(2^n\mu x)$ is fuzzy continuous. Then the functions $\mu\to A(\mu x)$ and $\mu\to C(\mu x)$ are fuzzy continuous for each $x\in X$ and $A(\gamma x)=\gamma A(x)$ and for each $x\in X$ and $\gamma\in$ R.

Proof: Define $\varphi(x,y)=[\|x\|_X^r\ \|y\|_X^s+(\|x\|_X^{r+s}+\|y\|_X^{r+s})]z_0$ for all $x,y\in X$. The proof can be done on the same lines as in Theorem 7.

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