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Full Length Research Paper

Differential transform method for nonlinear fractional gas dynamics equation

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In the present study, an analytic solution of nonlinear fractional gas dynamics equation was deduced with the help of the powerful differential transform method (DTM). To illustrate the ability and efficiency of the method, two special cases of the equation has been solved

Key words: Differential transform method, nonlinear fractional gas dynamics equation.

INTRODUCTION

In recent years, considerable interest in fractional differential equations has been stimulated due to their numerous applications in the areas of physics and engineering (West, 2003). Many important phenomena in electromagnetic, acoustics, viscoelasticity, and electrochemistry and material science are well described by differential equations of fractional order (Miller, 1993; Samko, 1993; Podlubny, 1999; Caputo, 1967). A homogeneous nonlinear fractional gas dynamics equation can be written as follows:

$$\frac{\partial^\alpha u}{\partial t^\alpha}+\frac{1}{2}(u^2)_x -u(1-u)=0, t>0, \quad 0<\alpha\le 1, \tag{1}$$

with initial condition $u(x,0)=g(x)$. In case of $\alpha=1,$, Equation (1) reduces to the classical gas dynamics equation (Adomian, 1994; Evans, 2002). The purpose of this paper is to obtain analytic solution of this equation by DTM. The differential transform method was first introduced by Zhou (1986) who solved linear and nonlinear initial value problems in electric circuit analysis. This method constructs an analytical solution in form of polynomial expressions such as Taylor series expansion. But procedure is easier than the traditional higher order Taylor series method, which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method is computationally expensive for higher orders. The differential transform is an iterative procedure for obtaining analytic Taylor series solution of ordinary or partial differential equations.

BASIC DEFINITIONS

Here, some basic definitions and properties of the fractional calculus theory which can be found in Podlubny (1999); Caputo (1967).

Definition 1. A real function $f(x), x>0,$ in the space $C_\mu, \mu\in R$ if there exists a real number $p>\mu,$ such that $f(x)=x^p f_1(x)$ where $f_1(x)\in C[0,\infty]$ and it is said to be in the space C_μ^m if $f^{(m)}\in C_\mu, m\in N.$

Definition 2. The Riemann–Liouville fractional integral operator of order $\alpha\ge 0,$ of a function $f\in C_\mu, \mu\ge -1,$ is defined as

$$J^\alpha f(x)=$$

$$\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)dt, \quad \alpha>0, x>0,$$

$$J^0 f(x)=f(x).$$

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The properties of the operator J^{α} can be found in Momani (2006), and we only mentioned the following (in this case, $f \in C_{\mu}, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma > -1$):

(1) $J^{\alpha} J^{\beta} f(x) = J^{\alpha+\beta} f(x),$

(2) $J^{\alpha} J^{\beta} f(x) = J^{\beta} J^{\alpha} f(x),$

(3) $J^{\alpha} x^{\gamma} = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}.$

The Riemann–Liouville derivative has certain disadvantages, in trying way to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D_{*}^{α} proposed by M. Caputo, in his work on the theory of viscoelasticity.

Definition 3. The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$D_{*}^{\alpha} f(x) = J^{m-\alpha} D^{m} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{0}^{x} (x-t)^{m-\alpha-1} f^{(m)}(t) dt,$$

for $m-1 < \alpha \leq m, m \in N, x > 0, f \in C_{-1}^{m}.$

The following two properties of this operator will be used in what follows;

Lemma 1. If $m-1 < \alpha \leq m,$ and $f \in C_{\mu}^{m}, \mu \geq -1,$ then $D_{*}^{\alpha} J^{\alpha} f(x) = f(x),$ and $J^{\alpha} D_{*}^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} f(0^{+}) \dfrac{x^{k}}{k!}, x > 0.$

The Caputo fractional derivative is considered here, because it allows traditional initial and boundary conditions to be included in the formulation of the problem. In this study, we have considered nonlinear fractional gas dynamics equation, where the unknown function $u = u(x,t)$ is assumed to be a causal function of fractional derivatives taken in Caputo sense as follows:

Definition 4. For m as the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha > 0,$ is defined as

$$D_{*t}^{\alpha} u(x,t) = \frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_{0}^{t} (t-\tau)^{m-\alpha-1} \frac{\partial^{m} u(x,\tau)}{\partial \tau^{m}} d\tau, m-1 < \alpha < m, \\ \dfrac{\partial^{m} u(x,t)}{\partial t^{m}}, \qquad \alpha = m \in N. \end{cases}$$

For more information on the mathematical properties of fractional derivatives and integrals, one can consult the mentioned references.

GENERALIZED TWO-DIMENSIONAL DIFFERENTIAL TRANSFORM METHOD

DTM is an analytic method based on the Taylor series expansion which constructs an analytical solution in the form of a polynomial. The traditional high order Taylor series method requires symbolic computation. However, the DTM obtains a polynomial series solution by means of an iterative procedure. The method is well addressed by Odibat (2006). The proposed method is based on the combination of the classical two dimensional DTM and generalized Taylor's Table 1 formula.

Consider a function of two variables $u(x, y)$, and suppose that it can be represented as a product of two single-variable functions, that is, $u(x, y) = f(x) g(y)$. Based on the properties of generalized two-dimensional differential transform (Jang, 2001; Kangalgil, 2009; Ravi, 2009; Arikoglu, 2009), the function $u(x, y)$ can be represented as:

$$u(x, y) = \sum_{k=0}^{\infty} F_{\alpha}(k)(x-x_0)^{k\alpha} \sum_{h=0}^{\infty} G_{\beta}(h)(y-y_0)^{h\beta} = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U_{\alpha,\beta}(k,h)(x-x_0)^{k\alpha}(y-y_0)^{h\beta}, \quad (2)$$

where $0 < \alpha, \beta \leq 1, U_{\alpha,\beta}(k,h) = F_{\alpha}(k) G_{\beta}(h)$ is called the spectrum of $u(x, y)$. The generalized two-dimensional differential transform of the function $u(x, y)$ is given by

$$U_{\alpha,\beta}(k,h) = \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)} \times \left[(D_{*x_0}^{\alpha})^{k} (D_{*y_0}^{\beta})^{h} u(x,y) \right]_{(x_0,y_0)}, \quad (3)$$

where $(D_{*x_0}^{\alpha})^{k} = \underbrace{D_{*x_0}^{\alpha} \ldots D_{*x_0}^{\alpha}}_{k}.$

In case of $\alpha = 1,$ and $\beta = 1$ the generalized two-dimensional differential transform (2) reduces to the classical two-dimensional differential transform. Let $U_{\alpha,\beta}(k,h), G_{\alpha,\beta}(k,h), V_{\alpha,\beta}(k,h)$ and $H_{\alpha,\beta}(k,h)$ are the differential transformations of the functions $u(x, y), g(x, y), v(x, y)$ and $h(x, y),$ from Equations (2) and (3), some basic properties of the two-dimensional differential transform are introduced in Table 1. Then, the

Table 1. The operations for the two-dimensional differential transform method**.**

Original function	Transformed function
$u(x,y)=g(x,y)\pm h(x,y)$	$U_{\alpha,\beta}(k,h)=G_{\alpha,\beta}(k,h)\pm H_{\alpha,\beta}(k,h)$
$u(x,y)=\lambda g(x,y)$	$U_{\alpha,\beta}(k,h)=\lambda G_{\alpha,\beta}(k,h)$
$u(x,y)=g(x,y)h(x,y)$	$U_{\alpha,\beta}(k,h)=\sum_{r=0}^{k}\sum_{s=0}^{h}G_{\alpha,\beta}(r,h-s)H_{\alpha,\beta}(k-r,s)$
$u(x,y)=(x-x_0)^{m\alpha}(y-y_0)^{n\beta}$	$U_{\alpha,\beta}(k,h)=\delta(k-m,h-n)=\begin{cases}1, & k=m,h=n\\ o & otherwise\end{cases}$
$u(x,y)=g(x,y)h(x,y)v(x,y)$	$U_{\alpha,\beta}(k,h)=\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}G_{\alpha,\beta}(r,h-s-p)H_{\alpha,\beta}(t,s)V_{\alpha,\beta}(k-r-t,p)$
$u(x,y)=D_{*x_0}^{\alpha}g(x,y), 0<\alpha\le 1$	$U_{\alpha,\beta}(k,h)=\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)}G_{\alpha,\beta}(k+1,h)$

generalized differential transform (3) becomes;

$$U_{\alpha,\beta}(k,h)=\frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}\left[D_{*x_0}^{\alpha k}(D_{*y_0}^{\beta})^h u(x,y\right]_{(x_0,y_0)}.$$

If $u(x,y)=D_{x_0}^{\gamma}v(x,y), m-1<\gamma\le m,$ and $v(x,y)=f(x)f(y),$ then

$$U_{\alpha,\beta}(k,h)=\frac{\Gamma(\alpha k+\gamma+1)}{\Gamma(\alpha k+1)}V_{\alpha,\beta}(k+\gamma/\alpha,h).$$

The proofs of these properties can be found in Momani (2006), Jang (2001), Momani (2007).

NUMERICAL EXAMPLE

Here, differential transform method (DTM) will be applied for solving nonlinear gas dynamics equation. The results reveal that the method is very effective and simple.

Example 1. Consider the following gas dynamics equation with the following initial condition

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}+\frac{1}{2}(u^2)_x-u(1-u)=0, \quad (4)$$

$$u(x,0)=e^{-x}. \quad (5)$$

Taking the differential transform of (4), leads to;

$$\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)}U_{\alpha,1}(k,h+1)$$
$$+\sum_{r=0}^{k}\sum_{s=0}^{h}(r+1)U_{\alpha,1}(r+1,h-s)U_{\alpha,1}(k-r,s)$$
$$-U_{\alpha,1}(k,h)+\sum_{r=0}^{k}\sum_{s=0}^{h}U_{\alpha,1}(r,h-s)U_{\alpha,1}(k-r,s).$$

From the initial condition given by Equation (5), we obtained;

$$U_{\alpha,1}(k,0)=\frac{(-1)^k}{k!}, \qquad k=0,1,2,\ldots$$

Substituting all $U(k,h)$ into Equation (2), the series solution form will be obtained;

$$u(x,t)=(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}\ldots)$$
$$\times(1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\ldots)$$
$$=e^{-x}\sum_{k=0}^{\infty}\frac{(t^{\alpha})^k}{\Gamma(k\alpha+1)}.$$

As $\alpha=1$, this series has the closed form e^{t-x}, which is an exact solution of the classical gas dynamics equation.

Example 2. In this example, consider inhomogeneous fractional gas dynamics equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{1}{2}(u^2)_x + (1+t)^2 u^2 = x^2, \tag{6}$$

with initial condition,

$$u(x,0) = x. \tag{7}$$

One can readily find the differential transform of (6), as follows,

$$\begin{aligned} &\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)} U_{\alpha,1}(k,h+1) \\ &+ \sum_{r=0}^{k}\sum_{s=0}^{h}(r+1)U_{\alpha,1}(r+1,h-s)U_{\alpha,1}(k-r,s) \\ &+ U_{\alpha,\beta}(k,h) + \sum_{r=0}^{k}\sum_{s=0}^{h}U_{\alpha,1}(r,h-s)U_{\alpha,1}(k-r,s) \\ &+ \sum_{r=0}^{k}\sum_{s=0}^{k-r}\sum_{t=0}^{h}\sum_{p=0}^{h-s}U_{\alpha,1}(r,h-s-p)U_{\alpha,1}(t,s)\delta(k-r-t,p-2) \\ &+ 2\sum_{r=0}^{k}\sum_{s=0}^{k-r}\sum_{t=0}^{h}\sum_{p=0}^{h-s}U_{\alpha,1}(r,h-s-p)U_{\alpha,1}(t,s)\delta(k-r-t,p-1) \\ &= \delta(k-2,h). \end{aligned}$$

From the initial condition (7) we can write;

$$U(k,0) = \begin{cases} 1, & for\, k = 1, \\ 0, & for\, k = 0,2,\ldots \end{cases}$$

Consequently, substituting all $U(k,h)$ into Equation (2), we obtain the series form solution of Equations (6) and (7) as;

$$\begin{aligned} u(x,t) &= (x)(1 - t^\alpha + t^{2\alpha} - t^{3\alpha} + \ldots) \\ &= x\sum_{k=0}^{\infty}(-t^\alpha)^k. \end{aligned}$$

For special case $\alpha = 1$, the solution will be as follows;

$$u(x,t) = \frac{x}{1+t},$$

which is an exact solution.

CONCLUSION

In this study, application of DTM to fractional gas dynamics equation has been presented successfully. The results show that differential transform method is a powerful and efficient technique for finding analytic solutions for nonlinear partial differential equations of fractional order. The obtained results reinforce the conclusions made by many researchers about the efficiency of DTM. In this study, we used the Maple Package, to calculate the series obtained by differential transform method.

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