International Journal of Physical Sciences Vol. 5(4), pp. 293-298, April 2010
Available online at http://www.academicjournals.org/IJPS
ISSN 1992 - 1950 © 2010 Academic Journals

Full Length Research Paper

Complete spacelike hypersurfaces with constant scalar curvature and sectional curvatures ≥ 1 in De Sitter space $S_1^6(1)$

Mehmet Erdogan* and Jeta Alo

Department of Mathematics and Computing, Faculty of Science and Letters, Beykent University, Ayazaga, 34396 Sisli, Istanbul, Turkey.

Accepted 22 January, 2010

Hypersurfaces with constant scalar curvature and two different principal curvatures isometrically immersed in an $(n+1)$-dimensional space form M^{n+1} (C) of constant curvature *c* and especially in $S^{n+1}(c)$ have been extensively investigated within the last four decades. In the present work, we study complete spacelike hypersurfaces with constant scalar curvature and have sectional curvatures $K(\pi)\geq 1$ in de Sitter space $S_1^6(1)$ and find a result on the type number of such a hypersurface.

Key words: Hypersurfaces, scalar curvature, type number, primary 53c40, secondary 53c15.

INTRODUCTION

Let $M_p^{n+p}(c)$ be a $(n+p)$-dimensional connected semi-Riemannian manifold of constant curvature *c* whose index is *p*. It is called an indefinite space form of index *p* or simply a Lorentzian space form when $p=1$. If $c>0$, we call it as a de sitter space of index *p* and denote it by $S_p^{n+p}(c)$. Now, let L^{n+2} be the $(n+2)$-dimensional Lorentz-Minkowski space, that is, the real vector space $\mathbb{R}^{n+2}$ endowed with the Lorentzian metric tensor and let $S_1^{n+1}\subset L^{n+2}$ be the $(n+1)$-dimensional unitary de Sitter space. For $n\geq 2$ the de Sitter space S_1^{n+1} is the standard simply connected Lorentzian space form of positive constant sectional curvature 1. A smooth immersion $\varphi: M^n \to S_1^{n+1}$ of an *n*-dimensional connected manifold M^n is said to be a spacelike hypersurface if the induced metric via φ is a Riemannian metric on M^n. A hypersurface in E^{n+1} is said to be of type number *p* if the rank of its second fundamental form is *p*. In 1979 B.Y. Chen introduced the isometric immersions in Euclidean spaces of finite type (Chen, 1979). Essentially these are submanifolds whose immersion into E^{n+1} is constructed by making use of a finite number of E^{n+1}-valuated eigenfunctions of their Laplacian. In terms of finite type terminology, a well-known result of Takahashi (Takahashi, 1966), affirms that a connected Euclidean submanifold is of 1-type, if and only if it is either minimal in E^{n+1} or minimal in some hypersphere of E^{n+1}.

Minimal and isoparametric hypersurfaces with distinct principal curvatures have been studied by many authors, (Chen, 1979; Chern, 1970; Erdogan, 2010; Itoh and Nakagawa, 1973; Lawson, 1969; Otsuki, 1970; Otsuki, 1978; Peng and Terng, 1983). T. Otsuki, T. Itoh and H. Nakagawa gave a lot of examples of complete hypersurfaces with type number 1 in $H^{n+1}(c)$, (Itoh and Nakagawa, 1973; Otsuki, 1970; Otsuki, 1978). On the other hand, there exist many hypersurfaces with type number ≤ 2 in E^{n+1} by the fundamental theorem for hypersurfaces (Sasaki, 1972).

In an early work, we studied hypersurfaces with constant scalar curvature and having sectional curvatures ≤ 1 in six-dimensional sphere $S^6(1)$, (Erdogan, 2010).

In the present paper, we study complete spacelike

*Corresponding author. E-mail: erdoganm@beykent.edu.tr.

hypersurfaces with constant scalar curvature and having sectional curvatures ≥ 1 in $S_1^6(1)$ and obtain a new result on the type number of the hypersurfaces. Let *M* be a complete spacelike hypersurface with second fundamental form h in $S_1^6(1)$. The eigenvalues $\lambda_i, 1 \leq i \leq 5$, of the second fundamental form *h* are the principal curvature functions over *M*. Our main result is the following:

Theorem

If a complete spacelike hypersurface M with constant scalar curvature in $S_1^6(1)$ has the sectional curvatures ≥ 1, then the type number of M is not greater than 1.

Preliminaries

Let M be a complete spacelike hypersurface and isometrically immersed in $S_1^6(1)$. We denote by ∇ (resp. ∇') the covariant differentiation on *M* (resp. $S_1^6(1)$). We choose a local field of Lorentzian orthonormal frames $e_1, e_2, ..., e_6$ in $S_1^6(1)$ such that at each point of *M*, $e_1, ..., e_5$ span the tangent space of *M* (and, consequently e_6 is normal to *M*). We use the following convention on the range of indices:

$$1 \leq A, B, C, ... \leq 6\ ;\ 1 \leq i, j, k, ... \leq 5\ ;\ \alpha = \beta = \gamma = 6$$

Let *B* be the set of all such frames in $S_1^6(1)$. With respect to the frame field of $S^6(1)$ chosen above, let $\omega_1, \omega_2, ..., \omega_6$ be the field of dual frames so that the Lorentzian metric of $S_1^6(1)$ is given by $d\tilde{s}^2 = \sum_i \omega_i^2 - \omega_6^2 = \sum_A \varepsilon_A \omega_A^2$, where $\varepsilon_i = 1$ and $\varepsilon_6 = -1$. Then the structural equations of $S_1^6(1)$ are given by

$$\left.\begin{aligned} d\omega_A &= \sum_{B=1}^{6} \varepsilon_B \omega_{AB} \wedge \omega_B,\ \omega_{AB} + \omega_{BA} = 0, \\ d\omega_{AB} &= \sum_{C=1}^{6} \varepsilon_C \omega_{AC} \wedge \omega_{CB} - \omega_A \wedge \omega_B \end{aligned}\right\} \tag{2.1}$$

where ω_{AB}'s are the connection forms on $S_1^6(1)$. The Ricci tensor and the scalar curvature of $S_1^6(1)$ are given respectively by

$$Ric_{AB} = Ric_{BA} = \sum_{C=1}^{6} R'_{ACBC} = 6\varepsilon_A \varepsilon_B \delta_{AB} \tag{2.2}$$

$$S' = \sum_{A=1}^{6} Ric_{AA} = \sum_{A,C=1}^{6} R'_{ACAC} = 30 \tag{2.3}$$

where R' is the Riemannian curvature tensor on $S_1^6(1)$ and its entries are given by $R'_{ABCD} = \varepsilon_A \varepsilon_B (\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC})$. The second fundamental form (the shape operator) *h* of the immersion is given by

$h(X,Y) = \nabla'_X Y - \nabla_X Y$, for tangent vectors *X* and *Y*, and it satisfies $h(X,Y) = h(Y,X)$. If we restrict these formulas to *M*, we have

$$\omega_6 = 0,\ 0 = d\omega_6 = \sum_{i=1}^{5} \omega_{6i} \wedge \omega_i,$$

and from Cartan's lemma we write

$$\omega_{i6} = \sum_{j=1}^{5} h_{ij} \omega_j \tag{2.4}$$

where $h_{ij} = h_{ji}$. The Riemann metric of *M* is written as $ds^2 = \sum_i \omega_i^2$ and we have the structure equations of *M* as follows:

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j,\ \omega_{ij} = -\omega_{ji} \tag{2.5}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l, \tag{2.6}$$

$$\begin{aligned} R_{ijkl} &= R'_{ijkl} + h_{ik}h_{jl} - h_{il}h_{jk} \\ &= \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + h_{il}h_{jk} - h_{ik}h_{jl} \end{aligned} \tag{2.7}$$

Where R is the Riemannian curvature tensor on the hypersurface *M*.

Then, the second fundamental form *h* can be written as

$$h(X,Y)=\sum_{j=1}^{5} h_{ij}\omega_i(X)\omega_j(Y)e_6 .$$

The covariant derivative $\nabla' h$ of *h*, with components h_{ijk}, is given by

$$\nabla' h=\sum_{i,j,k} h_{ijk}\omega_i\omega_j\omega_k$$

and

$$\sum_k h_{ijk}\omega_k = dh_{ij}+\sum_r h_{rj}\omega_{ri}+\sum_r h_{ir}\omega_{rj} . \tag{2.8}$$

Then we have $h_{ijk}=h_{ikj}$ for any *i, j* and *k*=1, 2,3,4,5, because $S_1^6(1)$ is of constant curvature 1.
Indeed, by exterior differentiating (2.4), we get

$$d\omega_{i6}=\sum_j dh_{ij}\wedge\omega_j+\sum_{jm} h_{im}\omega_{mj}\wedge\omega_j$$

or

$$d\omega_{i6}=\sum_m \omega_{im}\wedge\omega_{m6}-\frac{1}{2}\sum_{ml} R_{iml6}\omega_m\wedge\omega_l .$$

We also have from (2.4) and (2.6)

$$d\omega_{i6}=-\sum_{jm} h_{mj}\omega_{mi}\wedge\omega_j .$$

Therefore,

$$\sum_j dh_{ij}\wedge\omega_j=-\sum_{jr} h_{rj}\omega_{ri}\wedge\omega_j-\sum_{jr} h_{ir}\omega_{rj}\wedge\omega_j . \tag{2.9}$$

So, we get

$$\sum_{kj} h_{ijk}\omega_k\wedge\omega_j=0 ,$$

Therefore, h_{ijk}'s are symmetric in all indices. Exterior differentiating the equation (2.8) and defining h_{ijkl} by

$$\sum_l h_{ijkl}\omega_l = dh_{ijk}+\sum_r h_{rjk}\omega_{ri}+\sum_r h_{irk}\omega_{rj}+\sum_r h_{ijr}\omega_{rk} \tag{2.10}$$

We have

$$\sum_{kl}(h_{ijkl}-\frac{1}{2}\sum_r h_{ir}R_{rjkl}-\frac{1}{2}\sum_r h_{rj}R_{rikl})\omega_k\wedge\omega_l=0 \tag{2.11}$$

and from this we obtain

$$h_{ijkl}-h_{ijlk}=\sum_r h_{ir}R_{rjkl}+\sum_r h_{rj}R_{rikl} . \tag{2.12}$$

Now, let us define the laplacian Δh of the second fundamental form *h* by

$$(\Delta h)_{ij}=\Delta h_{ij}=\sum_k h_{ijkk} \tag{2.13}$$

From (2.12) and (2.13) we obtain

$$\sum_k h_{ijkk}=\sum_k h_{kijk}$$

and so

$$\Delta h_{ij}=\sum_k h_{kijk} .$$

Then, from (2.11) we find

$$\Delta h_{ij}=\sum_k h_{kikj}+\sum_k(\sum_r h_{ri}R_{rkjk}+\sum_r h_{kr}R_{rijk}) \tag{2.14}$$

Proof of the theorem

Let *M* be a complete spacelike hypersurface with constant scalar curvature in $S_1^6(1)$. We suppose that the sectional curvature $K(\pi)$ of *M* is not smaller than 1, that is

$$K(\pi)\geq 1 . \tag{3.1}$$

For a plane π in the tangent space T_xM at $x\in M$ to *M*, the sectional curvature $K(\pi)$ for π is defined by

$$K(p)=1+g'(h(X,Y),h(X,Y))-g'(h(X,X),h(Y,Y)), \tag{3.2}$$

Where *g'* is the Riemannian metric of $S_1^6(1)$ and *X*, *Y* is a pair of orthonormal vectors in T_xM .

Let $\lambda_1, \lambda_2, ..., \lambda_5$ be the principal curvatures of *M*, then by (2.6) and (2.7) the sectional curvature $K(\pi_{ij})$ for the plane spanned by e_i and e_j is expressed as follows:

$$K(\pi_{ij}) = 1 - \lambda_i \lambda_j ,$$

Which, together with (3.1) implies that the following

Lemma

The type number of M is not greater than 2 at each point of M

Now let N be the set of all points at which the type number of *M* is 2. If N is not empty, then N is an open subset of *M*. Suppose that there exists a point of *M* at which the type number is greater than 1. Then by Lemma, N is a non - empty open subset of *M*. Hence there is a neighborhood U of a point $x \in N$ where we can choose a frame field $\{e_1, ..., e_5\}$ such that

$$\omega_{16} = \lambda \omega_1, \omega_{26} = \mu \omega_2, \mu < 0 < \lambda, \tag{3.3}$$

$$\omega_{k6} = 0, k = 3, 4, 5. \tag{3.4}$$

Where λ and μ are differentiable functions on U, because we have $R_{1212} = 1 - \lambda\mu \geq 1,$ i.e., $\lambda\mu < 0$ by (2.5)-(2.8) and (3.1)-(3.4). Using (2.5)-(2.7), from (3.3) and (3.4) we have

$$\lambda \omega_{1k} = \lambda_k \omega_1 + h_{12k} \omega_2, k = 3, 4, 5, \tag{3.5}$$

$$m w_{2k} = h_{12k} w_1 + m_k w_2, k = 3, 4, 5, \quad \text{....(3.6)}$$

$$w_{12} = \frac{1}{l - m}(l_2 w_1 + m_1 w_2 + \mathring{a}_{k=3}^{5} h_{12k} w_k), \tag{3.7}$$

$$h_{klj} = 0, k, l = 3, 4, 5, \; j = 1, ..., 5, \tag{3.8}$$

where $d\lambda = \sum_{i=1}^{5} \lambda_i \omega_i$. Furthermore, making use of (2.10), (3.3)-(3.8), from (2.12) and (2.14) we obtain

$$\sum_{i,j=1}^{5} h_{ij} \Delta h_{ij} = \lambda \sum_{r=1}^{5} h_{rr11} + \mu \sum_{r=1}^{5} h_{rr22}$$

$$+ \lambda\mu (R_{2112} + R_{1221}) + \lambda^2 \sum_{r=1}^{5} R_{1r1r} + \mu^2 \sum_{r=1}^{5} R_{2r2r}$$

$$= \lambda \sum_{r=1}^{5} h_{rr11} + \mu \sum_{r=1}^{5} h_{rr22} + (\lambda^2 + \mu^2)(4 + \lambda\mu) - 2(\lambda^2\mu^2 + \lambda\mu) \tag{3.9}$$

Using (3.5)-(3.8), from (2.10) we have

$$\left.\begin{aligned} h_{1111} &= \lambda_{11} - \frac{2\lambda_2^2}{\lambda - \mu} - \frac{2}{\lambda}(\lambda_3^2 + \lambda_4^2 + \lambda_5^2), \\ h_{1122} &= \lambda_{22} - \frac{2\mu_1^2}{\lambda - \mu} - \frac{2}{\lambda}(h_{123}^2 + h_{124}^2 + h_{125}^2), \\ h_{2211} &= \mu_{11} + \frac{2\lambda_2^2}{\lambda - \mu} - \frac{2}{\mu}(h_{123}^2 + h_{124}^2 + h_{125}^2), \\ h_{2222} &= \mu_{22} + \frac{2\mu_1^2}{\lambda - \mu} - \frac{2}{\mu}(\mu_3^2 + \mu_4^2 + \mu_5^2), \end{aligned}\right\} \tag{3.10}$$

where $d\lambda_i = \sum_{j=1}^{5} \lambda_{ij}\omega_j + \sum_{j=1}^{5} \lambda_j \omega_{ij}$ for any $i = 1, ..., 5.$ On the other hand, we can write

$$\left.\begin{aligned} h_{kk11} &= \frac{2}{\lambda}\lambda_k^2 + \frac{2}{\mu} h_{12k}^2, \\ h_{kk22} &= \frac{2}{\mu}\mu_k^2 + \frac{2}{\lambda} h_{12k}^2, k = 3, 4, 5. \end{aligned}\right\} \tag{3.11}$$

It follows from (3.9)-(3.11) that we have

$$\begin{aligned} \sum_{i,j=1}^{5} h_{ij}\Delta h_{ij} &= \lambda(\lambda_{11} + \mu_{11}) + \mu(\lambda_{22} + \mu_{22}) \\ &\quad - 2\lambda\mu(1 + \lambda\mu) + (\lambda^2 + \mu^2)(4 + \lambda\mu). \end{aligned} \tag{3.12}$$

Besides, we may write

$$\sum_{i,j=1}^{5} h_{ij}\Delta h_{ij} = \lambda \sum_{r=1}^{5} h_{11rr} + \mu \sum_{r=1}^{5} h_{22rr} \tag{3.13}$$

and

$$\left.\begin{aligned} h_{11kk} &= \lambda_{kk} - \frac{2}{\lambda-\mu} h_{12k}^2, \\ h_{22kk} &= \mu_{kk} + \frac{2}{\lambda-\mu} h_{12k}^2, k = 3,4,5. \end{aligned}\right\} \quad (3.14)$$

Now, it follows, from (3.10), (3.13) and (3.14), that

$$\begin{aligned} \sum_{i,j=1}^{5} h_{ij}\Delta h_{ij} &= \lambda\Delta\lambda + \mu\Delta\mu - 2\sum_{i=1}^{5}(\lambda_i^2 + \mu_i^2) \\ &\quad + 2(\lambda_1^2 + \mu_2^2) - 6\sum_{k=1}^{5} h_{12k}^2. \end{aligned} \quad (3.15)$$

Using (3.12) and (3.15) we write that

$$\begin{aligned} &(\lambda-\mu)(\mu_{11} - \lambda_{22}) - 2\lambda\mu(1+\lambda\mu) \\ &\quad + (\lambda^2 + \mu^2)(4 + \lambda\mu) \\ &= \lambda(\lambda_{33} + \lambda_{44} + \lambda_{55}) + \mu(\mu_{33} + \mu_{44} + \mu_{55}) \\ &\quad - 2\sum_{i=1}^{5}(\lambda_i^2 + \mu_i^2) + 2(\lambda_1^2 + \mu_2^2) - 6\sum_{K=3}^{5} h_{12k}^2. \end{aligned} \quad (3.16)$$

On the other hand, using (3.3) and (3.4), from (2.12) we have

$$h_{1212} = \lambda_{22} - \frac{2\mu_1^2}{\lambda-\mu} - \frac{2}{\lambda}(h_{123}^2 + h_{124}^2 + h_{125}^2),$$

$$h_{1212} = h_{2211} + (\lambda-\mu)(1+\lambda\mu),$$

$$h_{2211} = \mu_{11} + \frac{2\lambda_2^2}{\lambda-\mu} - \frac{2}{\mu}(h_{123}^2 + h_{124}^2 + h_{125}^2),$$

Which implies that

$$(\lambda-\mu)(\mu_{11}-\lambda_{22}) = \frac{(\lambda-\mu)^2}{\lambda\mu}\sum_{k=3}^{5} h_{12k}^2 - (\lambda-\mu)^2(1+\lambda\mu) - 2(\lambda_2^2 + \mu_1^2). \quad (3.17)$$

Thus, from (3.16) and (3.17) we get

$$\begin{aligned} &\lambda(\lambda_{33}+\lambda_{44}+\lambda_{55}) + \mu(\mu_{33}+\mu_{44}+\mu_{55}) = \left\{\frac{(\lambda-\mu)^2}{\lambda\mu} + 6\right\}(h_{123}^2 + h_{124}^2 + h_{125}^2) \\ &+ 3(\lambda^2+\mu^2) + 2\sum_{i=1}^{5}(\lambda_i^2 + \mu_i^2) - 2(\lambda_i^2 + \lambda_2^2 + \mu_1^2 + \mu_2^2). \end{aligned} \quad (3.18)$$

Now, by using (2.10) and (2.12) we write that

$$h_{kl11} = h_{11kl} - \lambda\delta_{kl}, h_{11kl} = \lambda_{kl} - \frac{2}{\lambda-\mu} h_{12k} h_{12l}$$

and

$$h_{kl11} = \frac{2}{\lambda}\lambda_k\lambda_l + \frac{2}{\mu} h_{12k} h_{12l}, k, l = 3,4,5.$$

which implies that

$$\lambda_{kl} = \frac{2}{\lambda}\lambda_k\lambda_l + \frac{2\lambda}{\mu(\lambda-\mu)} h_{12k} h_{12l} + \lambda\delta_{kl}, k, l = 3,4,5. \quad (3.19)$$

It follows from (3.18) and (3.19) that

$$\begin{aligned} &\lambda(\lambda_{33}+\lambda_{44}+\lambda_{55}) + \mu(\mu_{33}+\mu_{44}+\mu_{55}) = 2\sum_{k=3}^{5}(\lambda_k^2 + {}_k^2) + 3(\lambda^2+\mu^2) \\ &+ \left\{\frac{2(\lambda^2+\mu^2+\lambda\mu)}{\lambda\mu}\right\}(h_{123}^2 + h_{124}^2 + h_{125}^2). \end{aligned} \quad (3.20)$$

Making use (3.18) and (3.20) we get

$$(\lambda-\mu)^2 \sum_{k=3}^{5} h_{12k}^2 = 0,$$

which implies that

$$h_{12k} = 0 \text{ for any } k = 3,4,5. \quad (3.21)$$

Now from (3.5)-(3.7) and (3.21) , we obtain that

$$\lambda\omega_{1k} = \lambda_k\omega_1, \mu\omega_{2k} = \mu_k\omega_2, k = 3,4,5. \quad (3.22)$$

and

$$(\lambda-\mu)\omega_{12} = \lambda_2\omega_1 + \mu_1\omega_2, \mu < 0 < \lambda. \quad (3.23)$$

In this case, the scalar curvature S is given by $S = 2(\lambda\mu + 10)$ which, together with the assumption $S = const.$, implies that

$$\lambda_i\mu + \lambda\mu_i = 0, i = 1, \ldots, 5 \quad (3.24)$$

and

$$\mu_{ij} + \frac{\mu}{\lambda}\lambda_{ij} = \frac{2\mu\lambda_i\lambda_j}{\lambda^2}, i, j = 1, ..., 5. \qquad (3.25)$$

Hence, from (3.19), (3.21) and (3.25) we have that

$$\lambda_k\lambda_l + \lambda^2\delta_{kl} = 0, k, l = 3, 4, 5$$

which implies $\lambda = 0$. This result contradicts the assumption $\lambda \neq 0$, therefore, it must be $N = \varnothing$.

This result shows that the type number of *M* is not greater than 1 at each point of *M* and so the theorem is proved.

ACKNOWLEDGEMENTS

The authors would like to express their thanks to the referees for their valuable comments and suggestions.

REFERENCES

Chen BY (1979). On the total curvature of immersed manifolds, IV, Bull. Inst. Math. Acad. Sinica 7: 301-311.

Chern SS, Carmo M, Kobayashi S (1970). Minimal submanifolds of the sphere with second fundamental form of constant length, Func. Anal. Related Fields, Ed. F. Browder, NewYork, Springer pp. 59-75.

Erdogan M, Alo J (2010). A characterization of hypersurfaces with sectional curvature ≤ 1 in six dimensional sphere, submitted.

Itoh T, Nakagawa H (1973). On certain hypersurfaces in a real space form, Tohoku. Math. J. 25: 445-450.

Lawson HB (1969). Local rigidity theorems for minimal hypersurfaces, Ann. Math. 89: 187-191.

Otsuki T (1970). Minimal hypersurfaces in a Riemannian manifold of constant curvature. Am. J. Math. 92: 145-173.

Otsuki T (1978). Minimal hypersurfaces with three principal curvature fields in S^{n+1}, Kodai Math. J. 1: 1-29.

Peng CK, Terng CL (1983). The scalar curvature of minimal hypersurfaces in spheres, Math. Ann. 266: 105-113.

Sasaki S (1972). A proof of the fundamental theorem of hypersurfaces in a space form. Tensor, N.S. 24: 363-373.

Takahashi T (1966). Minimal immersions of Riemannian manifolds. J. Math. Soc. Japan 18: 380-385.