Vol. 15(3), pp. 112-119, July-September, 2020
DOI: 10.5897/IJPS2020.4879
Article Number: E5FAA0464393
ISSN 1992-1950
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International Journal of Physical Sciences

Full Length Research Paper

Application of the alternative variational iteration method to solve delay differential equations

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Received 22 March, 2020; Accepted 14 July, 2020

In this paper, alternative variational iteration method (AVIM) is presented to solve linear and nonlinear delay differential equations (DDEs). A general lagrange multiplier is used to construct a correction functional. The proposed series solutions are found to converge to exact solution rapidly. The computation of three test examples of DDEs was presented to confirm the efficacy and validity of AVIM. The results obtained show that the AVIM is very simple and efficient.

Key words: Alternative variational iteration method (AVIM), Lagrange multiplier, Delay differential equations (DDEs).

INTRODUCTION

In several physical phenomena, science, engineering and applied mathematics, differential equations appear both linear and nonlinear latency. The introduction of delay in the model enriches its dynamics of these models and allows for a precise description of the phenomena of real life. Delay differential equations (DDEs) are proved to be useful in Fridman et al. (2000), control systems. In many physical phenomena, lasers, traffic models Davis (2002), metal cutting, epidemiology, neuroscience, population dynamics Kuange (1993), chemical kinetics Epstein and Luo (1991). In disparate ordinary differential equation (ODE), where the initial conditions are defined at preliminary point, DDEs given device history as the initial conditions over the delay interval. Different kinds of vigorous techniques have been developed in recent years to find an approximate solution for these such delay differential equations, as the Sumudu variational iteration method (SVIM) Vilu et al. (2019), the Laplace variational iteration method (LVIM) Biala et al. (2014), the Optimal homotopy asymptotic method (OHAM) Anakira et al. (2013), the Differential transform method (DTM) Karakoc and Bereketoglu (2009), Rashidi et al. (2011), the Homotopy perturbation method (HPM) Shakeri and Dehghan (2008), the Adomian decomposition method (ADM) Evans and Raslan (2007) and the Homotopy analysis method (HAM) Alomari et al. (2009), Hassan and Rashidi (2011). The method of variational iteration method (VIM) was developed for solving nonlinear problems by He (1999, 2000, 2004, 2007) He and Wang (2007) and He and Wu (2007). Subsequent Batiha et al. (2007), Mahdy et al. (2015), Noor and Mohyu-Din (2008), Wazwaz (2009) and Wu (2013) reflect the VIM

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procedure's versatilliy, reliability and performance. The aim of this paper is to use alternative method of variational iteration (AVIM) (Sakar and Ergoren, 2015; Singh and Kumar, 2017) to solve linear and nonlinear differential equations of delay (Biala et al., 2014; Vilu et al., 2019; Alomari et al., 2009). We show that by standard of variational iteration the approximate solution thus obtained converges more easily relative to the approximate solutions. Several illustrative examples have been presented.

ALTERNATIVE VARIATIONAL ITERATION METHOD (AVIM)

To illustrate basic idea of this method, we consider a general differential equation (Sakar and Ergoren, 2015; Singh and Kumar, 2017)

$$L\left[x(t)\right]+R\left[x(t)\right]+N\left[x(t)\right]=g(t),\quad x^{(k)}(0)=c_k,\ k=0,\ 1,\dots,\ m-1, \tag{1}$$

where c_k are real numbers, R is a linear operator, N is a nonlinear operator, $L=d^m x(t)/dt^m$ is the term of maximum order derivative, and $g(t)$ is a known analytic function.

The correctional function (1) can be constructed using AVIM as defined in Odibat (2010) as:

$$x_{k+1}(t)=x_k(t)+\int_0^t \lambda(\tau)\left[L\left[x(t)\right]+R\left[\tilde{x}_k(\tau)\right]+N\left[\tilde{x}_k(\tau)\right]-g(\tau)\right]d\tau, \tag{2}$$

Where the Lagrange multiplier $\lambda(\tau)$ can be optimally defined by way of variation theory. Generally, the following Lagrange multipliers are used:

$$\lambda(\tau)=\frac{(-1)^m}{(m-1)!}(t-\tau)^{m-1},\quad m\geq 1. \tag{3}$$

Equations 3 and 2 yield the following iteration formula

$$y_0(t)=\sum_{k=0}^{m-1}\frac{c_k}{k!}t^m,\quad y_{k+1}(t)=\frac{(-1)^m}{(m-1)!}\int_0^t (t-\tau)^{m-1}\left(L\left[y_k(\tau)\right]+R\left[\tilde{y}_k(\tau)\right]+N\left[\tilde{y}_k(\tau)\right]-g(\tau)\right)d\tau. \tag{4}$$

Now, we apply the variational iteration solution $x_k(t)=\sum_{k=0}^{\infty}y_k(t)$, in the present framework is obtained by the following iteration formula for $m\in N$.

Then we have $x(t)=\lim_{k\to\infty}x_k(t)=\sum_{k=0}^{\infty}y_k(t).$

APPLICATIONS

In this part, three examples are provided that illustrate the method of Biala et al. (2014), Vilu et al. (2019), Alomari et al. (2009).

Example 1. Consider a first order nonlinear delay differential equation as given:

$$x'(t)=1-2x^2\left(\frac{t}{2}\right),\quad x(0)=0. \tag{5}$$

Keeping Equation 4 in mind, the iteration formula of Equation 5 at $m=1$ can be constructed as

$$y_0(t)=0,\quad y_{k+1}(t)=-\int_0^t\left(\frac{d}{d\tau}\left[y_k(\tau)\right]+2\left(y_k\left(\frac{\tau}{2}\right)\right)^2-1\right)d\tau. \tag{6}$$

Applying the iteration formula of Equation 6, we attain

$$y_0(t)=0,$$

$$y_1(t)=\int_0^t d\tau=t,$$

$$y_2(t)=-\frac{1}{2}\int_0^t\tau^2\,d\tau=-\frac{t^3}{6},$$

$$y_3(t)=-\int_0^t\left(-\frac{\tau^4}{24}+\frac{\tau^6}{1152}\right)d\tau=\frac{t^5}{120}-\frac{t^7}{8064},$$

$$y_4(t)=-\int_0^t\left(\frac{\tau^6}{1920}-\frac{61\tau^8}{2580480}+\frac{67\tau^{10}}{309657600}-\frac{\tau^{12}}{990904320}+\frac{\tau^{14}}{532790162432}\right)d\tau$$
$$=-\frac{t^7}{13440}+\frac{61t^9}{23224320}-\frac{67t^{11}}{3406233600}+\frac{t^{13}}{12881756160}-\frac{t^{15}}{7990652436480}.$$

The solution of Equation 5 is

$$x(t)=\sum_{k=0}^{\infty}y_k(t). \tag{7}$$

Which is closed form to the exact solution and the results

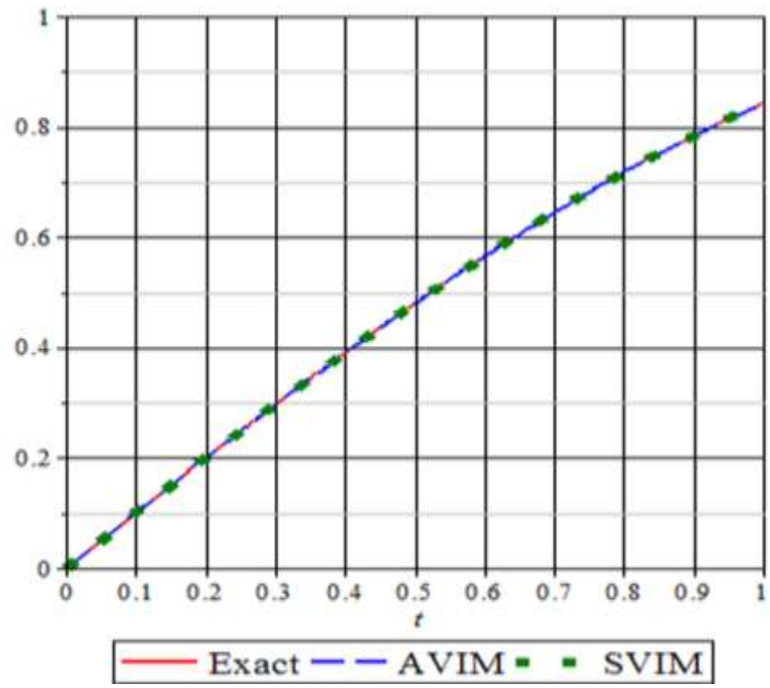

Figure 1. The solution behavior of AVIM, SVIM and exact solutions $x(t)$ of Example 1 for $t \in [0,1]$.

Table 1. Approximate AVIM solution of Example 1.

t	Exact	AVIM approx.	E_{abs}
0	0	0	0
0.2	0.1986693308	0.1986693309	0
0.4	0.3894183423	0.3894183422	$1E-10$
0.6	0.5646424734	0.5646424721	$1.3E-09$
0.8	0.7173560909	0.7173560740	$1.69E-08$
1.0	0.8414709848	0.8414708609	$1.239E-07$

due to Biala et al. (2014) and Vilu et al. (2019) and Alomari et al. (2009). The solution behavior of $x(t)$ for different time $t \in [0,1]$ depicted in Figure 1. In particular, the Solution 7 reduces to

$$x(t)=t-\frac{t^3}{6}+\frac{t^5}{120}-\frac{t^7}{5040}+\frac{61t^9}{23224320}-\frac{67t^{11}}{3406233600}+\frac{t^{13}}{12881756160}-\frac{t^{15}}{7990652436480}$$

Which is same as obtained by SVIM Vilu et al. (2019), LVIM Biala et al. (2014), HPM Alomari et al. (2009) and is closed form of the exact solution $x(t)=\sin(t)$. The approximate AVIM solution and the absolute values of AVIM are reported in Table 1 while the error behavior between SVIM Vilu et al. (2019) and AVIM in Example 1 is shown in Figure 2.

This confirms that the proposed results agreed well with solutions obtained by SVIM Vilu et al. (2019), LVIM Biala et al. (2014), Alomari et al. (2009) and approach exact solution.

Example 2. Consider a second order linear delay

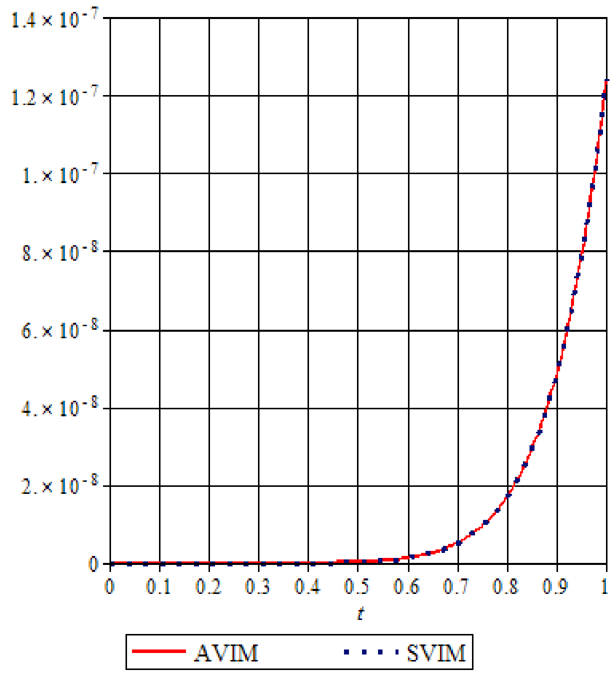

Figure 2. Plots of the absolute errors for some values of t with SVIM (Vilu et al., 2019) and AVIM in Example 1.

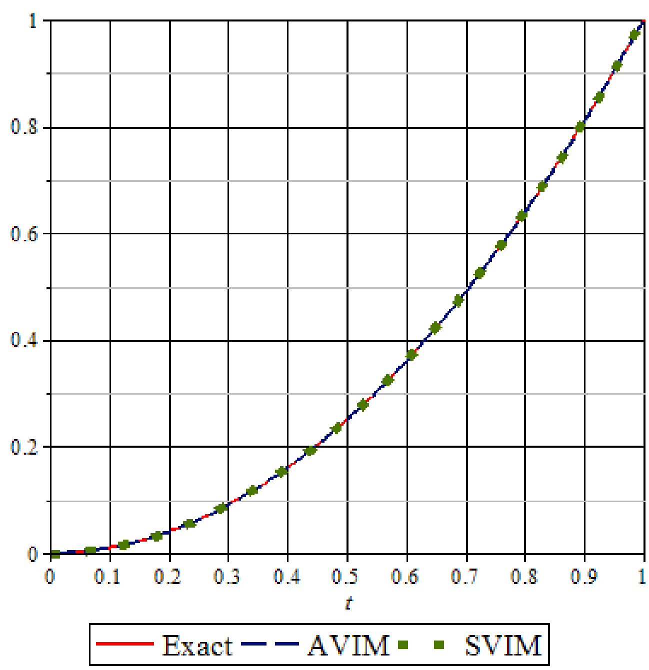

Figure 3. The solution behavior of AVIM, SVIM and exact solutions $x(t)$ of Example 2 for $t \in [0,1]$.

differential equation as given:

$$x''(t) = \frac{3}{4}x(t) + x\left(\frac{t}{2}\right) - t^2 + 2, \quad x(0) = 0, \quad x'(0) = 0. \tag{8}$$

Keeping (4) in mind, the iteration formula of Equation 8 at $m = 2$ can be constructed as:

$$\begin{aligned} &y_0(t) = 0, \\ &y_{k+1}(t) = \int_0^t (\tau - t)\left(\frac{d^2}{d\tau^2}[y_k(\tau)] - \frac{3}{4}(y_k(\tau)) - y_k\left(\frac{\tau}{2}\right) + \tau^2 - 2\right) d\tau. \end{aligned} \tag{9}$$

Applying the iteration formula of equation (9) above, we attain

$$y_0(t) = 0,$$

$$y_1(t) = \int_0^t (\tau - t)\left(\tau^2 - 2\right) d\tau = t^2 - \frac{t^4}{12},$$

$$y_2(t) = \int_0^t (\tau - t)\left(\frac{13\tau^4}{192} - \tau^2\right) d\tau = \frac{t^4}{12} - \frac{13\tau^6}{5760},$$

$$y_3(t) = \int_0^t (\tau - t)\left(-\frac{13\tau^4}{192} + \frac{637\tau^6}{368640}\right) d\tau = \frac{13t^6}{5760} - \frac{91t^8}{2949120},$$

$$y_4(t) = \int_0^t (\tau - t)\left(-\frac{637\tau^6}{368640} - \frac{17563\tau^8}{754974720}\right) d\tau = \frac{91t^8}{2949120} - \frac{17563t^{10}}{67947724800}$$

The solution of Equation 8 is

$$x(t) = \sum_{k=0}^{\infty} y_k(t). \tag{10}$$

Which is closed form to the exact solution and the results due to Biala et al. (2014) and Vilu et al. (2019) and Alomari et al. (2009). The solution behavior of $x(t)$ for different time $t \in [0,1]$ depicted in Figure 3. In particular, the solution 10 reduces to

$$x(t) = t^2 - \frac{17563t^{10}}{67947724800}.$$

Which is same as obtained by SVIM Vilu et al. (2019), LVIM Biala et al. (2014), HPM Alomari et al. (2009) and is closed form of the exact solution $x(t) = t^2$. The approximate AVIM solution and the absolute values of AVIM are reported in Table 2 while the error behavior between SVIM Vilu et al. (2019) and AVIM in Example 1 is shown in Figure 4. This confirms that the proposed

Table 2. Approximate AVIM solution of Example 2.

t	Exact	AVIM Approx.	E_{abs}
0	0	0	0
0.2	0.04	0.04	0
0.4	0.16	0.16	0
0.6	0.36	0.3599999984	$1.6E-09$
0.8	0.64	0.6399999722	$2.78E-08$
1.0	1	0.9999997415	$2.585E-07$

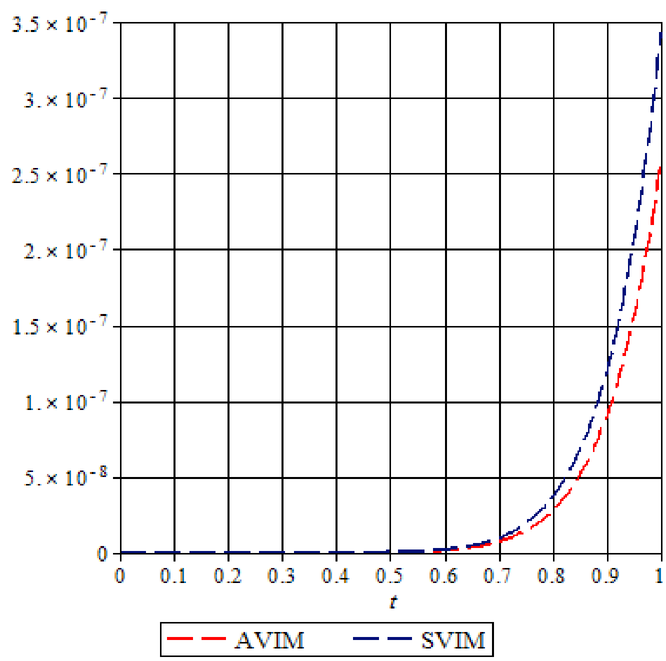

Figure 4. Plots of the absolute errors for some values of t with SVIM [Vilu et al. (2019)] and AVIM in Example 2.

results agreed well with solutions obtained by SVIM Vilu et al. (2019), LVIM Biala et al. (2014), Alomari et al. (2009) and approach exact solution.

Example 3. Consider the third order nonlinear delay differential equation as given:

$$x'''(t) = -1 + 2x^2\left(\frac{t}{2}\right), \quad x(0)=0,\ x'(0)=1,\ x''(0)=0. \tag{11}$$

Keeping (4) in mind, the iteration formula of equation (11) at $m = 2$ can be constructed as:

$$\begin{aligned} &y_0(t) = t, \\ &y_{k+1}(t) = -\frac{1}{2}\int_0^t (\tau-t)^2\left(\frac{d^3}{d\tau^3}[y_k(\tau)] - 2\left(y_k\left(\frac{\tau}{2}\right)\right)^2 + 1\right)d\tau. \end{aligned} \tag{12}$$

Applying the iteration formula of equation (12) above, we attain:

$$\begin{aligned} y_0(t) &= t, \\ y_1(t) &= -\frac{1}{2}\int_0^t (\tau-t)^2\left(1-\frac{\tau^2}{2}\right)d\tau = -\frac{t^3}{6}+\frac{t^5}{120}, \\ y_2(t) &= -\frac{1}{2}\int_0^t (\tau-t)^2\left(\frac{\tau^4}{24}-\frac{\tau^6}{720}+\frac{\tau^8}{46080}-\frac{\tau^{10}}{7372800}\right)d\tau \\ &= -\frac{t^7}{5040}+\frac{t^9}{362880}-\frac{t^{11}}{45619200}+\frac{t^{13}}{12651724800}, \end{aligned}$$

$$y_3(t) = -\int_0^t (\tau-t)^2\left(\begin{array}{l} \frac{\tau^8}{322560}-\frac{13\tau^{10}}{92897280}+2.0846\times10^{-9}\tau^{12}-1.1323\times10^{-11}\tau^{14} \\ +4.5326\times10^{-14}\tau^{16}-1.3435\times10^{-16}\tau^{18}+2.9026\times10^{-19}\tau^{20} \\ -4.3685\times10^{-22}\tau^{22}+4.1309\times10^{-25}\tau^{24}-1.8619\times10^{-28}\tau^{26} \end{array}\right)d\tau$$

$$\begin{aligned} &= -\frac{t^{11}}{319334400}+\frac{t^{13}}{12262440960}-7.6471\times10^{-13}t^{15}+2.7753\times10^{-15}t^{17}- \\ &7.7960\times10^{-18}t^{19}+1.6836\times10^{-20}t^{21}-2.7316\times10^{-23}t^{23}+3.1656\times10^{-26}t^{25} \\ &-2.3538\times10^{-29}t^{27}+8.4924\times10^{-33}t^{29}. \end{aligned}$$

The solution of Equation 11 is

$$x(t) = \sum_{k=0}^{\infty} y_k(t). \tag{13}$$

Which is closed form to the exact solution and the results

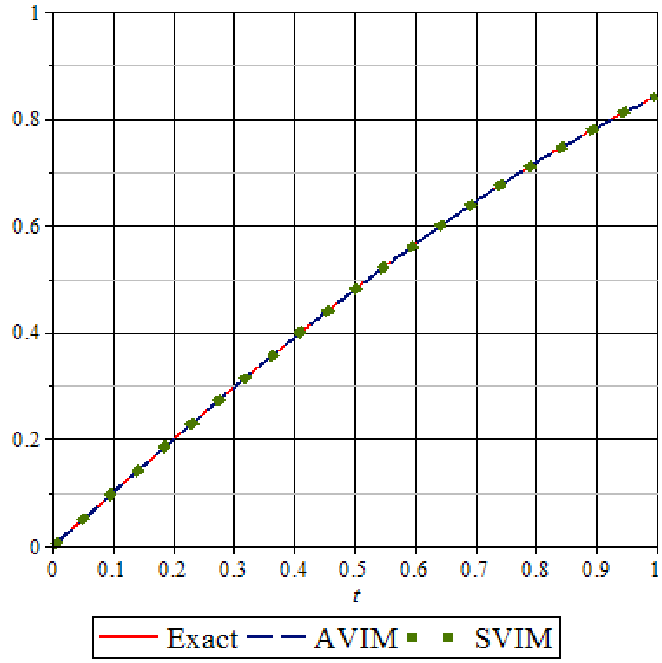

Figure 5 The solution behavior of AVIM, SVIM and exact solutions $x(t)$ of Example 3 for $t \in [0,1]$.

Table 3. Approximate AVIM solution of Example 3.

t	**Exact**	**AVIM Approx.**	E_{abs}
0	0	0	0
0.2	0.1986693308	0.1986693309	1E −10
0.4	0.3894183423	0.3894183422	1E −10
0.6	0.5646424734	0.5646424734	0
0.8	0.7173560909	0.7173560909	0
1.0	0.8414709848	0.8414709848	0

due to Biala et al. (2014) and Vilu et al. (2019) and Alomari et al. (2009). The solution behavior of $x(t)$ for different time $t \in [0,1]$ is depicted in Figure 5. In particular, Solution 13 reduces to

$$x(t) = t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040} + \frac{t^9}{362880} - \frac{t^{11}}{39916800} + \frac{t^{13}}{6227020800} - 7.6471\times10^{-13} t^{15} + 2.7753\times10^{-15} t^{17} - 7.7960\times10^{-18} t^{19} + 1.6836\times10^{-20} t^{21} - 2.7316\times10^{-23} t^{23} + 3.1656\times10^{-26} t^{25} - 2.3538\times10^{-29} t^{27} + 8.4924\times10^{-33} t^{29}.$$

Which is same as obtained by SVIM Vilu et al. (2019), LVIM Biala et al. (2014), HPM Alomari et al. (2009) and is closed form of the exact solution $x(t) = \sin(t)$. The approximate AVIM solution and the absolute values of AVIM are reported in Table 3 while the error behavior between SVIM Vilu et al. (2019) and AVIM in Example 1 is shown in Figure 6. This confirms that the proposed results agreed well with solutions obtained by SVIM Vilu et al. (2019), LVIM Biala et al. (2014) and Alomari et al. (2009) and approach exact solution.

Conclusion

In this paper, AVIM has been applied for finding the

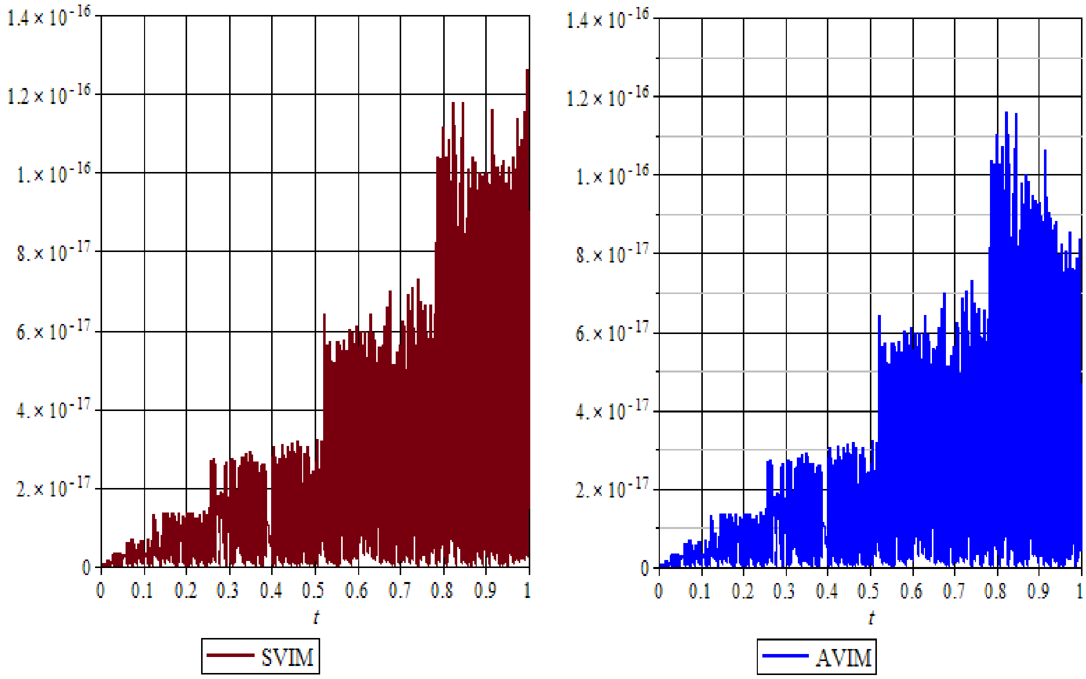

Figure 6. Plots of the absolute errors for some values of t with SVIM [Vilu et al. (2019)] and AVIM in Example 3.

approximate solutions of the linear and nonlinear DDE. The method gives more realistic series solutions that converge quickly in physical problems. The numerical results have been given in terms of the power series which converges to the exact solutions. The computation of three test examples of DDEs was presented to confirm the efficacy and validity of AVIM. The proposed agreement was excellent with Biala et al. (2014) and Vilu et al. (2019). These approximate solutions are obtained without any perturbation, discretization or restrictive conditions.

CONFLICT OF INTERESTS

The authors have not declared any conflict of interests.

ACKNOWLEDGEMENT

The authors wish to thank the referees for their comments.

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