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Review

Auto-Bäcklund transformation and singular solutions for the generalized coupled Camassa-Holm equation

Lu Sun* and Lixin Tian

Nonlinear Scientific Research Center, Postdoctor Research Station of Management Science and Engineering, Jiangsu University, Zhenjiang, Jiangsu, 212013, People Republic of China.

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Auto-Bäcklund transformation and Weiss-Tabor-Carnevale (WTC)-kruskal algorithm are applied for the coupled Camassa-Holm equation in this paper. Four regular solitons including blow-up waves and compactons are given by the ansatz method. Then, an auto-Bäcklund transformation of the coupled equation is obtained. Thus, we observe that based on the solitons we get before, some singular solitons are derived which we have not found before.

Key words: Coupled Camassa-Holm equation, Weiss-Tabor-Carnevale (WTC) test, auto-Bäcklund transformation.

INTRODUCTION

Lots of investigations are implemented into the generalized form of the coupled Camassa-Holm shallow water system, which includes integrable problems, solitonic problems, well-posedness problems, blow-up phenomena, Cauchy problems, etc. A thorough study is presented on solitonic structure problems for generalized form of the coupled Camassa-Holm shallow water system:

$$\begin{cases} m_t + 2u_x m + u m_x + \sigma\rho\rho_x = 0, t > 0, x \in R, \\ \rho_t + (u\rho)_x = 0, t > 0, x \in R, \end{cases} \quad (1)$$

where $m = u - u_{xx}, \sigma = \pm 1.$

Equation 1 was recently derived by Constantin and Ivanov (2008) in the context of shallow water theory. The variable $u(x,t)$ describes the horizontal velocity of the fluid and the variable $\rho(x,t)$ is in connection with the horizontal deviation of the surface from equilibrium, all measured in dimensionless units. Partha and Peter (2006) investigated the geodesic flow equation. Ming et al. (2006) derived a coupled generalization of the Camassa-Holm equation and its solutions. Henrik et al. (2006) discussed the negative flow of the Ablowitz-Kaup-Newell-Segur (AKNS) hierarchy and its relation to a two component Camassa-Holm equation. Ziemowit (2006) investigated a two component supersymmetric Camassa-Holm equation.

Escher et al. (2007) investigated the well-posedness problems and the blow-up phenomena for the two component Camassa-Holm equation. Constantin and Ivannov (2008) investigated an integrable two component Camassa-Holm shallow water system. Guilong and Yue (2003) discussed the Cauchy problem of the two component Camassa-Holm system. Hu and YIN (2010) investigated the well posedness and blow up phenomena for the periodic two component Camassa-Holm equation. Qiaoyi and Zhaoyang (2011) discussed the global existence and the blow up phenomena of the periodic two component Camassa-Holm equation. Guo and Zhou (2010) derived solutions of a two component generalized Camassa-Holm system. Guo (2010) derived blow-up and global solutions to a new integrable model with two components. Guan and Yin (2010) described the globle existence and blow-up phenomena for an integrable coupled Camassa-Holm shallow water system. Jingjing and Zhaoyang (2011) discussed the blow-up and global

*Corresponding author. E-mail: flowersl@yahoo.cn.

existence for a modified two component Camassa-Holm equation. Chunxia et al. (2010) described the well-posedness and blow-up phenomena for a modified coupled Camassa-Holm equation. Guilong and Yue (2010) investigated the global existence and wave breaking criteria for the two components Camassa-Holm.

Chuanxia and Zhaoyang (2011) investigated the global weak solutions for a two component Camassa-Holm shallow water system. Kai and Zhaoyang (2005) discussed the analytic solutions of the Cauchy problem for coupled shallow water system]. Ming et al. (2006) investigated the two component generalization of the Camassa-Holm equation and its solutions. Jibin and Yishen (2008) discussed the bifurcations of travelling wave solutions for a two component Camassa-Holm equation. Zhang and Liu (2010) investigated the stability of solitary waves and wave-breaking phenomena for the tow component Camassa-Holm system. David (2009) investigated the infinite propagation speed for a two component Camassa-Holm equation. Manwai (2010) investigated the self similar blowup solutions to the two component Camassa-Holm equation, he also discussed the perturbational blow up solutions to the two component Camassa-Holm equations. The modified Kudryashov method is employed for the nonlinear heat conduction equations. By utilization of low energy high-resolution (LEHR) and low energy general-purpose (LEGP) collimators, hardware and software filters on the image quality are shown in the work of Alireza (2001).

The goal of this paper is to implement the WTC-kruskal algorithm for dealing with solitons problems and finding solitons that have not described before. Our objective is to find various types of solutions existing in the following generalized systems.

$$\begin{cases} f_1(t)m_t = f_2(t)mu_x + f_3(t)m_x u + f_4(t)(nv)_x \\ f_1(t)n_t = f_2(t)nv_x + f_3(t)n_x v + f_4(t)(mu)_x \end{cases} \quad (2)$$

With $m = u - u_{xx}, n = v - v_{xx}$. As for Equation 2, the normalized equation contains four different solitons systems. Respectively, they are periodic blow-up waves, a new form of compactons which we entitled floating compacton, a blow-up wave with finite amplitude and singular solitary waves. Take the systems as the initial condition; it is necessary to launch auto-Bäcklund transformations. Through the transformation and via the four soliton systems given earlier, we extended the soliton systems to a much extensive and complex range in the case of soliton initial conditions. Periodic blow-up waves with compacton or anti-compacton structures in its wave base are obtained. We also find double collapse to coexists in kinks and regular solitons.

SOLITONIC STRUCTURES FOR MODEL 2

We now analyze generalized model 2 and try to find the singular solitonic patterns for existing regular solitons. In order to do this, we begin with an ansatz method for achieving regular solitons of model 2. The traveling wave solution is determined by the expressions $\begin{cases} u(x,t) = u(\xi) = u(x - Dt) \\ v(x,t) = v(\xi) = v(x - Dt) \end{cases}$ with constant t and the velocity D. Thus, we can simplify model 2 into an ordinary differential equation.

$$\begin{cases} Df_1(t)(u_\xi - u_{\xi\xi\xi}) - f_2(t)uu_\xi + f_2(t)u_\xi u_{\xi\xi} - f_3(t)uu_\xi + f_3(t)uu_{\xi\xi\xi} + f_4(t)v^2 + f_4(t)vv_{\xi\xi} - f_4(t)vv_\xi + f_4(t)v_\xi v_{\xi\xi} = 0 \\ Df_1(t)(v_\xi - v_{\xi\xi\xi}) - f_2(t)vv_\xi + f_2(t)v_\xi v_{\xi\xi} - f_3(t)vv_\xi + f_3(t)vv_{\xi\xi\xi} + f_4(t)u^2 + f_4(t)uu_{\xi\xi} - f_4(t)uu_\xi + f_4(t)u_\xi u_{\xi\xi} = 0 \end{cases} \quad (3)$$

We introduce the assumption that model 2 has the compactons, and solitary pattern solutions taken as:

Ansatz 1: $\begin{cases} u(\xi) = A\cos^\beta B\xi \\ v(\xi) = C\cos^\beta E\xi \end{cases}$;

Ansatz 2: $\begin{cases} u(\xi) = A\sin^\beta B\xi \\ v(\xi) = C\sin^\beta E\xi \end{cases}$;

Ansatz 3: $\begin{cases} u(\xi) = A\cosh^\beta B\xi \\ v(\xi) = C\cosh^\beta E\xi \end{cases}$;

Ansatz 4: $\begin{cases} u(\xi) = A\sinh^\beta B\xi \\ v(\xi) = C\sinh^\beta E\xi \end{cases}$.

Ansatz 1

Substituting $\begin{cases} u(\xi) = A\cos^\beta B\xi \\ v(\xi) = C\cos^\beta E\xi \end{cases}$ into ODE 3 yields a model for both parameters A ,B, C and E.

$$\begin{cases}[D-k-(B^2\beta^2+4\beta-2B^2\beta-2)B^2Df_1(t)]\cos^{\beta-1}B\xi+B^4(\beta-1)(\beta-2)Df_1(t)\\ \cos^{\beta-3}B\xi-2Af_2(t)\cos^{2\beta-1}B\xi+2A^2B^4[4\beta^2f_4(t)-3\beta f_4(t)+2f_4(t)+\beta]\cos^{3\beta-1}B\xi\\ +8A^2B^4(\beta-1)f_4(t)\cos^{3\beta-2}B\xi-4A^2B^4(\beta-1)(2\beta-1)f_4(t)\cos^{3\beta-3}B\xi-3A^3B^4\beta\\ [2\beta f_3(t)+\beta-1]\cos^{4\beta-3}B\xi+3A^3B^4\beta[2\beta f_3(t)+f_3(t)+\beta-1]\cos^{4\beta-1}B\xi=0\\ [D-k-(E^2\beta^2+4\beta-2E^2\beta-2)E^2Df_1(t)]\cos^{\beta-1}E\xi+E^4(\beta-1)(\beta-2)Df_1(t)\\ \cos^{\beta-3}E\xi-2Cf_2(t)\cos^{2\beta-1}E\xi+2C^2E^4[4\beta^2f_4(t)-3\beta f_4(t)+2f_4(t)+\beta]\cos^{3\beta-1}E\xi\\ +8C^2E^4(\beta-1)f_4(t)\cos^{3\beta-2}E\xi-4C^2E^4(\beta-1)(2\beta-1)f_4(t)\cos^{3\beta-3}E\xi-3C^3E^4\beta\\ [2\beta f_3(t)+\beta-1]\cos^{4\beta-3}E\xi+3C^3E^4\beta[2\beta f_3(t)+f_3(t)+\beta-1]\cos^{4\beta-1}E\xi=0\end{cases}$$

From here, it is easy to find the solution:

$$\begin{cases}u(x,t)=\dfrac{10f_4(t)}{3f_3(t)+3}\sec^2\sqrt[4]{\dfrac{5f_2(t)f_4(t)}{9Df_1(t)[f_3(t)+1]}}(x-Dt)\\ v(x,t)=\dfrac{10f_4(t)}{3f_3(t)+3}\sec^2\sqrt[4]{\dfrac{5f_2(t)f_4(t)}{9Df_1(t)[f_3(t)+1]}}(x-Dt)\end{cases} \quad (4)$$

which shows periodic properties. They are defined in infinite space sector, and have the form of blow up phenomena.

Ansatz 2

Appling the ansatz form 2 and solving the differential system, it is easy to find the following solitary wave solution.

$$\begin{cases}u(x,t)=\dfrac{10f_4(t)}{3f_3(t)+3}\csc^2\sqrt[4]{\dfrac{5f_2(t)f_4(t)}{9Df_1(t)[f_3(t)+1]}}(x-Dt)\\ v(x,t)=\dfrac{10f_4(t)}{3f_3(t)+3}\csc^2\sqrt[4]{\dfrac{5f_2(t)f_4(t)}{9Df_1(t)[f_3(t)+1]}}(x-Dt)\end{cases} \quad (5)$$

Here, Equation 5 denotes the periodic blow-up waves of generalized model 2.

Ansatz 3

After substituting ansatz expression $\begin{cases}u(\xi)=A\cosh^{\beta}B\xi\\ v(\xi)=C\cosh^{\beta}E\xi\end{cases}$ into ODE 3, yields a system for both A , B, C and E.

$$\begin{cases}2Af_2(t)\cosh^{\beta}B\xi+[k-D+B^4D\beta^2f_1(t)-6B^4D\beta f_1(t)+4B^4Df_1(t)]\cosh^{\beta-1}B\xi\\ -B^4D(\beta-1)(\beta-2)f_1(t)\cosh^{\beta-3}B\xi-B^2(\beta^2-6\beta+4)\cosh^{2\beta-1}B\xi\\ +B^2(\beta-1)(\beta-2)\cosh^{2\beta-3}B\xi-6A^2B^4\beta(\beta-2)f_4(t)\cosh^{3\beta-1}B\xi\\ +6A^2B^4\beta(\beta-1)f_4(t)\cosh^{3\beta-3}B\xi-3A^3B^4\beta(3\beta-2)f_3(t)\cosh^{4\beta-1}B\xi\\ +3A^3B^4\beta(3\beta-1)f_3(t)\cosh^{4\beta-3}B\xi=0\\ 2Cf_2(t)\cosh^{\beta}E\xi+[k-D+B^4D\beta^2f_1(t)-6E^4D\beta f_1(t)+4E^4Df_1(t)]\cosh^{\beta-1}E\xi\\ -E^4D(\beta-1)(\beta-2)f_1(t)\cosh^{\beta-3}E\xi-E^2(\beta^2-6\beta+4)\cosh^{2\beta-1}E\xi\\ +E^2(\beta-1)(\beta-2)\cosh^{2\beta-3}E\xi-6C^2E^4\beta(\beta-2)f_4(t)\cosh^{3\beta-1}E\xi\\ +6C^2E^4\beta(\beta-1)f_4(t)\cosh^{3\beta-3}E\xi-3C^3E^4\beta(3\beta-2)f_3(t)\cosh^{4\beta-1}E\xi\\ +3C^3E^4\beta(3\beta-1)f_3(t)\cosh^{4\beta-3}E\xi=0\end{cases}$$

which admits the solution:

$$\begin{cases}u(x,t)=\dfrac{f_4(t)}{4f_3(t)}\sec h^2[2f_3(t)f_4^{-\frac{3}{2}}(t)](x-Dt)\\ v(x,t)=\dfrac{f_4(t)}{4f_3(t)}\sec h^2[2f_3(t)f_4^{-\frac{3}{2}}(t)](x-Dt)\end{cases} \quad (6)$$

It is easy to see that there is a floating compacton of model 2.

Ansatz 4

Finally, solitonic structures can be obtained by calculating the differential system.

$$\begin{cases} u(x,t)=\dfrac{f_4(t)}{4f_3(t)}\operatorname{csc}h^2[2f_3(t)f_4^{-\frac{3}{2}}(t)](x-Dt) \\ v(x,t)=\dfrac{f_4(t)}{4f_3(t)}\operatorname{csc}h^2[2f_3(t)f_4^{-\frac{3}{2}}(t)](x-Dt) \end{cases} \tag{7}$$

WTC TEST

According to the idea of Weiss, Tabor and Carnevale procedure (Weiss et al., 1983), we seek the auto-Bäcklund transformation of Equation 2. The singular manifold is defined by $\begin{cases} F(\varphi(x,t))=0 \\ Q(\psi(x,t))=0 \end{cases}$ and $\begin{cases} u=u(x,t) \\ v=v(x,t) \end{cases}$ is a solution of the partial differential equation. Inserting a formal ansatz of the form:

$$\begin{cases} u(x,t)=\sum_{i=0}^{N} F^{(i)}(\varphi(x,t)) \\ v(x,t)=\sum_{i=0}^{N} Q^{(i)}(\psi(x,t)) \end{cases}$$

into model 2, where t is in the neighborhood of the manifold. When balancing the nonlinear term and the highest partial derivatives term, therefore we may choose:

$$\begin{cases} u(x,t)=F''(\varphi)\varphi_x^2(x,t)+F'(\varphi)\varphi_{xx}(x,t)+U(x,t) \\ v(x,t)=Q''(\psi)\psi_x^2(x,t)+Q'(\psi)\psi_{xx}(x,t)+V(x,t) \end{cases} \tag{8}$$

where $\begin{cases} '=\partial/\partial\varphi \\ '=\partial/\partial\psi \end{cases}, \begin{cases} F^{(r)}=\partial^r F/\partial\varphi^r \\ Q^{(r)}=\partial^r Q/\partial\psi^r \end{cases}$

and $\begin{cases} \varphi=\varphi(x,t) \\ \psi=\psi(x,t) \end{cases}, \begin{cases} U(x,t) \\ V(x,t) \end{cases}$ are the two solutions of Equation 2. Substituting Equation 8 into Equation 2, we obtain:

$$\begin{cases}
k(U_x+2F''\varphi_x\varphi_{xx}+F'\varphi_{xxx})+2f_2(t)(U+F''\varphi_x^2+F'\varphi_{xx})(U_x+2F''\varphi_x\varphi_{xx}+F'\varphi_{xxx}) \\
+f_3(t)(U_x+2F''\varphi_x\varphi_{xx}+F'\varphi_{xxx})(6U+6F''\varphi_x^2+6F'\varphi_{xx})(U_x+2F''\varphi_x\varphi_{xx}+F'\varphi_{xxx}^2) \\
+(3U+3F''\varphi_x^2+3F'\varphi_{xx})^2(U_{xx}+2F''\varphi_{xx}^2+2F''\varphi_x\varphi_{xxx}+F'\varphi_{xxxx}) \\
+f_1(t)(U_{xxt}+4F''\varphi_{xx}\varphi_{xxt}+2F''\varphi_{xt}\varphi_{xxx}+2F''\varphi_x\varphi_{xxxt}+F'\varphi_{xxxxt})+f_4(t)(U+F''\varphi_x^2+F'\varphi_{xx}) \\
(6U_x+12F''\varphi_x\varphi_{xx}+6F'\varphi_{xxx})(U_{xx}+2F''\varphi_{xx}^2+2F''\varphi_x\varphi_{xxx}+F'\varphi_{xxxx}) \\
+(2U+2F''\varphi_x^2+2F'\varphi_{xx})(U_{xxx}+6F''\varphi_{xx}\varphi_{xxx}+2F''\varphi_x\varphi_{xxxx}+F'\varphi_{xxxxx})) \\
+f_5(t)((24U+24F''\varphi_x^2+24F'\varphi_{xx})(U_x+2F''\varphi_x\varphi_{xx}+F'\varphi_{xxx})^3+36(U+F''\varphi_x^2+F'\varphi_{xx})^2 \\
(U_x+2F''\varphi_x\varphi_{xx}+F'\varphi_{xxx})(U_{xx}+2F''\varphi_{xx}^2+2F''\varphi_x\varphi_{xxx}+F'\varphi_{xxxx})+4(U+F''\varphi_x^2+F'\varphi_{xx})^3 \\
(U_{xxx}+6F''\varphi_{xx}\varphi_{xxx}+2F''\varphi_x\varphi_{xxxx}+F'\varphi_{xxxxx})) \\
K(V_x+2Q''\psi_x\psi_{xx}+Q'\psi_{xxx})+2f_2(t)(V+Q''\psi_x^2+Q'\psi_{xx})(V_x+2Q''\psi_x\psi_{xx}+Q'\psi_{xxx}) \\
+f_3(t)(V_x+2Q''_x\psi_{xx}+Q'\psi_{xxx})(6V+6Q''\psi_x^2+6Q'\psi_{xx})(V_x+2Q''\psi_x\psi_{xx}+Q'\psi_{xxx}^2) \\
+(3V+3Q''\psi_x^2+3Q'\psi_{xx})^2(V_{xx}+2Q''\psi_{xx}^2+2Q''\psi_x\psi_{xxx}+Q'\psi_{xxxx}) \\
+f_1(t)(V_{xxt}+4Q''\psi_{xx}\psi_{xxt}+2Q''\psi_{xt}\psi_{xxx}+2Q''\psi_x\psi_{xxxt}+Q'\psi_{xxxxt})+f_4(t)(V+Q''\psi_x^2+Q'\psi_{xx}) \\
(6V_x+12Q''\psi_x\psi_{xx}+6Q'\psi_{xxx})(V_{xx}+2Q''\psi_{xx}^2+2Q''\psi_x\psi_{xxx}+Q'\psi_{xxxx}) \\
+(2V+2Q''\psi_x^2+2Q'\psi_{xx})(V_{xxx}+6Q''\psi_{xx}\psi_{xxx}+2Q''\psi_x\psi_{xxxx}+Q'\psi_{xxxxx})) \\
+f_5(t)((24V+24Q''\psi_x^2+24Q'\psi_{xx})(V_x+2Q''\psi_x\psi_{xx}+Q'\psi_{xxx})^3+36(V+Q''\psi_x^2+Q'\psi_{xx})^2 \\
(V_x+2Q''\psi_x\psi_{xx}+Q'\psi_{xxx})(V_{xx}+2Q''\psi_{xx}^2+2Q''\psi_x\psi_{xxx}+Q'\psi_{xxxx})+4(V+Q''\psi_x^2+Q'\psi_{xx})^3 \\
(V_{xxx}+6Q''\psi_{xx}\psi_{xxx}+2Q''\psi_x\psi_{xxxx}+Q'\psi_{xxxxx}))
\end{cases} \tag{9}$$

Setting the coefficients of $\begin{cases}\varphi_x^6\\ \psi_x^6\end{cases}$ in system Equation 9 to zero yields an ordinary differential equation for F which indicates that the nonlinear terms and the third derivative term in Equation 2 have been partially balanced. By solving it, we have solutions $\begin{cases}F(\varphi)=k\ln\varphi\\ Q(\psi)=K\ln\psi\end{cases}$, where $\begin{cases}k\\ K\end{cases}$ are arbitrary constants, it brings:

$$\begin{cases}F''F'=-\frac{k}{2}F''',F'F'F'F'=-\frac{k^3}{6}F^{(4)},F^{(2)}F^{(2)}F^{(2)}=\frac{k^2}{120}F^{(6)},F'F^{(2)}F^{(2)}F^{(2)}=-\frac{k^3}{720}F^{(7)},\\ F'F'F^{(2)}=\frac{k^2}{6}F^{(4)},F'F^{(2)}F^{(2)}=\frac{k^2}{24}F^{(5)},F'F'F'F^{(2)}=-\frac{k^3}{24}F^{(5)},F'F^{(2)}F'F^{(2)}=-\frac{k^3}{120}F^{(6)},\\ F''F''=-\frac{k}{6}F^{(4)},F'F'=-kF'',F^{(2)}F^{(2)}F^{(2)}F^{(2)}=\frac{k^3}{5040}F^{(8)},F'F'F'=\frac{k^2}{2}F^{(3)}\\ Q''Q'=-\frac{k}{2}Q''',Q'Q'Q'Q'=-\frac{k^3}{6}Q^{(4)},Q^{(2)}Q^{(2)}Q^{(2)}=\frac{k^2}{120}Q^{(6)},Q'Q^{(2)}Q^{(2)}Q^{(2)}=-\frac{k^3}{720}Q^{(7)},\\ Q'Q'Q^{(2)}=\frac{k^2}{6}Q^{(4)},Q'Q^{(2)}Q^{(2)}=\frac{k^2}{24}Q^{(5)},Q'Q'Q'Q^{(2)}=-\frac{k^3}{24}Q^{(5)},Q'Q^{(2)}Q'Q^{(2)}=-\frac{k^3}{120}Q^{(6)},\\ Q''Q''=-\frac{k}{6}Q^{(4)},Q'Q'=-kQ'',Q^{(2)}Q^{(2)}Q^{(2)}Q^{(2)}=\frac{k^3}{5040}Q^{(8)},Q'Q'Q'=\frac{k^2}{2}Q^{(3)}\end{cases} \tag{10}$$

By using Equation 10, formula 9 can be simplified to a linear polynomial of $F',F'',F''',\cdots$, the setting coefficients of $F',F'',F''',\cdots$ to vanish, from the set of PB models, an auto-Bäcklund transformation of nonlinear coupled Camassa-Holm model constitutes:

$$\begin{cases}u(x,t)=k(\ln\varphi)_{xx}+U(x,t)\\ v(x,t)=K(\ln\psi)_{xx}+V(x,t)\end{cases} \tag{11}$$

where $\begin{cases}\varphi=\varphi(x,t)\\ \psi=\psi(x,t)\end{cases}$ satisfies the PB system , $\begin{cases}U(x,t)\\ V(x,t)\end{cases}$ is a solution of Equation 2.

SINGULAR SOLITON STRUCTURES

By applying the solitary wave solutions 4, 5, 6 and 7 to the transformations yield a set of partial differential equations (PDEs).

$$\begin{cases}[2f_2(t)U+k]\varphi_{xxx}+2f_4(t)U^2\varphi_{xxxxx}+f_1(t)\varphi_{xxxt}=0\\ [2f_2(t)V+K]\psi_{xxx}+2f_4(t)V^2\psi_{xxxxx}+f_1(t)\psi_{xxxt}=0\end{cases} \tag{12}$$

$$\begin{cases}[k+2f_2(t)U]\varphi_x\varphi_{xx}+f_1(t)\varphi_{xt}\varphi_{xxx}+f_1(t)\varphi_x\varphi_{xxt}+2f_4(t)U^2\varphi_x\varphi_{xxxx}\\ +[6f_4(t)U^2-kf_2(t)]\varphi_{xx}\varphi_{xxx}+2f_1(t)\varphi_{xx}\varphi_{xxt}-2kf_4(t)U\varphi_{xx}\varphi_{xxxxx}=0\\ [K+2f_2(t)V]\psi_x\psi_{xx}+f_1(t)\psi_{xt}\psi_{xxx}+f_1(t)\psi_x\psi_{xxt}+2f_4(t)V^2\psi_x\psi_{xxxx}\\ +[6f_4(t)V^2-Kf_2(t)]\psi_{xx}\psi_{xxx}+2f_1(t)\psi_{xx}\psi_{xxt}-2Kf_4(t)V\psi_{xx}\psi_{xxxxx}=0\end{cases} \tag{13}$$

$$\begin{cases}2f_2(t)\varphi_x\varphi_{xx}^2+f_2(t)\varphi_x^2\varphi_{xxx}-[3kUf_3(t)-3f_4(t)]\varphi_{xx}\varphi_{xxx}\varphi_{xxxx}+[Uf_3(t)+6f_4(t)]3U\varphi_{xx}^2\varphi_{xxx}\\ +[3Uf_3(t)+6f_4(t)]U\varphi_x\varphi_{xxx}^2-3f_3(t)kU\varphi_{xxx}^3-f_4(t)k\varphi_{xx}^2\varphi_{xxxxx}=0\\ 2f_2(t)\psi_x\psi_{xx}^2+f_2(t)\psi_x^2\psi_{xxx}-[3KVf_3(t)-3f_4(t)]\psi_{xx}\psi_{xxx}\psi_{xxxx}+[Vf_3(t)+6f_4(t)]3V\psi_{xx}^2\psi_{xxx}\\ +[3Vf_3(t)+6f_4(t)]V\psi_x\psi_{xxx}^2-3f_3(t)KV\psi_{xxx}^3-f_4(t)K\psi_{xx}^2\psi_{xxxxx}=0\end{cases} \tag{14}$$

$$\begin{cases}[12Uf_3(t)+24f_4(t)]k\varphi_{xx}^3\varphi_{xxx}-4f_2(t)\varphi_x^3\varphi_{xx}+4f_4(t)k\varphi_{xx}\varphi_{xxxx}\varphi_x^2+[48Uf_3(t)\\ +12f_4(t)]k\varphi_x\varphi_{xx}\varphi_{xxx}^2+[12Uf_3(t)+16f_4(t)]k\varphi_x\varphi_{xxxx}\varphi_{xx}^2+[6Uf_3(t)+6f_4(t)]k\varphi_{xxx}\varphi_{xxxx}\varphi_x^2\\ -[12Uf_3(t)-24f_4(t)]U\varphi_x\varphi_{xx}^3-8f_4(t)U\varphi_x^3\varphi_{xxxx}-[12Uf_3(t)-48f_4(t)]U\varphi_x^2\varphi_{xx}\varphi_{xxx}\\ -6f_3(t)k^2\varphi_{xxx}^3\varphi_{xx}-3f_3(t)k^2\varphi_{xx}^2\varphi_{xxx}\varphi_{xxxx}=0\\ [12Vf_3(t)+24f_4(t)]K\psi_{xx}^3\psi_{xxx}-4f_2(t)\psi_x^3\psi_{xx}+4f_4(t)K\psi_{xx}\psi_{xxxx}\psi_x^2+[48Vf_3(t)\\ +12f_4(t)]K\psi_x\psi_{xx}\psi_{xxx}^2+[12Vf_3(t)+16f_4(t)]K\psi_x\psi_{xxxx}\psi_{xx}^2+[6Vf_3(t)+6f_4(t)]K\psi_{xxx}\psi_{xxxx}\psi_x^2\\ -[12Vf_3(t)-24f_4(t)]V\psi_x\psi_{xx}^3-8f_4(t)V\psi_x^3\psi_{xxxx}-[12Vf_3(t)-48f_4(t)]V\psi_x^2\psi_{xx}\psi_{xxx}\\ -6f_3(t)K^2\psi_{xxx}^3\psi_{xx}-3f_3(t)K^2\psi_{xx}^2\psi_{xxx}\psi_{xxxx}=0\end{cases} \tag{15}$$

$$\begin{cases}[f_3(t)U+f_4(t)]12\varphi_x\varphi_{xx}^4+[54f_3(t)U+30f_4(t)]\varphi_x^2\varphi_{xx}^2\varphi_{xxx}+[f_3(t)U+f_4(t)]6\varphi_x^3\varphi_{xxx}^2\\ +[6f_3(t)U+10f_4(t)]\varphi_x^3\varphi_{xx}\varphi_{xxxx}+f_4(t)\varphi_x^4\varphi_{xxxxx}-3f_3(t)k-21f_3(t)k\varphi_x\varphi_{xx}^2\varphi_{xxx}^2\\ -3f_3(t)kU\varphi_x^2\varphi_{xxx}^3-3f_3(t)\varphi_x\varphi_{xx}^3\varphi_{xxxx}-3f_3(t)k\varphi_x^2\varphi_{xx}\varphi_{xxx}\varphi_{xxxx}=0\\ [f_3(t)V+f_4(t)]12\psi_x\psi_{xx}^4+[54f_3(t)V+30f_4(t)]\psi_x^2\psi_{xx}^2\psi_{xxx}+[f_3(t)V+f_4(t)]6\psi_x^3\psi_{xxx}^2\\ +[6f_3(t)V+10f_4(t)]\psi_x^3\psi_{xx}\psi_{xxxx}+f_4(t)\psi_x^4\psi_{xxxxx}-3f_3(t)K-21f_3(t)K\psi_x\psi_{xx}^2\psi_{xxx}^2\\ -3f_3(t)KV\psi_x^2\psi_{xxx}^3-3f_3(t)\psi_x\psi_{xx}^3\psi_{xxxx}-3f_3(t)K\psi_x^2\psi_{xx}\psi_{xxx}\psi_{xxxx}=0\end{cases} \tag{16}$$

$$\begin{cases}24f_4(t)\varphi_x^3\varphi_{xx}^2+72f_3(t)U\varphi_x^3\varphi_{xxx}^3-12f_3(t)k\varphi_x\varphi_{xx}^5+[24f_3(t)U+36f_4(t)]\varphi_x^4\varphi_{xx}\varphi_{xxx}\\ +4f_4(t)\varphi_x^5\varphi_{xxxx}-48f_3(t)k\varphi_x^3\varphi_{xx}\varphi_{xxx}^2-12f_3(t)k\varphi_x^3\varphi_{xx}^2\varphi_{xxxx}-[3f_3(t)U+6f_4(t)]kU\varphi_{xxx}\varphi_{xxxx}\\ -3f_3(t)k\varphi_x^4\varphi_{xxx}\varphi_{xxxx}-12k\varphi_x^4\varphi_{xx}\varphi_{xxxxx}-96f_3(t)k\varphi_x^2\varphi_{xx}^3\varphi_{xxx}=0\\ 24f_4(t)\psi_x^3\psi_{xx}^2+72f_3(t)V\psi_x^3\psi_{xxx}^3-12f_3(t)K\psi_x\psi_{xx}^5+[24f_3(t)V+36f_4(t)]\psi_x^4\psi_{xx}\psi_{xxx}\\ +4f_4(t)\psi_x^5\psi_{xxxx}-48f_3(t)K\psi_x^3\psi_{xx}\psi_{xxx}^2-12f_3(t)K\psi_x^3\psi_{xx}^2\psi_{xxxx}-[3f_3(t)V+6f_4(t)]KV\psi_{xxx}\psi_{xxxx}\\ -3f_3(t)K\psi_x^4\psi_{xxx}\psi_{xxxx}-12K\psi_x^4\psi_{xx}\psi_{xxxxx}-96f_3(t)K\psi_x^2\psi_{xx}^3\psi_{xxx}=0\end{cases} \tag{17}$$

As for the solution $\begin{cases}\varphi(x,t)\\ \psi(x,t)\end{cases}$ of system Equations 12 to 17, we suppose that:

$$\begin{cases}\varphi(x,t)=A+B\exp\alpha(x-\lambda t)\\ \psi(x,t)=C+D\exp\beta(x-\delta t)\end{cases},$$

where parameters A, B, C and D are given as arbitrary constants, $\alpha,\beta,\lambda,\delta$ will be given exact values in the following by detailed analysis. Substituting this ansatz form into Equations 12 to 17 brings Equation 12, 13 and 14 which can help with the determination of the values of parameters $\alpha,\beta,\lambda,\delta$.

$$\begin{cases}k-\alpha^2\lambda f_1(t)+2f_2(t)U+2\alpha^2U^2f_4(t)=0\\ K-\beta^2\delta f_1(t)+2f_2(t)V+2\beta^2V^2f_4(t)=0\end{cases} \tag{18}$$

$$\begin{cases}\dfrac{3f_2(t)}{f_4(t)}+(3-k)\alpha^4+24U\alpha^2=0\\ \dfrac{3f_2(t)}{f_4(t)}+(3-K)\beta^4+24V\beta^2=0\end{cases} \tag{19}$$

Via Equation 19, we get the value of α,β with the constraint model $\begin{cases}3kUf_3(t)=0\\ 3KVf_3(t)=0\end{cases}$.

$$\begin{cases}\alpha_1=-\left[\dfrac{12Uf_4(t)-\sqrt{3f_4(t)H}}{kf_4(t)-3f_4(t)}\right]^{\frac{1}{2}}\\ \beta_1=-\left[\dfrac{12Vf_4(t)-\sqrt{3f_4(t)h}}{Kf_4(t)-3f_4(t)}\right]^{\frac{1}{2}}\end{cases},\quad \begin{cases}\alpha_2=\left[\dfrac{12Uf_4(t)-\sqrt{3f_4(t)H}}{kf_4(t)-3f_4(t)}\right]^{\frac{1}{2}}\\ \beta_2=\left[\dfrac{12Vf_4(t)-\sqrt{3f_4(t)h}}{Kf_4(t)-3f_4(t)}\right]^{\frac{1}{2}}\end{cases},$$

$$\begin{cases}\alpha_3=-\left[\dfrac{12Uf_4(t)+\sqrt{3f_4(t)H}}{kf_4(t)-3f_4(t)}\right]^{\frac{1}{2}}\\ \beta_3=-\left[\dfrac{12Vf_4(t)+\sqrt{3f_4(t)h}}{Kf_4(t)-3f_4(t)}\right]^{\frac{1}{2}}\end{cases},\ \begin{cases}\alpha_4=\left[\dfrac{12Uf_4(t)+\sqrt{3f_4(t)H}}{kf_4(t)-3f_4(t)}\right]^{\frac{1}{2}}\\ \beta_4=\left[\dfrac{12Vf_4(t)+\sqrt{3f_4(t)h}}{Kf_4(t)-3f_4(t)}\right]^{\frac{1}{2}}\end{cases}.$$

Using Equation 18, we have the relation of $\alpha,\beta,\lambda,\delta$. This relation is regarded as determining λ in terms of

$$\begin{cases}\lambda=\dfrac{k+2f_2(t)U+2\alpha^2U^2f_4(t)}{\alpha^2 f_1(t)}\\ \delta=\dfrac{K+2f_2(t)V+2\beta^2V^2f_4(t)}{\beta^2 f_1(t)}\end{cases}$$

α and δ in terms of β, so that the exact value for λ and δ can be obtained:

$$\begin{cases}\lambda=\dfrac{\sqrt{f_4(t)}\{k^2-3k+(2k-6)f_2(t)U+24f_4(t)U^3-\sqrt{8f_4(t)[(k-3)f_2(t)+48f_4(t)U^2]}U^2\}}{f_1(t)[12U\sqrt{f_4(t)}-\sqrt{(3k-9)f_2(t)+144f_4(t)U^2}]}\\ \delta=\dfrac{\sqrt{f_4(t)}\{K^2-3K+(2K-6)f_2(t)V+24f_4(t)V^3-\sqrt{8f_4(t)[(K-3)f_2(t)+48f_4(t)V^2]}V^2\}}{f_1(t)[12V\sqrt{f_4(t)}-\sqrt{(3K-9)f_2(t)+144f_4(t)V^2}]}\end{cases}$$

By the application of the transformation, we generate a hierarchy of solutions with increasing complexity where solitary wave solutions are given:

$$\begin{cases}u_1(x,t)=-k\left\{\left[\dfrac{12Uf_4(t)-\sqrt{3f_4(t)h}}{kf_4(t)-3f_4(t)}\right]^{\frac{1}{2}}+1\right\}\left\{\dfrac{\left[\dfrac{12Uf_4(t)-\sqrt{3f_4(t)h}}{kf_4(t)-3f_4(t)}\right]^{\frac{1}{2}}B}{Ae^{\left[\frac{12Uf_4(t)-\sqrt{3f_4(t)h}}{kf_4(t)-3f_4(t)}\right]^{\frac{1}{2}}(x-\lambda t)}+B}\right\}^2+U(x,t)\\ v_1(x,t)=-K\left\{\left[\dfrac{12Vf_4(t)-\sqrt{3f_4(t)H}}{Kf_4(t)-3f_4(t)}\right]^{\frac{1}{2}}+1\right\}\left\{\dfrac{\left[\dfrac{12Vf_4(t)-\sqrt{3f_4(t)H}}{Kf_4(t)-3f_4(t)}\right]^{\frac{1}{2}}D}{Ce^{\left[\frac{12Vf_4(t)-\sqrt{3f_4(t)H}}{Kf_4(t)-3f_4(t)}\right]^{\frac{1}{2}}(x-\delta t)}+D}\right\}^2+V(x,t)\end{cases}$$

In other parameter values, similar singular periodic blow up wave in different expression is obtained.

$$\begin{cases}u_2(x,t)=k\left\{\left[\dfrac{12Uf_4(t)-\sqrt{3f_4(t)H}}{kf_4(t)-3f_4(t)}\right]^{\frac{1}{2}}-1\right\}\left\{\dfrac{\left[\dfrac{12Uf_4(t)-\sqrt{3f_4(t)H}}{kf_4(t)-3f_4(t)}\right]^{\frac{1}{2}}Be^{\left[\frac{12Uf_4(t)-\sqrt{3f_4(t)H}}{kf_4(t)-3f_4(t)}\right]^{\frac{1}{2}}(x-\lambda t)}}{A+Be^{\left[\frac{12Uf_4(t)-\sqrt{3f_4(t)H}}{kf_4(t)-3f_4(t)}\right]^{\frac{1}{2}}(x-\lambda t)}}\right\}^2+U(x,t)\\ v_2(x,t)=K\left\{\left[\dfrac{12Vf_4(t)-\sqrt{3f_4(t)h}}{Kf_4(t)-3f_4(t)}\right]^{\frac{1}{2}}-1\right\}\left\{\dfrac{\left[\dfrac{12Vf_4(t)-\sqrt{3f_4(t)h}}{Kf_4(t)-3f_4(t)}\right]^{\frac{1}{2}}Ce^{\left[\frac{12Vf_4(t)-\sqrt{3f_4(t)h}}{Df_4(t)-3f_4(t)}\right]^{\frac{1}{2}}(x-\delta t)}}{C+De^{\left[\frac{12Vf_4(t)-\sqrt{3f_4(t)h}}{Kf_4(t)-3f_4(t)}\right]^{\frac{1}{2}}(x-\delta t)}}\right\}^2+V(x,t)\end{cases}$$

Another hierarchy of solutions with increasing complexity is generated which should satisfy the constraint $\begin{cases} 3kUf_3(t)=0 \\ 3KVf_3(t)=0 \end{cases}$.

$$\begin{cases} u_3(x,t)=-k\left\{\left[\dfrac{12Uf_4(t)+\sqrt{3f_4(t)H}}{kf_4(t)-3f_4(t)}\right]^{\frac{1}{2}}+1\right\}\left\{\dfrac{\left[\dfrac{12Uf_4(t)+\sqrt{3f_4(t)H}}{kf_4(t)-3f_4(t)}\right]^{\frac{1}{2}}B}{Ae^{\left[\frac{12Uf_4(t)+\sqrt{3f_4(t)H}}{kf_4(t)-3f_4(t)}\right]^{\frac{1}{2}}(x-\lambda t)}+B}\right\}^2+U(x,t) \\ v_3(x,t)=-K\left\{\left[\dfrac{12Vf_4(t)+\sqrt{3f_4(t)h}}{Kf_4(t)-3f_4(t)}\right]^{\frac{1}{2}}+1\right\}\left\{\dfrac{\left[\dfrac{12Vf_4(t)+\sqrt{3f_4(t)h}}{Kf_4(t)-3f_4(t)}\right]^{\frac{1}{2}}D}{Ce^{\left[\frac{12Vf_4(t)+\sqrt{3f_4(t)h}}{Kf_4(t)-3f_4(t)}\right]^{\frac{1}{2}}(x-\delta t)}+D}\right\}^2+V(x,t) \end{cases}$$

In this way, we have the type of singular kink waves.

$$\begin{cases} u_4(x,t)=k\left\{\left[\dfrac{12Uf_4(t)+\sqrt{3f_4(t)H}}{kf_4(t)-3f_4(t)}\right]^{\frac{1}{2}}-1\right\}\left\{\dfrac{\left[\dfrac{12Uf_4(t)+\sqrt{3f_4(t)H}}{kf_4(t)-3f_4(t)}\right]^{\frac{1}{2}}Be^{\left[\frac{12Uf_4(t)+\sqrt{3f_4(t)H}}{kf_4(t)-3f_4(t)}\right]^{\frac{1}{2}}(x-\lambda t)}}{A+Be^{\left[\frac{12Uf_4(t)+\sqrt{3f_4(t)H}}{kf_4(t)-3f_4(t)}\right]^{\frac{1}{2}}(x-\lambda t)}}\right\}^2+U(x,t) \\ v_4(x,t)=K\left\{\left[\dfrac{12Vf_4(t)+\sqrt{3f_4(t)h}}{Kf_4(t)-3f_4(t)}\right]^{\frac{1}{2}}-1\right\}\left\{\dfrac{\left[\dfrac{12Vf_4(t)+\sqrt{3f_4(t)h}}{Kf_4(t)-3f_4(t)}\right]^{\frac{1}{2}}Ce^{\left[\frac{12Vf_4(t)+\sqrt{3f_4(t)h}}{Df_4(t)-3f_4(t)}\right]^{\frac{1}{2}}(x-\delta t)}}{C+De^{\left[\frac{12Vf_4(t)+\sqrt{3f_4(t)h}}{Kf_4(t)-3f_4(t)}\right]^{\frac{1}{2}}(x-\delta t)}}\right\}^2+V(x,t) \end{cases}$$

Conclusions

In this paper, we presented auto-Bäcklund transformation and WTC test. An implementation of the methods was given by coupled Camassa-Holm equation. We obtain regular solitons, such as compactons and blow-up solutions at first. By using them as “seed” solutions for the auto-Bäcklund transformation, a lot of singular solutions are obtained. The method can be used in many other nonlinear equations or coupled ones.

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APPENDIX

$$\begin{cases} H = -3f_2(t) + kf_2(t) + 48U^2 f_4(t) \\ h = -3f_2(t) + Kf_2(t) + 48V^2 f_4(t) \end{cases}$$