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Full Length Research Paper

Solutions of nonlinear singular initial value problems by modified homotopy perturbation method

M. Jafari[1], M. M. Hosseini[1] and Syed Tauseef Mohyud-Din[2]

[1]Faculty of Mathematics, Yazd University, P. O. Box 89195-74, Yazd, Iran.
[2]Department of Mathematics, HITEC University, Taxila Cantt, Pakistan.

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In this paper we present an efficient modification of the homotopy perturbation method. We apply the method to solve a class of singular initial value problems (IVPs) in the second-order ordinary differential equations (ODEs). Modified homotopy perturbation method (MHPM) yields solutions in convergent series forms with easily computable terms, and in some cases, yields exact solutions in one iteration. Comparisons with the exact solutions and the solutions obtained by standard HPM show the efficiency and accuracy of MHPM in solving singular problems.

Key words: Singular initial value problems, Homotopy perturbation method, ordinary differential equations.

INTRODUCTION

The homotopy perturbation method (HPM) was proposed for solving linear and nonlinear differential and integral equations first by He (1999, 2000). The method, which is a coupling of the traditional perturbation method and homotopy in topology, deforms continuously to a simple problem which can be solved easily. The HPM is applied to Volterra's integro-differential equation (El-Shahed, 2005); nonlinear oscillators (He, 2004); bifurcation of nonlinear problems (He, 2005); bifurcation of delay-differential equations (He, 2005); nonlinear wave equations (He, 2005); boundary value problems (He, 2006); quadratic Riccati differential equation of fractional order (Golbabai and Sayevand, 2010; Odibat and Momani, 2008); singular differential equations (Chowdhury and Hashim, 2007; Yıldırım and Ozi, 2007) and other fields (Abbasbandy, 2006; Ariel, 2010; He, 2003; Siddiqui et al., 2006). In recent years some modifications have been done on HPM for solving different kinds of differential equations (Ghorbani and Saberi-Nadjafi, 2008; Lu, 2009; Odibat, 2007; Siddiqui et al., 2009).

In Wazwaz (2002), a convenient modification of Adomian decomposition method has been proposed by introducing a new differential operator to solve singular Lane–Emden equations. In Hosseini and Nasabzadeh (2007), this new operator has been extended to solve specific singular ODEs. In Hosseini and Jafari (2009), this operator has been applicable for almost all kinds of singular differential equations.

It is the purpose of the present paper to introduce a new reliable modification on the HPM according to the defined operator mentioned in Hosseini and Jafari (2009). The new modification demonstrates a rapid convergence of the series solution if it compared with standard HPM. The obtained results suggest that this newly improvement technique introduces a powerful improvement for solving nonlinear singular problems.

The HPM and MHPM

The principles of the HPM and its applications for various kinds of differential equations are given in (Golbabai and Sayevand, 2010; Abbasbandy, 2006; Ariel, 2010; He, 2003; Siddiqui et al., 2006). For convenience of the reader, we will present a review of the HPM (He, 1999, 2000) and then we will present the algorithm of the new modification of the HPM. To achieve our goal, we

*Corresponding author. E-mail: syedtauseefs@hotmail.com.

consider the nonlinear differential equation:

$$Ly+Ny=f(r), \qquad r\in \Omega, \tag{1}$$

with boundary conditions

$$B\left(y,\frac{dy}{dn}\right)=0, \qquad r\in \Gamma, \tag{2}$$

where L is a linear operator while N is nonlinear operator, B is a boundary operator, Γ is the boundary of the domain Ω and $f(r)$ is a known analytic function.

Homotopy perturbation method (HPM)

The He's homotopy perturbation technique defines the homotopy $y(r.p):\Omega\times[0,1]\rightarrow R$ which satisfies

$$H(y,p)=(1-p)[L(y)+L(y_0)]+p\,[A(y)\text{-}f(r)]=0, \quad p\in[0,1], \quad r\in\Omega, \tag{3}$$

which is equivalent to

$$H(y,p)=L(y)-L(y_0)+pL(y_0)+p[N(y)\text{-}f(r)]=0, \tag{4}$$

where $p\in[0,1]$ is an embedding parameter and y_0 is an initial approximation which satisfies the initial conditions. It follows from equations (3) and (4) that

$$H(y,0)=L(y)-L(y_0)=0 \text{ and } H(y,1)=A(y)-f(r)=0. \tag{5}$$

Thus, the changing process of p from 0 to 1 is just that of $y(r,p)$ from $y_0(r)$ to $y(r)$. In topology, this is called deformation and $L(y)-L(y_0)$, $A(y)-f(r)$ are called homotopic. Here the embedding parameter is introduced much more naturally, unaffected by artificial factors; further it can be considered as a small parameter for $0\leq p\leq 1$. So it is very natural to assume that the solution of (3) and (4) can be expressed as

$$y(x)=y_0(x)+p\,y_1(x)+p^2y_2(x)+\cdots. \tag{6}$$

The approximate solution of Eq. (1), therefore, can be readily obtained,

$$y=lim_{p\rightarrow 1}\, y(x)=y_0(x)+y_1(x)+y_2(x)+\cdots. \tag{7}$$

The convergence of the series (7) has been proved by He 1999, 2000.

Modified homotopy perturbation method

Consider the singular initial value problems in the form:

$$y''+p(x)y'+N(x,y)=g(x), \tag{8}$$

subject to the initial conditions

$$y(a)=A, \qquad y'(a)=B, \tag{9}$$

where $N(x,y)$ is a continuous real value function and $g(x)$ and $p(x)$ are given functions and A,B are constants.

In addition, suppose that,

$$p(x)=\frac{h(x)}{(x-a)}, \tag{10}$$

where $h(x)$ has Taylor series in $x=a$. In Hosseini and Nasabzadeh 2007, a new differential operator has been presented, as below:

$$L(.)=e^{-\int p(x)dx}\frac{d}{dx}(e^{\int p(x)dx}\frac{d}{dx}(.)). \tag{11}$$

So, the problem (8) can be written as,

$$Ly+Ny=g(x), \tag{12}$$

The inverse operator L^{-1} is therefore considered a two-fold integral operator, as below,

$$L^{-1}(.)=\int_a^x e^{-\int p(x)dx}\int_a^x e^{\int p(x)dx}(.)dxdx. \tag{13}$$

According to HPM, and by (13) we can determine the components $y_n(x)$, and the series solution of $y(x)$ in equation (7) can be obtained. For numerical purposes, the n-term approximant,

$$\Psi_n=\sum_{n=0}^{n-1}y_n(x), \tag{14}$$

can be used to approximate the exact solution.

To continue, the performance of modified HPM (above proposed method) is considered. By attention to equations (7) and (13), computing $y_n(x)$'s may be difficult, especially, when $p(x)$ has singularity at initial point, $x=a$. To remove this difficulty, we attempt to approximate $e^{\int p(x)dx}$ and $e^{-\int p(x)dx}$ by polynomials. Thus, in this case, $y_n(x)$'s (which are presented in equation

(7)), will be easily computed. For this reason, we substitute the Taylor series of $h(x)$, at $x=a$, in equation (10). Now, for arbitrary natural number m, we put,

$$p(x)=\frac{1}{(x-a)}\sum_{k=0}^{m}\frac{(x-a)^k}{k!}h^k(a).$$

Thus, we have,

$$\int p(x)dx \approx \ln(x-a)^{h(a)}+(x-a)h'(a)+\cdots+\frac{(x-a)^m}{m\times m!}h^m(a).$$

and

$$e^{\int p(x)dx}\approx (x-a)^{h(a)}S(x), \tag{15}$$

where,

$$S(x)=e^{(x-a)h'(a)+\cdots+\frac{(x-a)^m}{m\times m!}h^m(a)}.$$

Again, for arbitrary natural number ν, we substitute the Taylor series of $S(x)$ in Equation (15),

$$e^{\int p(x)dx}\approx (x-a)^{h(a)}\{S(a)+(x-a)S'(a)+\cdots+\frac{(x-a)^\nu}{\nu!}S^\nu(a)\}, \tag{16}$$

and with a similar way, we have,

$$e^{-\int p(x)dx}\approx (x-a)^{-h(a)}\{\tilde{S}(a)+(x-a)\tilde{S}'(a)+\cdots+\frac{(x-a)^\nu}{\nu!}\tilde{S}^\nu(a)\}, \tag{17}$$

where,

$$\tilde{S}(x)=e^{-(x-a)h'(a)-\cdots-\frac{(x-a)^m}{m\times m!}h^m(a)}.$$

It will be shown that substitution of the approximations (16) and (17) in L^{-1} operator (13), is convenient and by using these approximations, we can easily compute $u_n(x)$'s.

Numerical examples

Example 1.

Consider the nonlinear singular IVP (Hosseini and Jafari, 2009),

$$y''+\frac{1+5x}{2x(x+1)}y'+y^3=g(x), \qquad y(0)=y'(0)=0, \tag{18}$$

where $g(x)$ is compatible to the exact solution

$$y(x)=\frac{2}{3}x^3+\frac{1}{2}x^4. \tag{19}$$

Standard homotopy perturbation method

We put

$$L(.)=\frac{d^2}{dx^2}(.),$$

So

$$L^{-1}(.)=\int_0^x\int_0^x(.)dxdx.$$

In an operator form, equation (18) yields

$$Ly+\frac{1+5x}{2x(x+1)}y'+y^3=g(x),$$

According to the HPM, we now construct a homotopy $y(r.p):R\times[0,1]\rightarrow R$ which satisfies the following equation

$$y''+p(\frac{1+5x}{2x(x+1)}y'+y^3-g(x))=0. \tag{20}$$

By applying the standard procedures of HPM, we have

$$y_0''=0, \qquad y_0(0)=y_0'(0)=0,$$
$$y_1''+\frac{1+5x}{2x(x+1)}y_0'+y_0^3=g(x), \qquad y_1(0)=y_1'(0)=0,$$
$$y_2''+\frac{1+5x}{2x(x+1)}y_1'+3y_1y_0^2=0, \qquad y_2(0)=y_2'(0)=0, \;\vdots$$

By solving above system we have,

$$y_0(x)=0,$$
$$y_0(x)+y_1(x)=\frac{5}{6}x^3+\frac{11}{12}x^4+\frac{4}{1485}x^{11}+\frac{1}{198}x^{12}+\cdots,$$
$$y_0(x)+y_1(x)+y_2(x)=\frac{5}{8}x^3+\frac{25}{72}x^4-\frac{7}{60}x^5+\frac{7}{90}x^6+\cdots; \;\vdots$$

$$y_0(x)+y_1(x)+\cdots+y_6(x)=\frac{1365}{2048}x^3+\frac{372575}{746496}x^4-\frac{41797}{19906560}x^5-\frac{10416077}{466560000}x^6+\cdots,$$

It is easy to see that the standard HPM converges to the exact solution (19) very slowly.

Modified homotopoy perturbation method (MHPM)

Here we use,

$$L^{-1}(.)=\int_a^x e^{-\int p(x)dx}\int_a^x e^{\int p(x)dx}(.)dxdx.$$

such that:

$$\int p(x)dx=\int\frac{1+5x}{2x(x+1)}dx=\ln(\sqrt{x}(1+x)^2),$$

$$e^{\int p(x)dx}=\sqrt{x}(1+x)^2=x^{\frac{1}{2}}+2x^{\frac{3}{2}}+x^{\frac{5}{2}},$$

and

$$e^{-\int p(x)dx}=\frac{1}{\sqrt{x}(1+x)^2}. \qquad (21)$$

Now by substituting the Taylor series of $\frac{1}{(x+1)^2}$ with order 9 into equation (21), we obtain,

$$e^{-\int p(x)dx}\approx x^{\frac{-1}{2}}(1-2x+3x^2+\cdots+9x^8).$$

In an operator form, equation (18) yields

$$Ly+y^3=g(x),$$

According to the HPM, we now construct a homotopy $y(r.p):R\times[0,1]\rightarrow R$ which satisfies the following equation

$$y''+\frac{1+5x}{2x(x+1)}y'+p(y^3-g(x))=0.$$

Now by applying the procedures of HPM, we have

$$y_0''+\frac{1+5x}{2x(x+1)}y_0'=0, \qquad y_0(0)=y_0'(0)=0,$$
$$y_1''+\frac{1+5x}{2x(x+1)}y_1'+y_0^3=g(x), \qquad y_1(0)=y_1'(0)=0,$$
$$y_2''+\frac{1+5x}{2x(x+1)}y_2'+3y_1y_0^2=0, \qquad y_2(0)=y_2'(0)=0, \vdots$$

By solving above system we have,

$$y_0(x)=0,$$
$$y_0(x)+y_1(x)=\frac{2}{3}x^3+\frac{1}{2}x^4+\frac{16}{6237}x^{11}+\frac{65378}{39123}x^{12}+\cdots,$$
$$y_0(x)+y_1(x)+y_2(x)=\frac{2}{3}x^3+\frac{1}{2}x^4+O(x^{12}),$$

which $y\approx y_0(x)+y_1(x)+y_2(x)$ is quite close to the exact solution (18).

Example 2

Consider the nonlinear singular IVP (Hosseini and Jafari 2009),

$$y''+\frac{1+5x}{2x(x+1)}y'+y\ln(y)=g(x), \qquad y(0)=1,\ y'(0)=0, \qquad (22)$$

where $g(x)$ is compatible to the exact solution

$$y(x)=(1-x+x^2)e^x. \qquad (23)$$

Here, it is convenient to use the Taylor series of $g(x)$ with order ν which by choosing $\nu=9$,

$$g(x)=\frac{3}{2}+7x+\frac{31}{4}x^2+\frac{14}{3}x^3+\frac{95}{48}x^4+\frac{7}{8}x^5+\frac{739}{1440}x^6+\frac{647}{2520}x^7+\frac{317}{11520}x^8.$$

Standard homotopy perturbation method

According to example 1 we have,

$$y''+p(\frac{1+5x}{2x(x+1)}y'+y\ln(y)-g(x))=0. \qquad (24)$$

By applying the standard procedures of HPM, we have

$$y_0''=0, \qquad y_0(0)=1,\ y_0'(0)=0,$$
$$y_1''+\frac{1+5x}{2x(x+1)}y_0'+y_0\ln(y_0)=g(x), \qquad y_1(0)=y_1'(0)=0,$$
$$y_2''+\frac{1+5x}{2x(x+1)}y_1'+y_1\ln(y_0)=0, \qquad y_2(0)=y_2'(0)=0, \vdots$$

By solving above system we have,

$$y_0(x)=1,$$

$$y_0(x)+y_1(x)=1+\frac{3}{4}x^2+\frac{7}{6}x^3+\frac{31}{48}x^4+\cdots,$$

$$y_0(x)+y_1(x)+y_2(x)=1+\frac{3}{8}x^2+\frac{3}{8}x^3+\frac{41}{288}x^4+\cdots,\vdots$$

$$y_0(x)+y_1(x)+\cdots+y_6(x)=1+\frac{63}{128}x^2+\frac{1323}{2048}x^3+\frac{270415}{746496}x^4+\cdots,$$

Here, the standard HPM converges to the exact solution (23) very slowly, since the Taylor series of the exact solution is as below,

$$y(x)=1+\frac{1}{2}x^2+\frac{2}{3}x^3+\frac{3}{8}x^4+\frac{2}{15}x^5+\frac{5}{144}x^6+\frac{1}{140}x^7+\frac{7}{5760}x^8+\frac{1}{5670}x^9+\frac{1}{40320}x^{10}+\cdots \quad (25)$$

Modified homotopoy perturbation method (MHPM)

According to example 1 we have,

$$y''+\frac{1+5x}{2x(x+1)}y'+p(y\ln(y)-g(x))=0.$$

Now by applying the procedures of HPM, we have

$$y_0''+\frac{1+5x}{2x(x+1)}y_0'=0, \qquad y_0(0)=1, \quad y_0'(0)=0,$$

$$y_1''+\frac{1+5x}{2x(x+1)}y_1'+y_0\ln(y_0)=g(x), \qquad y_1(0)=y_1'(0)=0,$$

$$y_2''+\frac{1+5x}{2x(x+1)}y_2'+y_1\ln(y_0)=0, \qquad y_2(0)=y_2'(0)=0, \vdots$$

By solving above system we have,

$$y_0(x)=1,$$

$$y_0(x)+y_1(x)=1+\frac{1}{2}x^2+\frac{2}{3}x^3+\frac{23}{56}x^4+\cdots,$$

$$y_0(x)+y_1(x)+y_2(x)=1+\frac{1}{2}x^2+\frac{2}{3}x^3+\frac{3}{8}x^4+\frac{2}{15}x^5+\frac{373}{11088}x^6+\cdots;\vdots$$

$$y_0(x)+y_1(x)+\cdots+y_6(x)=1+\frac{1}{2}x^2+\frac{2}{3}x^3+\frac{3}{8}x^4+\frac{2}{15}x^5+\frac{5}{144}x^6+\frac{1}{140}x^7+\frac{7}{5760}x^8+\frac{1}{5670}x^9+\frac{1}{40320}x^{10}+\cdots;$$

The comparison between the above obtained result and equation (25) shows that the rate of convergence of the modified homotopy perturbation method is faster than the standard homotopy perturbation method for this problem.

Example 3

Consider the nonlinear singular IVP (Hosseini and Jafari 2009),

$$y''+\frac{\sin(x)}{x}y'+y^2=(x^2+\frac{1}{x})\sin^2(x)+\frac{1}{2}\sin(2x)-x\sin(x)+2\cos(x) \quad (26)$$

which has the exact solution

$$y(x)=x\sin(x). \quad (27)$$

Here, it is convenient to use the Taylor series of $g(x)$ with order ν which by choosing $\nu=9$,

$$g(x)=2+2x-2x^2-x^3+\frac{5}{4}x^4+\frac{8}{45}x^5-\frac{31}{90}x^6-\frac{1}{63}x^7+\frac{901}{20160}x^8.$$

In addition, to compute $y_n(x)$'s the power series of $\frac{\sin(x)}{x}$ with order 9, i.e.,

$$\frac{\sin(x)}{x}\approx 1-\frac{1}{6}x^2+\frac{1}{120}x^4-\frac{1}{5040}x^6+\frac{1}{362880}x^8, \quad (28)$$

is used.

Standard homotopy perturbation method

According to example 1 we have,

$$y''+p(\frac{\sin(x)}{x}y'+y^2-g(x))=0. \quad (29)$$

By applying the standard procedures of HPM, we have

$$y_0''=0, \qquad y_0(0)=y_0'(0)=0,$$

$$y_1''+\frac{\sin(x)}{x}y_0'+y_0^2=g(x), \qquad y_1(0)=y_1'(0)=0,$$

$$y_2''+\frac{\sin(x)}{x}y_1'+2y_1y_0=0, \qquad y_2(0)=y_2'(0)=0, \vdots$$

By solving above system we have,

$$y_0(x)=0,$$
$$y_0(x)+y_1(x)=x^2+\frac{1}{3}x^3-\frac{1}{6}x^4-\frac{1}{120}x^5+\cdots,$$
$$y_0(x)+y_1(x)+y_2(x)=x^2-\frac{1}{4}x^4+\frac{1}{45}x^6-\frac{13}{630}x^7+\cdots, \vdots$$

$$y_0(x)+y_1(x)+\cdots+y_9(x)=x^2-\frac{1}{6}x^4+\frac{1}{120}x^6-\frac{1}{5040}x^8+\frac{1}{362880}x^{10}-\frac{41}{33264000}x^{11}+\cdots,$$

Here, the standard HPM converges to the exact solution (27) very slowly, since the Taylor series of the exact solution is as below,

$$y(x)=x^2-\frac{1}{6}x^4+\frac{1}{120}x^6-\frac{1}{5040}x^8+\frac{1}{362880}x^{10}-\frac{1}{26127360}x^{12}+\cdots. \quad (30)$$

Modified homotopoy perturbation method (MHPM)

Here we use,

$$L^{-1}(.)=\int_a^x e^{-\int p(x)dx}\int_a^x e^{\int p(x)dx}(.)dxdx. \quad (31)$$

which by using equation (28) we have,

$$e^{\int p(x)dx}=e^{\int\frac{\sin(x)}{x}dx}\approx e^{x-\frac{1}{18}x^3+\frac{1}{600}x^5-\frac{1}{35280}x^7+\frac{1}{2665920}x^9}, \quad (32)$$

And

$$e^{-\int p(x)dx}=e^{-\int\frac{\sin(x)}{x}dx}\approx e^{-x+\frac{1}{18}x^3-\frac{1}{600}x^5+\frac{1}{35280}x^7-\frac{1}{2665920}x^9}, \quad (33)$$

Now by substituting the Taylor series of (32) and (33) with order 8 into Equation (31) we obtain,

$$L^{-1}(.)=\int_a^x v(x)\int_a^x w(x)(.)dxdx,$$

where,

$$v(x)=1-x+\frac{1}{2}x^2-\frac{1}{9}x^3-\frac{1}{72}x^4+\frac{4}{225}x^5-\frac{151}{32400}x^6-\frac{23}{99225}x^7,$$

and

$$w(x)=1+x+\frac{1}{2}x^2+\frac{1}{9}x^3-\frac{1}{72}x^4-\frac{4}{225}x^5-\frac{151}{32400}x^6+\frac{23}{99225}x^7.$$

According to the HPM, we now construct a homotopy $y(r.p):R\times[0,1]\to R$ which satisfies the following equation

$$y''+\frac{\sin(x)}{x}y'+p(y^2-g(x))=0.$$

Now by applying the procedures of HPM, we have

$$y_0''+\frac{\sin(x)}{x}y_0'=0, \qquad y_0(0)=y_0'(0)=0,$$
$$y_1''+\frac{\sin(x)}{x}y_1'+y_0^2=g(x), \qquad y_1(0)=y_1'(0)=0,$$
$$y_2''+\frac{\sin(x)}{x}y_2'+2y_1y_0=0, \qquad y_2(0)=y_2'(0)=0, \vdots$$

By solving above system we have,

$$y_0(x)=0,$$
$$y_0(x)+y_1(x)=x^2-\frac{1}{6}x^4+\frac{1}{24}x^6-\frac{1}{120}x^7+\cdots,$$
$$y_0(x)+y_1(x)+y_2(x)=x^2-\frac{1}{6}x^4+\frac{1}{120}x^6-\frac{1}{5040}x^8+\frac{1339}{1814400}x^{10}+\cdots,$$

$$y_0(x)+y_1(x)+y_2(x)+y_3(x)=x^2-\frac{1}{6}x^4+\frac{1}{120}x^6-\frac{1}{5040}x^8+\frac{1}{362880}x^{10}-\frac{1}{779625}x^{11}+\cdots,$$

The obtained results illustrate the advantages of using the proposed method in this paper, for these kinds of equations.

Conclusion

In this work, we proposed an efficient modification of the HPM which achieves to the exact or approximate solution of the singular nonlinear differential equations with less computational work comparing with the standard HPM. The new modification can provide the exact solution by using only a minimal number of iterations and improves the performance of the standard HPM. Several examples were tested by applying the modified HPM and the results have shown remarkable performance.

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