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Full Length Research Paper

Extended general bifunction variational inequalities

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In this paper, we introduce and consider a new class of variational inequalities which is called the mixed extended general bifunction variational inequality. We suggest and analyze some proximal methods for solving the mixed extended general bifunction variational inequalities using the auxiliary principle technique. Convergence of these methods is considered under some mild suitable conditions. Several cases are also discussed. Results in this paper include some new and known results as special cases.

Key words: General variational inequality, implicit method, convergence, auxiliary principle technique.

INTRODUCTION

Variational inequalities, which were introduced and studied in the sixties, have appeared as an important and interesting branch of mathematical sciences with applications in industry, regional sciences, and pure and applied sciences. Variational inequalities can be viewed as natural extension of the variational principles. It is a well known fact that the optimality conditions for the minimum of a differentiable convex function on a convex set can be characterized by the variational inequalities. Noor (2000) has shown that the optimality conditions of a differentiable non-convex functions on the non-convex set can characterize a by class of variational inequalities, which is called the general variational inequality. It has been shown that a wide class of odd-order and non-symmetric boundary value problems can be studied in the unified and general framework of the general variational inequalities. These problems have been extended and generalized in several direction using novel and innovative ideas and techniques (Giannessi and Maugeri, 1995, Giannessi et al., 2001; Glowinski et al., 1981; Noor, 1988, 2004, 2006, 2009; Noor et al., 1993, 2011, 2011a, 2011b, 2011c, 2011d).

Motivated and inspired by the research going on in this dynamic and interesting filed, we introduce and study a new class of general variational inequalities, which is called the extended general bifunction variational inequality. This class includes the general bifunction variational inequalities and its variant forms as special cases. One can easily show that the minimum of a directionally (Gateaux) differentiable non-convex functions can be characterized by this new class of extended general bifunction variational inequality. We note that the projection method and its variant form, resolvent method cannot be used to suggest some iterative methods for solving the mixed general equilibrium bifunction variational inequalities. This fact motivated us to use the technique of auxiliary principle of Glowinski et al. (1981) to suggest and analyze some implicit iterative methods for solving the mixed general equilibrium bifunction variational inequalities (Algorithms 1 and 2). We also consider the convergence criterion of the proposed methods (Algorithm 1) under suitable mild conditions, which are the main results (Theorems1 and 2) of this paper. Several special cases of our main results are also considered. Results obtained in this paper may be viewed as an improvement and refinement of the previously known results. The comparison of these methods with other technique is an open problem.

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PRELIMINARIES

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$, respectively. Let

$B(.,.)$: $\mathrm{H}\times\mathrm{H}\rightarrow R$ be a bifunction and let $\varphi(.)$ be a continuous function. We recall the following well-known results and concepts.

For given bifunction $B(.,.)$: $\mathrm{H}\times\mathrm{H}\rightarrow R$ and non-linear operators $g,h:H\rightarrow H$, we consider the problem of finding $u\in H$ such that:

$$B(u,g(v)-h(u))+\phi(g(v))-\phi(h(u))\geq 0,\ \forall v\in H. \quad (1)$$

An inequality of the type (1) is called the mixed extended general bifunction variational inequality. We now show that the minimum of a sum of directionally (Gateaux) differentiable non-convex function and non-differentiable non-convex function can be characterized by a class of mixed general equilibrium bifunction variational inequality of type (1). For this purpose, we recall the following well known concepts, which are mainly due to Noor (2009).

Definition 1

Let K be subset in the real Hilbert space H. The set K is said to be gh-convex set with respect to the operators $g,h:H\rightarrow H$, if and only if

$$(1-t)h(u)+tg(v)\in K,\ \ \forall u,v\in K: h(u),g(v)\in K,\ \ t\in[0,1].$$

It is clear that every convex set is a gh-convex set, but the converse is not true (Noor, 2009). Note that, if $g\equiv h$, then gh-convex set is a g-convex set.

Definition 2

Let K be a gh-convex set in H. A function f on the gh-convex set K is said to g-convex function, if and only if,

$$f((1-t)h(u)+tg(v))\leq(1-t)f(h(u))+tf(g(v),\ \forall u,v\in H: h(u),g(v)\in K,\ t\in[0,1].$$

It is known that every convex function is a gh-convex function, but the converse is not true (Noor, 2009; Youness, 1999).

We now show that the minimum of a sum of directionally differentiable and non-differentiable gh-convex functions can be characterized by the problem of type (1). For this purpose, we consider the functional $I[v]$, which is defined as:

$$I[v]=f(v)+\varphi(v).$$

If $f(.)$ is directionally (Gateaux0 differentiable gh-convex and $\varphi(.)$ is non-differentiable gh-convex function, then, using the technique of Noor (2009), one can easily show that the minimum of $I[v]$ on the gh-convex set K can be characterized by the problem of the type (1) with

$$f'(g(u),g(v)-g(u))=B(u,g(v)-g(u)).$$

This shows that problem of the type (1) can be used to characterize the optimum of non-convex (gh-convex) functions. Similarly, other problems, which arise in different branches of pure and applied sciences can be studied via the general and unified framework of the problem (1) and its variant forms.

Special cases

(1). If $\phi(.)$ is an indicator function of a closed convex set K in H, then problem (1) is equivalent to finding $u\in H: h(u)\in K$ satisfying

$$B(u,g(v)-h(u))\geq 0,\quad \forall v\in H: g(v)\in K, \quad (2)$$

which is known as the extended general bifunction variational inequality and appears to be a new one.

(2). If $g=h$ then the extended general bifunction variational inequality (2) is equivalent to finding $u\in H: g(u)\in K$ such that

$$B(u,g(v)-g(u))\geq 0,\quad \forall v\in H: g(v)\in K, \quad (3)$$

which is called the classical general bifunction variational inequalities. It can be shown that the minimum of a directionally differentiable of a non-convex function on a non-convex set in a Hilbert space can be characterized by the general bifucntion variational inequality of type (3). This shows that the general bifunction variational inequalities have the same relationship with non-convex function as the bifunction variational inequalities have with the directional differentiable convex function.

(3). If $g=h=I$, the identity operator, the extended general variational inequality (2) is equivalent to finding $u\in K$ such that

$$B(u,v-u)\geq 0,\quad \forall v\in K, \quad (4)$$

which is called the bifunction variational inequality. For the formulation, applications, numerical methods and other aspects of the bifunction variational inequalities (4), (Crespi et al., 2005, 2008; Fang and Hu, 2007; Noor, 2006).

(4). If $B(u,v-u)=\langle Tu,v-u\rangle$, then problem (4) reduces to finding $u\in K$ such that

$$\langle Tu,v-u\rangle \geq 0, \quad \forall v\in K, \tag{5}$$

which is called the classical variational inequality. For the recent applications, numerical methods and other aspects of the variational inequalities and related optimization problems (Noor, 1988, 2004, 2006, 2009; Noor et al., 1993, 2011, 2011a, 2011b, 2011c, 2011d, 2011e, 2011f).

Definition 3

The bifunction $B(.,.):H\times H\rightarrow R$ is said to be monotone with respect to the operator g, if and only if,

$$B(u,g(v)-h(u))+B(v,g(u)-h(v))\leq 0, \quad \forall u,v\in H.$$

MAIN RESULTS

In this section, we use the auxiliary principle technique to suggest and analyze some implicit iterative methods for solving the mixed extended general bifunction variational inequality (1). This technique is due to Glowinski et al. (1981).

For a given $u\in H$, consider the problem of finding $w\in H$ such that

$$\rho B(w,g(v)-h(w))+\langle h(w)-g(u),g(v)-h(w)\rangle+\rho\phi(g(v))-\rho\phi(h(w))\geq 0, \ \forall v\in H, \tag{6}$$

which is called the auxiliary the mixed extended general bifunction variational inequality.

Note that if $w=u$, then $w\in K$ is a solution of (1). This observation enables us to suggest and analyze the following iterative method for solving the mixed extended general bifunction variational inequality (1).

Algorithm 1

For a given $u_0\in H$, compute $u_{n+1}\in H$ from the iterative scheme

$$\begin{aligned}\rho B(u_{n+1},g(v)-h(u_{n+1}))+\langle h(u_{n+1})-g(u_n),g(v)-g(u_{n+1})\rangle \\ \geq \rho\phi(h(u_{n+1}))-\rho\phi(g(v)), \quad \forall v\in H.\end{aligned} \tag{7}$$

If $\phi(.)$ is an indicator function on the closed convex set K in H, then Algorithm 1 can be used to find the approximate solution of problem (4) and which appears to be new one.

Algorithm 2

For a given $u_0\in H$, compute $u_{n+1}\in H$ from the iterative scheme

$$\rho B(u_{n+1},g(v)-h(u_{n+1}))+\langle h(u_{n+1})-g(u_n),g(v)-h(u_{n+1})\rangle\geq 0, \quad \forall v\in H: g(v)\in K.$$

For suitable and appropriate choice of $B(.,.)$, $\varphi(.)$ and spaces, one can define iterative algorithms as special cases of Algorithms 1 and 2 to find the solutions to different classes of equilibrium problems and variational inequalities.

We now study the convergence analysis of Algorithm 1. Convergence analysis of other algorithms can be proved using the same technique.

Theorem 1

Let $u\in H$ be a solution of (1) and $u_{n+1}\in H$ be an approximate solution obtained from Algorithm 1. If the bifunctions $B(.,.)$ is gh-monotone, then

$$\|g(u)-h(u_{n+1})\|^2\leq\|g(u)-g(u_n)\|^2-\|h(u_{n+1})-g(u_n)\|^2. \tag{8}$$

Proof

Let $u\in H$ be a solution of (1). Then, replacing v by u_{n+1} in (1), we have

$$\rho B(u,h(u_{n+1})-g(u))+\rho\phi(h(u_{n+1}))-\rho\phi(g(u))\geq 0. \quad (9)$$

Let $u_{n+1}\in H$ be the approximate solution obtained from Algorithm 1. Taking $v=u$ in (7), we have

$$\rho B(u_{n+1},g(u)-h(u_{n+1}))+\rho\phi(g(u))-\rho\phi(h(u_{n+1}))+\langle h(u_{n+1})-g(u_n),g(u)-h(u_{n+1})\rangle\geq 0. \quad (10)$$

Adding (9) and (10), we have

$$\rho B(u,h(u_{n+1})-g(u))+\rho B(u_{n+1},g(u)-h(u_{n+1}))+\langle h(u_{n+1})-g(u_n),g(u)-h(u_{n+1})\rangle\geq 0$$

which implies that

$$\begin{aligned}&\langle h(u_{n+1})-g(u_n),g(u)-h(u_{n+1})\rangle\\ &\quad\geq \rho B(u,h(u_{n+1})-g(u))+\rho B(u_{n+1},g(u)-h(u_{n+1}))\ \geq 0,\end{aligned} \quad (11)$$

where, we have used the gh-monotonicity of bifunction $B(.,.)$. .

Using the relation $2\langle u,v\rangle=\|u+v\|^2-\|u\|^2-\|v\|^2, \quad \forall u,v\in H$ in (11),

we have

$$\|g(u)-h(u_{n+1})\|^2\leq\|g(u)-g(u_n)\|^2-\|h(u_{n+1})-g(u_n)\|^2,$$

which is the required result (8).

Theorem 2

Let H be a finite dimensional space. Let u_{n+1} be the approximate solution obtained from Algorithm 1 and $u\in H$ be a solution of problem (1). If $g^{-1}h=I$, then $\lim\limits_{n\to\infty}u_n=u$.

Proof

Let $u\in H$ be a solution of (1). Then, we see that the sequence $\{\|g(u)-g(u_n)\|\}$ is non-increasing and consequently $\{u_n\}$ is bounded. Also from (8), we have

$$\sum_{n=0}^{\infty}\|h(u_{n+1})-g(u_n)\|^2\leq\|g(u)-g(u_0)\|^2,$$

which implies that

$$\lim_{n\to\infty}\|u_{n+1}-u_n\|=0, \quad (11)$$

since $g^{-1}h=I$. Let $\hat{u}$ be the cluster point of $\{u_n\}$ and the sub sequence $\{u_{n_j}\}$ of this sequence converges to $\hat{u}\in H$. Replacing u_n by u_{n_j} in (8) and taking the limit $n_j\to\infty$ and using (11), we have

$$B(\hat{u},g(v)-h(\hat{u}))+\phi(g(v))-\phi(h(\hat{u}))\geq 0, \qquad \forall v\in H,$$

which shows $\hat{u}$ solves the mixed extended general bifunction variational inequality (1) and

$$\|h(u_{n+1})-g(\hat{u})\|^2\leq\|g(u_n)-g(\hat{u})\|^2$$

Thus, it follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point and $\lim\limits_{n\to\infty}u_n=\hat{u}$, the required result.

CONCLUSION

In this paper, we have introduced and considered a new class of the bifunction variational inequalities, which is called the mixed extended general bifunction variational inequalities. It is shown that the optimum of sum of directionally differentiable non-convex and non-differentiable non-convex functions can be characterized by this class. We have used the auxiliary principle technique to suggest some proximal iterative methods for solving the mixed extended general bifunction variational inequality. We have analyzed the convergence criteria of these new proximal methods under some very mild conditions on the involved operators. Some special cases are also discussed. The technique and ideas used in obtaining the results of this paper inspire further research in these filed and related areas.

RECOMMENDATIONS

We would like to emphasize that one can study the sensitivity analysis and dynamical system associated with the extended general bifunction variational inequalities. This is another aspect of this problem. Results obtained in this paper may be extended and generalized for the multi-valued extended general bifunction variational inequalities and its variant forms. It is an interesting problem from both application point of view and numerical analysis to verify the implementation and efficiency of the proposed iterative methods for solving the mixed equilibrium variational inequalities. Recently, much attention has been given to study the existence of a solution of the equilibrium problems and variational inequalities in the topological vector spaces using the KKM mapping theorem. This is an open problem to study the existence of a solution of the mixed extended general bifunction variational inequalities in the Banach and topological spaces. It is known that the variational inequalities can be characterized by system of variational equations using the penalty methods. One may try to use the penalty method to characterize the bifunction variational inequalities by a system of bifunction variational equations. This is another aspect of the future research in this fast developing field. The interested readers are advised to explore these aspects along with some novel and innovative applications.

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