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*Full Length Research Paper*

# Some properties of certain classes of analytic functions related with uniformly convex functions

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**In this paper, we extend the concept of $k$-uniform convexity. Making use of the principle of subordination and a family of integral operators, we introduce and investigate some new subclasses of analytic functions. Several inclusion results with some interesting consequences are proved. The rate of growth of Hankel determinant for the functions in these classes is also studied.**

**Key words:** $k$-uniformly convex, univalent, starlike, conic regions, subordination, Hankel determinant, bounded boundary rotation.

## INTRODUCTION

Let $A$ denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (1)$$

which are analytic in the open unit disc $E = \{z : |z| < 1\}$. Let $S$ be the subclass of $A$ which consists of functions which are univalent in $E$. Also let $S^*(\gamma)$ and $C(\gamma)$ be the subclasses of $S$ which contain starlike and convex functions of order $\gamma$ $(0 \le \gamma < 1)$ respectively.

In Komatu (1990), the following two parameter family of integral operator $Q_a^\sigma$ for $f \in A$, is defined by:

$$Q_a^\sigma f(z) = \frac{a^\sigma}{\Gamma(\sigma) z^{a-1}} \int_0^z t^{a-z} \left( \log \frac{z}{t} \right)^{\sigma-1} f(t)dt, \qquad (2)$$

where $z \in E$, $a > 0$, $\sigma \ge 0$, see also Chun and Srivastava (2004).

If $f(z)$ is given by (1), then from (2), we can write:

$$Q_a^\sigma f(z) = z + \sum_{n=2}^{\infty} \left( \frac{a}{a+n-1} \right)^\sigma a_n z^n, \qquad (a > 0, \sigma \ge 0). \qquad (3)$$

Some special cases of $Q_a^\sigma$ have been discussed; for example, see Flett (1972), Goodman (1983) and Jung et al. (1993). From (3) we can easily derive the following:

$$z\left(Q_a^{\sigma+1} f(z)\right)' = aQ_a^\sigma f(z) - (a-1)Q_a^{\sigma+1} f(z). \qquad (4)$$

In this paper we shall use the operator $Q_a^\sigma$ to introduce the generalized concept of $k$-uniformly convexity.

For $k \in [0, \infty)$, the domain $\Omega_k$ is defined as follows (Kanas, 2003):

$$\Omega_k = \left\{ u + iv : u > k\sqrt{(u-1)^2 + v^2} \right\}. \qquad (5)$$

For fixed $k$, $\Omega_k$ represents the conic region bounded, successively, by the imaginary axis $(k = 0)$, a parabola $(k = 1)$, the right branch of hyperbola $(0 < k < 1)$ and an ellipse $(k > 1)$. Also, we note that, for no choices of $k\,(k > 1)$, $\Omega_k$ reduces to a disc. We define the domain $\Omega_{k,\gamma}$ as follows (Noor et al., 2009):

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$$\Omega_{k,\gamma} = (1-\gamma)\Omega_k + \gamma, \qquad (0 \le \gamma < 1). \tag{6}$$

The following functions, denoted by $p_{k,\gamma}(z)$, are univalent in $E$ and map $E$ on $\Omega_{k,\gamma}$ such that $p_{k,\gamma}(0)=1$ and $p'_{k,\gamma}(0)>0$:

$$p_{k,\gamma}(z)=\begin{cases} \dfrac{1+(1-2\gamma)z}{1-z}, & (k=0) \\ 1+\dfrac{2(1-\gamma)}{\pi^2}\left(\log\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & (k=1) \\ 1+\dfrac{2(1-\gamma)}{1-k^2}\sinh^2\left[\left(\dfrac{2}{\pi}\arccos k\right)\operatorname{arctanh}\sqrt{z}\right], & (0<k<1) \\ 1+\dfrac{(1-\gamma)}{k^2-1}\sin^2\left(\dfrac{\pi}{2R(t)}\displaystyle\int_0^{\frac{u(z)}{\sqrt{t}}}\dfrac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}dx\right)+\dfrac{(1-\gamma)}{k^2-1}, & (k>1), \end{cases} \tag{7}$$

where $u(z)=\dfrac{z-\sqrt{t}}{1-\sqrt{t}\,z}$, $t\in(0,1)$, $z\in E$ and $z$ is chosen such that $k=\cosh\left(\dfrac{\pi R'(t)}{4R(t)}\right)$, $R(t)$ is the Legendre's complete elliptic integral of the first kind and $R'(t)$ is complementarily integral of $R(t)$, (Kanas et al., 1998, 1999; Noor at el., 2009). The function $p_{k,\gamma}(z)$ is continuous as regard to $k$ and has real coefficients for $k\in[0,\infty)$.

We now define the following subclasses of analytic functions related with the class $P$ of Caratheodory functions (Noor, 2011).

Definition 1.1: Let $P(p_{k,\gamma})\subset P$ denote the class of functions $p(z)$ which are analytic in $E$ with $p(0)=1$ and which are subordinate to $p_{k,\gamma}(z)$, written as $p\prec p_{k,\gamma}$, where $p_{k,\gamma}(z)$ is given by (7) and $p(E)\subset p_{k,\gamma}(E)$.

We note that $P(p_{0,0})=P$ and $P(p_{0,\gamma})=P(\gamma)$, where $p\in P(\gamma)$ implies $\operatorname{Re} p(z)>\gamma$, $z\in E$. It can easily be verified that $P(p_{k,\gamma})$ is a convex set.

We extend the class $P(p_{k,\gamma})$ as follows:

Definition 1.2: Let $p(z)$ be analytic in $E$ with $p(0)=1$. Then $p\in P_m(p_{k,\gamma})$ if and only if, for $m\ge 2$, $0\le\gamma<1$, $k\in[0,\infty)$, $z\in E$, we have:

$$p(z)=\left(\frac{m}{4}+\frac{1}{2}\right)p_1(z)-\left(\frac{m}{4}-\frac{1}{2}\right)p_2(z), \qquad p_1,p_2\in P(p_{k,\gamma}). \tag{8}$$

From (18), it can easily be seen that the class $P_m(p_{k,\gamma})$ coincides with the class $P(p_{k,\gamma})$ for $m=2$. Also $P_m(p_{0,\gamma})=P_m(\gamma)$, see Noor (2011). For the class $P_m(p_{0,0})=P_m$, we refer to Goodman (1983).

Related to the class $P_m(p_{k,\gamma})$, we define the following function classes. For these classes we assume that $m\ge 2$, $k\in[0,\infty)$, $0\le\gamma<1$ and $z\in E$.

$$k-\cup R_m(\gamma)=\left\{f:\ f\in A \text{ and } \frac{zf'}{f}\in P_m(p_{k,\gamma})\right\}, \tag{9}$$

$$k-\cup V_m(\gamma)=\left\{f:\ f\in A \text{ and } \frac{(zf')'}{f'}\in P_m(p_{k,\gamma})\right\}, \tag{10}$$

and, for $\alpha\ge 0$,

$$k-\cup M_m^{\alpha}(\gamma)=\left\{f:\ f\in A \text{ and } \left\{(1-\alpha)\frac{zf'}{f}+\alpha\frac{(zf')'}{f'}\right\}\in P_m(p_{k,\gamma})\right\}. \tag{11}$$

For $k=0$, these classes reduce to the known classes $R_m(\gamma)$, $V_m(\gamma)$, and $M_m^{\alpha}(\gamma)$ which, respectively, contain the functions of bounded radius, bounded boundary and bounded Mocanu rotation of order $\gamma$, see Goodman (1983) and Noor (2011).

We now define some new subclasses of $A$.

Definition 1.3: Let $f\in A$. Then $f\in k-\cup R_m(\gamma,\sigma,a)$ if and only if $Q_a^{\sigma}f\in k-\cup R_m(\gamma)$ for $k\in[0,\infty)$, $0\le\gamma<1$, $m\ge 2$, $a>0$, $\sigma\ge 0$ and $z\in E$. The class $k-\cup V_m(\gamma,\sigma,a)$ can be defined by the relation given as:

$f\in k-\cup V_m(\gamma,\sigma,a)$ if and only if

$$zf'\in k-\cup R_m(\gamma,\sigma,a). \tag{12}$$

It can easily be seen that $f\in k-\cup V_m(\gamma,\sigma,a)$ if and only if $Q_a^\sigma f \in k-\cup V_m(\gamma)$.

Definition 1.4: Let $f\in A$. Then $f\in k-\cup M_m^\alpha(\gamma,\sigma,a)$ if and only if $Q_a^\sigma f\in k-\cup M_m^\alpha(\gamma)$ for $z\in E$. For different permissible choices of parameters, we obtain several known as well as new subclasses of $A$ as special cases.

**PRELIMINARY RESULTS**

The following lemma is a generalized version of a result proved in Kansa (2003).

Lemma 2.1: Noor (2011). Let $0\le k<\infty$ and let $\beta,\ \delta$ be any complex numbers with $\beta\neq 0$ and $\mathrm{Re}\left(\frac{\beta k}{k+1}+\delta\right)>\gamma.$ If $h(z)$ is analytic in $E$, $h(0)=1$ and satisfies:

$$\left\{h(z)+\frac{zh'(z)}{\beta h(z)+\delta}\right\}\prec p_{k,\gamma}(z), \qquad (13)$$

and $q_{k,\gamma}(z)$ is an analytic solution of:

$$\left\{q_{k,\gamma}(z)+\frac{zq'(z)}{\beta q_{k,\gamma}(z)+\delta}\right\}=p_{k,\gamma}(z), \qquad (14)$$

then $q_{k,\gamma}(z)$ is univalent, $h(z)\prec q_{k,\gamma}(z)\prec p_{k,\gamma}(z)$, and $q_{k,\gamma}(z)$ is the best dominant of (13).

Lemma 2.2: Miller (1975). Let $u=u_1+iu_2$ and $v=v_1+iv_2$ and let $\psi(u,v)$ be complex-valued function satisfying the following conditions:

(i) $\psi(u,v)$ is continuous in a domain $D\subset\mathbb{C}^2$,

(ii) $(1,0)\in D$ and $\psi(1,0)>0$,

(iii) $\mathrm{Re}\,\psi(iu_2,v_1)\le 0$ whenever $(iu_2,v_1)\in D$ and $v_1\le -\frac{1}{2}(1+u_2^2)$.

If $h(z)=1+\sum_{n=2}^{\infty}c_nz^n$ is a function analytic in $E$ such that $(h(z),zh'(z))\in D$, and $\mathrm{Re}\left[\psi(h(z),zh'(z))\right]>0$ for $z\in E$, then $\mathrm{Re}(h(z))>0$ for $z\in E$.

Lemma 2.3: Let $h(z)$ be analytic in $E$ with $h(0)=1$ and let:

$$\left\{h(z)+\frac{zh'(z)}{h(z)+\beta}\right\}\in P(\gamma),\ z\in E. \qquad (15)$$

Then $h\in P(\gamma_1)$, where $z\in E$ and $\gamma_1$ is given as:

$$\gamma_1=\left\{\frac{2(2\beta\gamma+1)}{(2\beta-2\gamma+1)+\sqrt{(2\beta-2\gamma+1)^2+8(2\beta\gamma+1)}}\right\}. \qquad (16)$$

Proof: We shall use Lemma 2.2 to prove this result. Let

$$h(z)=(1-\gamma_1)H(z)+\gamma_1. \qquad (17)$$

Then $h(z)$ is analytic in $E$ with $h(0)=1$. From (17), we have:

$$h(z)+\frac{zh'(z)}{h(z)+\beta}=(1-\gamma_1)H(z)+\gamma_1+\frac{(1-\gamma_1)zH'(z)}{(1-\gamma_1)H(z)+(\beta+\gamma_1)},$$

and from (15) it follows that:

$$\mathrm{Re}\left\{(1-\gamma_1)H(z)+(\gamma_1-\gamma)+\frac{(1-\gamma_1)zH'(z)}{(1-\gamma_1)H(z)+(\beta+\gamma_1)}\right\}>0 \text{ in } E.$$

We form the functional $\psi(u,v)$ by taking $u=H(z)$, $v=zH'(z)$ as:

$$\psi(u,v)=(1-\gamma_1)u+(\gamma_1-\gamma)+\frac{(1-\gamma_1)v}{(1-\gamma_1)u+(\beta+\gamma_1)}.$$

The first two conditions of Lemma 2.2 are easily satisfied. We verify condition (iii) as follows:

$$\begin{aligned}\mathrm{Re}\,\psi(iu_2,v_1)&=(\gamma_1-\gamma)+\mathrm{Re}\frac{(1-\gamma_1)v_1}{(1-\gamma_1)iu_2+(\beta+\gamma_1)},\\ &=(\gamma_1-\gamma)+\frac{(\beta+\gamma_1)(1-\gamma_1)v_1}{(1-\gamma_1)^2u_2^2+(\beta+\gamma_1)^2},\\ &\le(\gamma_1-\gamma)-\frac{(\beta+\gamma_1)(1-\gamma_1)(1+u_2^2)}{2\left[(1-\gamma_1)^2u_2^2+(\beta+\gamma_1)^2\right]},\ \text{for } v_1\le\frac{-(1+u_2^2)}{2},\\ &=\frac{A+Bu_2^2}{2C},\end{aligned}$$

where

$A=2(\gamma_1-\gamma)(\beta+\gamma_1)^2-(\beta+\gamma_1)(1-\gamma_1),$

$B=2(\gamma_1-\gamma)(1-\gamma_1)^2-(\beta+\gamma_1)(1-\gamma_1),$ and $C=\left[(1-\gamma_1)^2u_2^2+(\beta+\gamma_1)^2\right]>0.$ It can easily be observed that $\operatorname{Re}\psi(iu_2,v_1)\le 0$ if and only if $A\le 0$ and $B\le 0.$ From $A\le 0,$ we obtain $\gamma_1$ as given by (16) and $B\le 0$ gives us $0\le\gamma_1<1.$ We now apply Lemma 2.2 to obtain $\operatorname{Re}H(z)>0$ and this implies $h\in P(\gamma_1)$ for $z\in E.$ This completes the proof. □

Lemma 2.4: Kanas (2003). Let $1<k<\infty$ and let $p(z)$ be analytic in $E,$ $p(0)=1$ and $p(z)$ satisfies (2.1). Then:

$$p(z)\prec\frac{z}{\left[(z-k)\log\left(1-\frac{z}{k}\right)\right]}, \tag{18}$$

and

$$\operatorname{Re}p(z)>\frac{1}{(k+1)\log\left(1+\frac{1}{k}\right)}, \tag{19}$$

For the following two results, we refer to Noor (1992).

Lemma 2.5: An analytic function $f\in V_m(\rho)$ if and only if there exists $f_1\in V_m$ such that:

$$f'(z)=(f_1'(z))^{1-\rho}.$$

Lemma 2.6: Let $f\in V_m(\rho)$ and let $h(z)=\frac{(zf'(z))'}{f'(z)}$ with $h(z)=1+\sum_{n=1}^{\infty}c_nz^n.$ Then, for $z=re^{i\theta},$ $z\in E$

(i) $$\frac{1}{2\pi}\int_0^{2\pi}|h(re^{i\theta})|^2d\theta\le\frac{1-\{m^2(1-\rho)^2-1\}r^2}{1-r^2},$$

(ii) $$\frac{1}{2\pi}\int_0^{2\pi}|h'(re^{i\theta})|d\theta\le\frac{m(1-\rho)}{1-r^2}.$$

Lemma 2.7: Goluzin (1946). Let $f(z)$ be univalent and $0\le r<1.$ Then there exists a number $z_1$ with $|z_1|=r$ such that for all $z,$ $|z|=r,$ we have:

$$|z-z_1||f(z)|\le\frac{2r^2}{1-r^2}.$$

## MAIN RESULTS

Theorem 3.1: Let $\sigma\ge 0,$ $\gamma\in[0,1),$ $a>\gamma+\frac{1}{k+1}$ and $k\in[0,\infty).$ Then:

$$k-\cup R_m(\gamma,\sigma,a)\subset k-\cup R_m(\gamma,\sigma+1,a).$$

Proof: Let $f\in k-\cup R_m(\gamma,\sigma,a).$ Set

$$\frac{z(Q_a^{\sigma+1}f(z))'}{Q_a^{\sigma+1}f(z)}=h(z)=\left(\frac{m}{4}+\frac{1}{2}\right)h_1(z)-\left(\frac{m}{4}-\frac{1}{2}\right)h_2(z). \tag{20}$$

Using (1.4), we obtain:

$$\frac{z(Q_a^{\sigma}f(z))'}{Q_a^{\sigma}f(z)}=\left\{h(z)+\frac{zh'(z)}{h(z)+a-1}\right\}\in P_m(p_{k,\gamma})\text{ in }E. \tag{21}$$

Define $\phi(z)=\sum_{n=1}^{\infty}\frac{a-1+n}{a}z^n.$ Then using convolution technique, we obtain from (20):

$$\left(h(z)*\frac{\phi(z)}{z}\right)=\left(\frac{m}{4}+\frac{1}{2}\right)\left(h_1(z)*\frac{\phi(z)}{z}\right)-\left(\frac{m}{4}-\frac{1}{2}\right)\left(h_2(z)*\frac{\phi(z)}{z}\right),$$

where symbol * denotes convolution. This gives us:

$$h(z)+\frac{zh'(z)}{h(z)+a-1}=\left(\frac{m}{4}+\frac{1}{2}\right)\left(h_1(z)+\frac{zh_1'(z)}{h_1(z)+a-1}\right)-\left(\frac{m}{4}-\frac{1}{2}\right)\left(h_2(z)+\frac{zh_2'(z)}{h_2(z)+a-1}\right).$$

Using (21), we have:

$$\left\{h_i(z)+\frac{zh_i'(z)}{h_i(z)+a-1}\right\}\prec p_{k,\gamma},\quad i=1,2. \tag{22}$$

We now apply Lemma 2.1 with $\beta=1,$ $\delta=a-1,$ $a>\gamma+\frac{1}{k+1}$ and obtain:

$$h_i(z)\prec q_{k,\gamma}(z)\prec p_{k,\gamma}(z),\quad i=1,2,$$

where $q_{k,\gamma}(z)$ is the best dominant of (22) and is given as:

$$q_{k,\gamma}(z)=\frac{1}{g_{k,\gamma}(z)}-(a-1),$$

$$g_{k,\gamma}(z)=\left\{\int_0^1\left(\exp\int_t^{tz}\frac{p_{k,\gamma}(u)-1}{u}du\right)dt\right\}^{-1}.$$

This implies $h_i\in P(p_{k,\gamma}),\ i=1,2$ and $z\in E.$ Consequently $h\in P_m(p_{k,\gamma})$ in $E,$ and the proof is complete.

We have the following special cases:

Corollary 3.1: For $k=0,\ a>0,$ we derive the result for $R_m(\gamma)$ by using Lemma 2.3 with $\beta=a-1.$ It follows that, if $f\in 0-\cup R_m(\gamma,\sigma,a),$ then $f\in 0-\cup R_m(\gamma_*,\sigma+1,a),$ for $z\in E$ and

$$\gamma_*=\left[\frac{2(2\gamma(a-1)+1)}{(2a-2\gamma-1)+\sqrt{(2a-2\gamma-1)^2+8(2\gamma(a-1)+1)}}\right].$$

When $a=1,\ Q_1^{\sigma}f\in R_m(\gamma)$ and from Theorem 3.1 it follows that $Q_1^{\sigma+1}f\in R_m(\gamma_1)$ in $E$ with

$$\gamma_1=\left[\frac{2}{(1-2\gamma)+\sqrt{(1-2\gamma)^2+8}}\right],$$

and $a=1,\ \sigma=1,\ \gamma=0$ gives us an interesting result that $f\in V_m$ implies $f\in R_m(\gamma_0),\ \gamma_0=\frac{2}{1+\sqrt{9}}=\frac{1}{2}.$ This leads to a well known result, for $m=2,$ that a convex function is starlike of order $\frac{1}{2}.$

Corollary 3.2: For $k=1,\ a=1$ and $\gamma=0,$ we note that

$$\left\{h_i(z)+\frac{zh_i'(z)}{h_i(z)}\right\}\in\Omega_1,\quad i=1,2.$$

Then, using Lemma 2.1, it follows that

$$h_i(z)\prec p_{1,0}(z)=1+\frac{2}{\pi^2}\log^2\frac{1+\sqrt{z}}{1-\sqrt{z}},$$

The branch of $\sqrt{z}$ is chosen such that $\operatorname{Im}\sqrt{z}\geq 0$ and $p_{1,0}(-1)=\frac{1}{2}.$ That is , $h\in P_m(p_{1,0}).$

Therefore, $Q_1^{\sigma}f\in 1-\cup R_m(0)$ implies $Q_1^{\sigma+1}f\in 1-\cup R_m(\gamma_0).$ In other words, $f\in 1-\cup R_m(0,\sigma,1)\Rightarrow f\in 1-\cup R_m(\gamma_0,\sigma+1,1),$ where $\gamma_0=\frac{1}{2}.$

Corollary 3.3: We take $\gamma=0,\ \ k>1,\ \ a=1.$ Let $f\in k-\cup R_m(0,\sigma,1).$ Then, $i=1,2,$

$$\left\{h_i(z)+\frac{zh_i'(z)}{h_i(z)}\right\}\prec p_{k,0}(z),$$

which, on using Lemma 2.4, gives us $h_i(z)\prec q_{k,0}(z)$ in $E,$ or

$$h_i(z)\prec\frac{z}{(z-k)\log\left(1-\frac{z}{k}\right)}.$$

This implies:

$$\operatorname{Re}\ h_i(z)>\frac{1}{(k+1)\log\left(1+\frac{1}{k}\right)}=\delta_k,\ \ i=1,2.$$

Thus $h\in P_m(\delta_k)$ and this gives us:

$$f\in k-\cup R_m(\delta_k,\sigma+1,1).$$

Corollary 3.4: Let $f\in 2-\cup R_m(0,\sigma,1).$ This gives us for $i=1,2$

$$\left\{h_i(z)+\frac{zh_i'(z)}{h_i(z)}\right\}\prec p_{2,0}(z)=\frac{1}{1-\frac{z}{2}}.$$

From this it follows that:

$$\operatorname{Re}\left[h_i(z)+\frac{zh_i'(z)}{h_i(z)}\right]>\frac{2}{3}\ \Longrightarrow\operatorname{Re}h_i(z)>\frac{1}{3\log\frac{3}{2}}\approx 0.813\ldots=\delta_3.$$

Consequently $f \in 2-\cup R_m(\delta_3, \sigma+1, 1)$ in $E$.

Theorem 3.2: Let $\sigma \geq 0,\ \gamma \in [0,1),\ a > \gamma + \frac{1}{k+1}$ and $k \in [0,\infty)$. Then:

$$k-\cup V_m(\gamma,\sigma,a) \subset k-\cup V_m(\gamma,\sigma+1,a) \text{ for } z \in E.$$

Proof: The proof of this result follows immediately when use relation (11) together with Theorem 3.1. ⊔

Theorem 3.3: For $k \in (0,\infty),\ m \geq 2,\ \alpha > 0,\ \sigma \geq 0,$

$$k-\cup M_m^{\alpha}(0,\sigma,a) \subset k-\cup R_m(0,\sigma,a)$$

Proof: The case $\alpha = 0$ is obvious. We suppose $\alpha > 0$. Let

$$\frac{z(Q_a^{\sigma} f(z))'}{Q_a^{\sigma} f(z)} = h(z) = \left(\frac{m}{4}+\frac{1}{2}\right)h_1(z) - \left(\frac{m}{4}-\frac{1}{2}\right)h_2(z). \quad (23)$$

We note that $h(z)$ is analytic in $E$ with $h(0)=1$. Then, from (23), we have:

$$\left\{(1-\alpha)\frac{z(Q_a^{\sigma} f(z))'}{Q_a^{\sigma} f(z)} + \alpha\frac{(z(Q_a^{\sigma} f(z))')'}{(Q_a^{\sigma} f(z))'}\right\} = \left\{h(z) + \alpha\frac{zh'(z)}{h(z)}\right\} \in P_m(p_{k,0}). \quad (24)$$

From (23), (24) and convolution technique, it follows that

$$\left\{h_i(z) + \frac{zh_i'(z)}{\frac{1}{\alpha}h_i(z)}\right\} \prec p_{k,0}(z), \quad i=1,2,\ z \in E.$$

With $\beta = \frac{1}{\alpha},\ \delta = 0,$ we apply Lemma 2.1 to have $h_i(z) \prec q_{k,0} \prec p_{k,0}(z),$ where $q_{k,0}(z)$ is the best dominant,

$$q_{k,0}(z) = \left[\frac{1}{\alpha}\int_0^1\left(t^{\frac{1}{\alpha}-1}\exp\int_t^{tz}\frac{p_{k,0}(u)-1}{u}du\right)^{\frac{1}{\alpha}}dt\right]^{-1},$$

and consequently $h \in P_m(p_{k,0})$ for $z \in E$. This proves the result. ⊔

We have the following special cases:

Corollary 3.5: We take $\alpha = 1,\quad k = 1,$ $p_{1,0}(z) = 1 + \frac{2}{\pi^2}\log^2\frac{1+\sqrt{z}}{1-\sqrt{z}}$, and the best dominant is given by:

$$q_{1,0}(z) = \left[\int_0^1\left(\exp\int_t^{tz}\frac{p_{1,0}(u)-1}{u}du\right)dt\right]^{-1}, \quad \text{with}$$

$q_{1,0}(-1) = \frac{1}{2}.$

Therefore, from Theorem 3.3, we have:

$$1-\cup V_m(0,\sigma,a) \subset 1-\cup R_m(\tfrac{1}{2},\sigma,a).$$

Corollary 3.6: Let, for $\gamma = 0,\quad \alpha = 1,\quad k \in (1,\infty),$ $f \in k-\cup M_m^1(0,\sigma,a)$. That is,

$$f \in k-\cup V_m(0,\sigma,a) \Rightarrow f \in k-\cup R_m(\gamma_k,\sigma,a),$$

where $\delta_k = q_{k,0}(-1) = \dfrac{1}{(k+1)\log(1+\frac{1}{k})}, \quad k \in (1,\infty).$

Corollary 3.7: With $\gamma = 0,\ \alpha = 1,\ k = 2,$ it follows from Theorem 3.3 that:

$$2-\cup V_m(0,\sigma,a) \subset 2-\cup R_m(\delta_3,\sigma,a), \qquad \delta_3 = \frac{1}{3\log\frac{3}{2}} \approx 0.813\ldots.$$

Theorem 3.4: For $0 \leq \alpha_2 < \alpha_1,$ $k-\cup M_m^{\alpha_1}(\gamma,\sigma,a) \subset k-\cup M_m^{\alpha_2}(\gamma,\sigma,a).$

Proof: For $\alpha_2 = 0,$ the proof is immediate from Theorem 3.3. Therefore, we suppose $\alpha_2 > 0,$ and $f \in k-\cup M_m^{\alpha_1}(\gamma,\sigma,a).$ There exist two analytic functions $H_1(z),\ H_2(z)$ in $P_m(p_{k,r})$ such that:

$$H_1(z) = \left\{(1-\alpha_1)\frac{z(Q_a^{\sigma} f(z))'}{Q_a^{\sigma} f(z)} + \alpha_1\frac{(z(Q_a^{\sigma} f(z))')'}{(Q_a^{\sigma} f(z))'}\right\},$$

$$H_2(z) = \frac{z(Q_a^{\sigma} f(z))'}{Q_a^{\sigma} f(z)} \in P_m(p_{k,r})$$

By Theorem 3.3. We use the fact that $P_m(p_{k,r})$ is a convex set, see Noor (2011), and since:

$$(1-\alpha_2)\frac{z(Q_a^\sigma f(z))'}{Q_a^\sigma f(z)}+\alpha_2\frac{(z(Q_a^\sigma f(z))')'}{(Q_a^\sigma f(z))'}=\left(1-\frac{\alpha_2}{\alpha_1}\right)H_1(z)+\frac{\alpha_2}{\alpha_1}H_2(z),$$

we obtain the required result that:

$f \in k-\cup M_m^{\alpha_2}(\gamma,\sigma,a)$ for $z\in E$. ⊔

Remark 3.1: Since $k-\cup V_m(\gamma)\subset V_m\left(\frac{k+\gamma}{1+k}\right)$, we can easily deduce that

$Q_a^\sigma f \in k-\cup V_m(\gamma)$ implies $f\in V_m\left(\frac{k+\gamma}{1+k}\right)$.

## HANKEL DETERMINANT PROBLEM

Let $f\in A$ and be given by (1.1). Let

$$Q_a^\sigma f(z)=z+\sum_{n=2}^{\infty}A_n z^n,\ \ A_n=\left(\frac{a}{a+n-1}\right)^\sigma a_n. \tag{25}$$

For $q\geq 1,\ n\geq 1,$ we define Hankel determinant $H_q(n)$ for a function $f(z)$, given by (1), as:

$$H_q(n)=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1}\\ a_{n+1} & a_{n+2} & \cdots & a_{n+q}\\ \vdots & & & \vdots\\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2}\end{vmatrix}. \tag{26}$$

The problem of determining the rate of growth of this determinant has been considered by several authors (Noonan et al., 1972; Noor, 1992; Pemmerenke, 1966, 1967). For $z_1$ a nonzero complex number, we define:

$$\Delta_j(n,z_1,f(z))=\Delta_{j-1}(n,z_1,f(z))-z_1\Delta_{j-1}(n+1,z_1,f(z)),\ j\geq 1, \tag{27}$$

with $\Delta_1(n,z_1,f(z))=a_n$.

To prove our main theorem here, we shall need the following two results, which are due to Noonan et al. (1972).

Lemma 4.1: Let $f\in A$ and be given by (1) and let the $q$th order of Hankel determinant of $f(z)$ be defined by (26). Then, writing $\Delta_j=\Delta_j(n,z_1,f(z))$, we have:

$$H_q(n)=\begin{vmatrix} \Delta_{2q-2}(n) & \Delta_{2q-3}(n+1) & \cdots & \Delta_{q-1}(n+q-1)\\ \Delta_{2q-3}(n+1) & \Delta_{2q-4}(n+2) & \cdots & \Delta_{q-2}(n+q)\\ \vdots & & & \vdots\\ \Delta_{q-1}(n+q-1) & \Delta_{q-2}(n+q) & \cdots & \Delta_0(n+2q-2)\end{vmatrix}. \tag{28}$$

Lemma 4.2: With $z_1=\frac{n}{n+1}y,$ and $\nu\geq 0$ any integer,

$$\Delta_j(n+\upsilon,z_1,zf'(z))=\sum_{l=0}^{j}\binom{j}{l}\frac{y^l(\upsilon-(l-1)n)}{(n+1)^l}\Delta_{j-1}(n+\upsilon+l,y,f(z)).$$

We shall also need the following remark given in Noonan et al. (1972).

Remark 4.1: Consider any determinant of the form

$$D=\begin{vmatrix} y_{2q-2} & y_{2q-3} & \cdots & y_{q-1}\\ y_{2q-3} & y_{2q-4} & \cdots & y_{q-2}\\ \vdots & & & \vdots\\ y_{q-1} & y_{q-2} & \cdots & y_0\end{vmatrix},$$

with $1\leq i,\ \ j\leq q$ and $\alpha_{i,j}=y_{2q-(i+j)},\ \ D=\det(\alpha_{i,j})$. Then:

$$D=\sum_{\upsilon_1\in S_q}(\operatorname{sgn}\upsilon_1)\prod_{j=1}^{q}y_{2q-(\nu_1(j)+j)},$$

where $S_q$ is the symmetric group on $q$ elements, and $\operatorname{sgn}\upsilon_1$ is either $+1$ or $-1$. Thus, in the expansion of $D$, each summand has $q$ factors, and the sum of the subscripts of the factors of each summand is $q^2-q$.

Now let $n$ be given and $H_q(n)$ be given as in Lemma 4.1, then each summand in the expansion of $H_q(n)$ is of the form:

$$\prod_{i=1}^{q}\Delta_{\upsilon(i)}(n+2q-2-\nu_1(i)),$$

where $\nu_1\in S_q$ and $\sum_{i=1}^{q}\upsilon(i)=q^2-q,\ \ 0\leq\nu(i)\leq 2q-2.$

We now prove:

Theorem 4.1: Let $f\in k-\cup V_m(\gamma,\sigma,a)$ and let the Hankel determinant of $f(z)$, for $q\geq 2,\ n\geq 1$ be defined by (26). Then, for $m>4\left\{\frac{1+k}{1-\gamma}(q-1)\right\}-2,$ we have:

$$H_q(n) = O(1)\ n^{cq-q^2}, \quad c = \left\{\left(\frac{m}{2}+1\right)\left(\frac{1-\gamma}{1+k}\right)-\sigma-1\right\},$$

and $O(1)$ depends only on $m,\ \gamma,\ k$ and $\sigma$.

Proof: Since $f \in k-\cup V_m(\gamma,\sigma,a),$ it follows from Remark 3.1 that:

$$Q_a^\sigma f \in k-\cup V_m(\gamma) \subset V_m\left(\frac{k+\gamma}{1+k}\right) \text{ for } z\in E.$$

Let $Q_a^\sigma f = F(z)$. then, using Lemma 2.5 and a result due to Brannan (1968, 1969), we can write:

$$\begin{aligned} F'(z) &= (F_1'(z))^{1-\frac{k+\gamma}{1+k}}, \quad F_1 \in V_m, \\ &= \left[\left(\frac{s_1(z)}{z}\right)^{\frac{m}{4}+\frac{1}{2}} \Big/ \left(\frac{s_2(z)}{z}\right)^{\frac{m}{4}-\frac{1}{2}}\right]^{\left(\frac{1-\gamma}{1+k}\right)}, \quad s_1, s_2 \in S^*. \end{aligned} \tag{29}$$

We can write:

$$(zF'(z))' = F'(z)h(z),\ h\in P_m\left(\frac{\gamma+k}{1+k}\right), \quad \text{and}$$

$$T(z) = (z(zF'(z))')' = F'(z)[h(z)+zh'(z)].$$

Now, for $z = re^{i\theta}$, $j \ge 1$, $z_1$ any nonzero complex number, we consider:

$$\begin{aligned} \left|\Delta_j(n,z_1,T(z))\right| &= \left|\frac{1}{2\pi r^{n+j}}\int_0^{2\pi}(z-z_1)^j T(z)e^{-i(n+j)\theta}d\theta\right|, \\ &\le \frac{1}{2\pi r^{n+j+1}}\int_0^{2\pi}|z-z_1|^j \frac{|s_1(z)|^{\left(\frac{m}{4}+\frac{1}{2}\right)\left(\frac{1-\gamma}{1+k}\right)}}{|s_2(z)|^{\left(\frac{m}{4}-\frac{1}{2}\right)\left(\frac{1-\gamma}{1+k}\right)}}\left|h^2(z)+zh'(z)\right|d\theta, \end{aligned} \tag{30}$$

where we have used (29).

Using Lemma 2.6, Lemma 2.7 and well-known distortion result for starlike functions in (4.6), we obtain for:

$$\left(\frac{m}{4}+\frac{1}{2}\right)\left(\frac{1-\gamma}{1+k}\right) > j, \quad j \ge 1$$

$$\left|\Delta_j(n,z_1,T(z))\right| \le \frac{1}{\gamma^{n+j+1}}\left[\left(\frac{4}{r}\right)^{\left(\frac{m}{4}-\frac{1}{2}\right)\left(\frac{1-\gamma}{1+k}\right)}\left(\frac{2r^2}{1-r^2}\right)^j\right]\left[\left(\frac{r}{1-r}\right)^{\left(\frac{m}{2}+1\right)\left(\frac{1-\gamma}{1+k}\right)-2j}\frac{B(m,\gamma,k)}{1-r}\right],$$

Where $B(m,\gamma,k)$ is a constant depending on $m,\ \gamma,\ k$ and $j$ only. Now applying Lemma 4.2 and putting $z_1 = \frac{n}{n+1}e^{i\theta_n}$, $(n\to\infty)$, we have for:

$$m > \left(\frac{2(1+k)}{1+\gamma}j-2\right),\ j\ge 1,\ \Delta_j(n,e^{i\theta_n},Q_a^\sigma f(z)) = O(1)\ n^{\left(\frac{m}{2}+1\right)\left(\frac{1-\gamma}{1+k}\right)-j-2},$$

and on using relation (4.1), we obtain:

$$\Delta_j(n,e^{i\theta_n},f(z)) = O(1)\ n^{\left(\frac{m}{2}+1\right)\left(\frac{1-\gamma}{1+k}\right)-j-\sigma-2}, \tag{31}$$

where $O(1)$ depends on $m,\ \gamma,\ k,\ j,\ a$ and $\sigma$ only.

For $q=1$, $H_1(n) = a_n = \Delta_0(n)$ and from (30) we have:

$$a_n = \Delta_0(n,e^{i\theta_n},f(z)) = O(1)\ n^{\left(\frac{m}{2}+1\right)\left(\frac{1-\gamma}{1+k}\right)-j-2}.$$

For $\sigma = 0$, this result reduces to one proved in Noor (2011). For $q \ge 2$, we use Lemma 4.1, Lemma 4.2, (31) and Remark 4.1 with similar argument due to Noonan et al. (1972) to have:

$$H_q(n) = O(1)\ n^{q\left[\left(\frac{m}{2}+1\right)\left(\frac{1-\gamma}{1+k}\right)-\sigma-2\right]-q^2}, \qquad m > \left(\frac{4(1+k)}{1+\gamma}(q-1)-2\right),$$

where $O(1)$ depends only on $m,\ q,\ \gamma,\ k,\ a$ and $\sigma$. This completes the proof. □

For $\sigma = k = \gamma = 0$, we obtain the rate of growth of Hankel determinant of functions of bounded boundary rotation. By choosing different permissible values of the parameters involved, we obtain several new and some known results as special cases of this result.

## CONCLUSION

In this paper, we have used the principle of subordination and a family of integral operators to introduce some new subclasses of analytic functions in the unit disc. We have obtained several results such as inclusions results and radius problems for these classes of analytic functions. The rate of growth of Hankel determinant for the functions in these new classes is also studied. We have also discussed some special cases of our results. These results may stimulate further research in this field.

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