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Spatial transformations of diffusion tensor magnetic resonance imaging using certain geometric operations

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The general classes of medical image processing and analysis procedures carried out over scalar images (viz. registration and visualization of X-ray, CT-scan, filtering, image segmentation, ultrasound, diffusion weighted MRI and others) need to be enlarged to diffusion tensor magnetic resonance imaging (DT-MRI) tensor fields in order to bring together qualitative and quantitative information, potentially advancing computer assisted diagnosis, following up best treatment and checking for major neuropsychiatric ailments and statistical analysis of structural and functional unpredictability of individual anatomy of human being. Moreover, DT-MRI offers a measurement of a symmetric second rank translational diffusion tensor D of water molecules for each voxel within an imaging volume. Recently, various results concerning to noise in the estimate of diffusion tensor via an ideal DT-MRI experiment using Gaussian distribution have been established by Pajevic et al. (1999; 2003). In this analysis, second order symmetric diffusion tensors were arranged in the form of 6×1 vector random variables. Forming such a vector, random variable's combination sometimes rise up the circumstances that do not preserve certain intrinsic algebraic relationships among the components of D and its geometric character with reference to laboratory co-ordinate systems in which it is measured. Here, our main object is to address the problem of applying spatial transformations (sometimes called image warps) to DT-MRI using certain geometric operations, viz. conformal collineation, affine collineation, isometric collineation and projective collineation which would most probably introduce some new dimensions in favor of DT-MRI studies. To study such spatial transformations, we put forward a natural interpretation of the "degree of connectivity" between two adjacent points of fiber or fibers in the manifold of human brain. This is because of the reason that diffusion operator $u_t = \nabla \cdot (D \nabla u)$ can naturally be associated with a Riemannian metric tensor G via the relation $G = D^{-1}$ and once we have the metric tensor G , we will be able to apply geometric operations of Riemannian geometry to DT-MRI study. Also, in the present article we shall discuss geodesic fibers of cortical brain manifold up to large extent.

Key words: Diffusion tensor, magnetic resonance, DT-MRI, spatial transformation, warp, Riemannian manifold, affine collineation, isometric, conformal, degree of connectivity.

INTRODUCTION

Diffusion tensor magnetic resonance imaging (DT-MRI) is a thrilling recent technique in neuroscience that paves the way to enumerate the self-diffusion of water molecules in biological tissues (especially in brain tissues). It is based

upon the phenomenon of molecular diffusion which refers to as random movement of molecules through space driven by their internal thermal energy, sometimes called Brownian motion or zig-zack motion named after the great English Botanist Robert Brown, who in 1827 observed the constant movement of corpuscle particles suspended within grains of pollen. Now, it is well known that the molecular self-transportation is affected by the features of the medium in which it occurs and that

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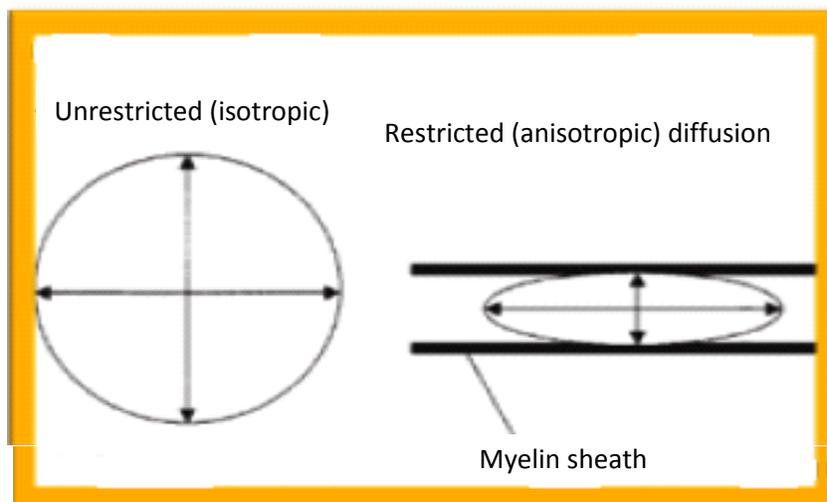


Figure 1. Difference between unrestricted (isotropic) and restricted (Anisotropic) diffusion within the brain. The shape of the tensor ellipsoid is determined by the strength of the diffusion along three principal directions (its eigenvectors). In nonrestrictive media such as cerebrospinal fluid, where diffusion is equal in all directions, the tensor can be visualized as a sphere. In restrictive media such as white matter, where diffusion is different in all directions, the tensor can be visualized as an ellipsoid (Kubicki et al., 2002).

diffusion along fiber tracts *in vivo* reflects both the tissue structure and architecture at the microscopic level. Equal or isotropic (Figure 1) diffusion occurs when a medium does not restrict the molecular self-transportation, as in case of the cerebrospinal fluid. Skewed or anisotropic (Figure 1) diffusion often seen in crystal and polymer films which is not equal in all direction occurs when a medium restricts molecular mobility. In case of brain, for instance, the grey matter is an isotropic environment in which the diffusion of water molecules is hindered symmetrically in all directions. Comparatively, brain white matter being formed of axon fibers (which are oriented in parallel and which are accompanied by glia cells) allows free diffusion along the fibers but hindered diffusion in direction perpendicular to fibers. Thus the technique particularly used to pursue white matter tracts in the brain involves skewed isotropic diffusion process of water molecules because the mobility of water molecules is restricted perpendicular to the axons oriented along the fiber tracts. The region for such perpendicular restriction is given by (Kubicki et al., 2002) that the concentric structure of multiple tightly packed myelin membranes enfolded around the axon fibers. Although, myelination is not essential for diffusion anisotropy of nerves, but myelin is basically considered to be the major barrier to diffusion in white matter tracts. Moreover, the diffusion coefficient involved in the study of DT-MRI is a measure of the molecular motion in the biological tissues and it can be determined by applying successive magnetic field gradient pulses and then measuring the change between the images obtained.

In isotropic (Kubicki et al., 2002) media, where diffusion along the three main axes is equal, the diffusion tensor is

symmetrical in all directions and is visualized as a sphere. In anisotropic media, where the diffusion is different along each axis, the diffusion tensor is visualized as an ellipsoid, with its longest axis indicating the greatest of the so-called principal directions of diffusion. The shape of the tensor ellipsoid (Figure 1) depends on the strength of the diffusion along the three principal directions (that is, its eigenvectors). Within myelinated white matter fiber tracts, the greatest principal direction of diffusion will always indicate the axonal trajectory, since perpendicular diffusion is restricted by myelin covering. The shape of the tensor ellipsoid therefore provides qualitative and quantitative measures of white matter tracts within the brain.

Basic principles and known results of DT-MRI

The philosophy of diffusion MRI was put forwarded in the mid-1980s by Le Dhand et al. (1995; Taylor et al., 1995), who merged nuclear magnetic imaging (NMR) principles with those introduced earlier to programmed molecular diffusion effect in NMR signals by using magnetic field gradient pulses (Stejskal et al., 1965) Molecular diffusion refers to random translation motion of molecules, also called Brownian motion, that produces from internal thermal energy carried by these molecules. As diffusion is the process by which molecules are transported from one medium to another. The flux of diffusion molecules is a result of their random Brownian motion in concentration gradients and is described by Fick's law. Diffusion tensor magnetic resonance imaging (DT-MRI) records the diffusion characteristics of water molecules along fiber

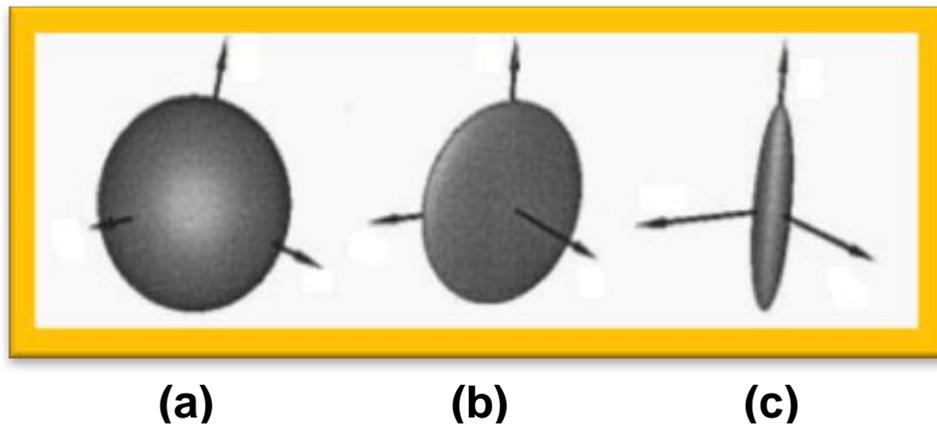


Figure 2. Illustration of the different shapes of DT ellipsoids and the corresponding distribution of eigenvalues. (a) Isotropic DT is represented by a spherical ellipsoid; these measurements arise when diffusion is unhindered in all directions, as in regions of cerebro-spinal fluid (CSF) in the brain, or when it is hindered equally in all directions, as in grey matter: $\lambda_1=\lambda_2=\lambda_3$ – Isotropic DT. Prevalent in CSF and grey matter regions of the brain. (b) Oblate DT is represented by a pancake shaped ellipsoid; this type of measurement can arise when tissue structure is planar, such as when white matter fibers intersect within a voxel: $\lambda_1=\lambda_2 \gg \lambda_3$ – oblate DT. Arise in white matter regions. (c) Prolate DT is represented (Alexander et al., 2001): $\lambda_1 \gg \lambda_2 = \lambda_3$ – Prolate DT. Prevalent in white matter regions

tracts. Furthermore, diffusion, truly being a three-dimensional process and hence molecular mobility in tissues might not be same in all direction. Therefore in DT-MRI, for each voxel of the 3-D image, there is assigned a rank three second order diffusion tensor, which forms a 3-D tensor field. Each individual tensor is expressed as a 3×3 positive semi-definite matrix with non-negative latent roots. The physical basis of DT-MRI is the assumption of normal or Gaussian distribution of water molecules random displacement using some suitable probability density function (PDF). However, the DT-MRI studies have limited capabilities of resolving multiple fiber orientations within one voxel, which is mainly due to second order diffusion tensor which is inadequate to characterize the diffusion process in biological tissues. In heterogeneous structure, such as brain white matter and other absorbent structures, the translational mobility of molecules should be affected by the confining boundaries of tissues. Thus, while imposing spatial transformations and restrictions on the diffusion process, the PDF of molecular random displacement should no longer be Gaussian. To study such a complicity, (Chunlei et al., 2004) have been proposed an approach of characterizing non-Gaussian diffusion by using generalized DT-MRI, in which they introduced an n -th order diffusion tensor together with a reconstructed PDF for the spin displacement. (Alexander et al., 2001) have addressed the problem of applying spatial transformations (or image warps) to DT-MRI and have discussed how to handle these MR images when they are transformed spatially during image registration. They proposed that size, shape and orientation of the ellipsoid

formed under spatial image transformation of an image are the three prime features (Figure 2) and may be characterized by the latent roots $(\lambda_1, \lambda_2 \text{ and } \lambda_3)$ and their corresponding latent vectors $e_1, e_2 \text{ and } e_3$ from the DT. Further, spatial normalization and reorientation of diffusion tensor field has been an extensive area of research in neuropsychology. Dongrong et al. (2003) have discussed that spatial normalization of DT requires an appropriate reorientation of the tensor on each voxel, in addition to its relocation into the standard space. This appropriate tensor reorientation is determined by the spatial normalized transformation and from an estimate of underlying fiber direction. Now, following are some known results on DT-MRI, which will be used in our proposed study under consideration.

The diffusion tensor

With plain diffusion MRI, diffusion is fully described by a single scalar parameter, the diffusion coefficient D (Denis et al., 2001). The influence of diffusion on MRI signal (most prominently a spin-echo signal) is a reduction A which has dependence on D and on the “ b factor”, which illustrates the gradient pulses (viz. timing, amplitude and shape) used in the MRI sequence (Le Bihan, 1991)

$$A = e^{(-bD)} \quad (1)$$

However, in the presence skewed-isotropy, diffusion of molecular mobility in the biological tissues of brain can no

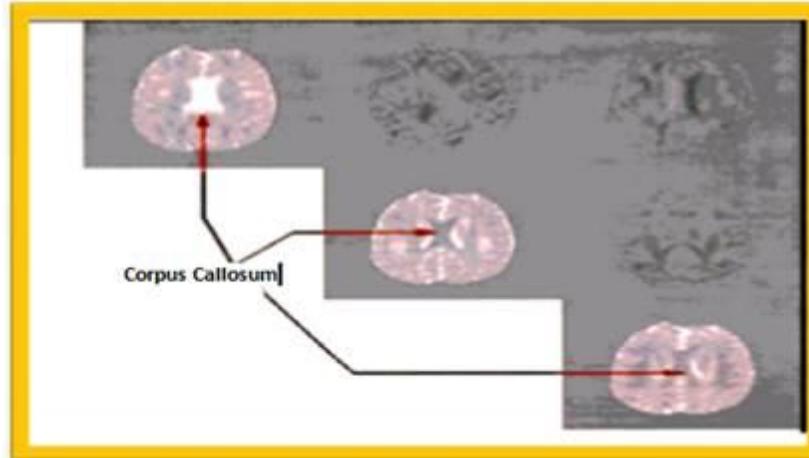


Figure 3. Corpus callosum: The diffusion tensor, axial slices. Diffusivity along x, y and z axes, shown in D_{xx} (a), D_{yy} (d) and D_{zz} (f) images respectively is clearly different in white matter, especially in the corpus callosum. Non-diagonal shadow images [D_{xy} (b), D_{xz} (c) and D_{yz} (e)] are not noise images because the x, y, z reference frame of the MRI scanner does not coincide with the diffusion reference frame of tissues in most voxels (Denis et al., 2001).

longer be described by a single scalar coefficient. For this purpose a tensor \underline{D} has been set up, which completely analyzes the hindered or restricted diffusion of water molecular mobility in all directions to the fibers [Figure 3] (Stejskal et al., 1965):

$$\underline{D} = \begin{bmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{yx} & D_{yy} & D_{yz} \\ D_{zx} & D_{zy} & D_{zz} \end{bmatrix} \quad (2)$$

This tensor is symmetric ($D_{ij} = D_{ji}$, with $i, j = x, y, z$). In a 3-dimensional continuum having co-ordinate system $[[x]', y', z']$ which coincides with the principal or self-directions of diffusivity, the off-diagonal elements do not come into existence and the tensor is reduced merely to its diagonal elements, $D_{x'x'}, D_{y'y'}, D_{z'z'}$, which represent molecular mobility along the new axes x', y' and z' respectively.

By virtue of this, the echo-reduction A becomes:

$$A = e^{(-b_{x'x'}D_{x'x'} - b_{y'y'}D_{y'y'} - b_{z'z'}D_{z'z'})} \quad (3)$$

where b_{ii} are the elements of matrix \underline{b} , which now replaces the b factor.

In practice, the diffusion measurements are made with reference to the frame $[x, y, z]$ of MRI scanner gradients, which usually does not coincide with the diffusion continuum of brain tissues and hence one must also consider the coupling of the off-diagonal elements, b_{ij} of

the matrix \underline{b} with the off-diagonal elements $D_{ij}, i \neq j$ of the diffusion tensor, which provides correlation between molecular transportation in perpendicular direction to fibers (Basser et al., 1994):

$$A = e^{\left[-\sum_{\substack{i=x,y,z \\ j=x,y,z}} [(b)_{ij}](D_{ij}) \right]} \quad (4)$$

or

$$A = \exp[-b_{xx}D_{xx} - b_{yy}D_{yy} - b_{zz}D_{zz} - 2b_{xy}D_{xy} - 2b_{yz}D_{yz} - 2b_{zx}D_{zx}] \quad (5)$$

Hence, it is remarkable that while using diffusion encoding gradient pulses along one direction only, say x, signal reduction not only depends on the diffusivity along that direction but may also include contribution from other directions, say y and z.

Also, this diffusion tensor, expressed as 3x3 matrix captures directional variation in the diffusion rate. As diffusion tensor is a second order symmetric tensor and thus expressible as 3x3 symmetric matrix, then by definition, it would have three real latent roots $\underline{\lambda}_1, \underline{\lambda}_2, \underline{\lambda}_3$ and each latent root would have corresponding latent vector e_1, e_2 and e_3 , respectively. By tensor visualization technique, it is emphasized that the diffusion tensor is geometrically equivalent to an ellipsoid, with the three latent vectors of such a matrix set as the radii of ellipsoid. The ellipsoid would match the shape that molecular mobility diffuses to from the data point in a fixed amount of time. Figure 4 illustrates three types of spatial diffusion.

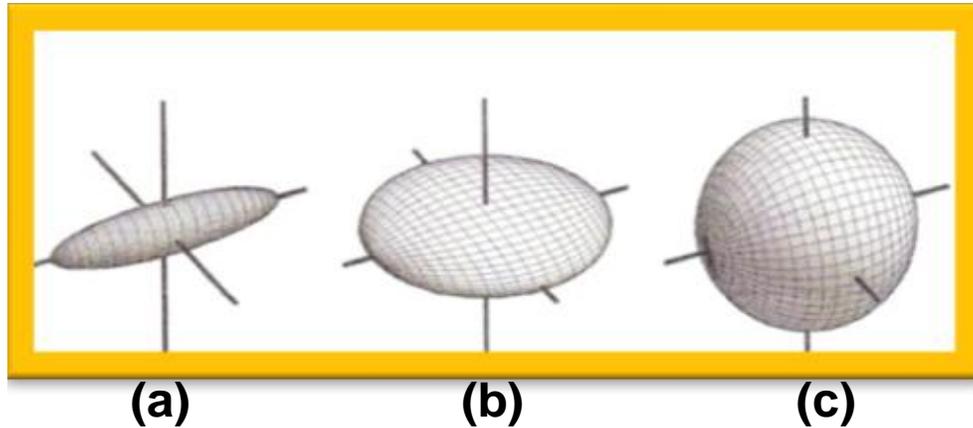


Figure 4. Ellipsoids illustrating diffusion tensor according to the latent system of the tensor, (a) linear skewed-isotropic diffusion (b) planar skewed-isotropic diffusion (c) isotropic diffusion.

However, the tracts of complex diffusion anisotropy, such as the area where two fiber bundles intersect or touch each other, a second order tensor could not accurately illustrate the diffusion process and hence these tracts of uncertainty would be addressed with a generalized diffusion technique that would have comparatively more degree of freedoms.

Mean diffusivity

In order to receive a complete evaluation of the diffusion in a tract of axon fiber, we must avoid skewed-isotropic diffusion effects and limit the result to an invariant, *that is*, a quantity which is independent of the orientation of the reference frame (Basser et al., 1994). Among various permutations of the tensor elements, the trace of the diffusion tensor,

$$\text{Trace } \underline{D} = D_{xx} + D_{yy} + D_{zz} \tag{6}$$

is one of the rotationally invariant. The mean diffusivity is then expressed as:

$$\langle D \rangle = (\text{Trace } \underline{D})/3 = (D_{xx} + D_{yy} + D_{zz})/3 = (\lambda_1 + \lambda_2 + \lambda_3)/3, \tag{7}$$

where λ_1, λ_2 and λ_3 are the latent roots (or principal diffusivity) of \underline{D} and D_{xx}, D_{yy} and D_{zz} are its leading diagonal entries measured in the laboratory reference Frame.

Proposition of Fick’s first law

In a real 3-dimensional manifold (Basser et al., 1994), the particle flux vector, that is, a tensor of order one is represented by:

$$F = F_i e_i = F_x e_x + F_y e_y + F_z e_z. \tag{8}$$

Here, e_i 's are a set of orthogonal unit vectors and $F_i (i = x, y, z)$ is the component of F along the direction of e_i . Also, in Equation (8), Einstein’s summation convention has been followed.

The macroscopic analysis of normal or Gaussian diffusion is based upon the proposition of Fick’s law, which states that “the flux F of the diffusion substances is directly proportional to the concentration gradient”, that is,

$$F_i = -D_{ij} \frac{\partial C}{\partial x_j}, (i, j = x, y, z) \tag{9}$$

Here, C is the concentration gradient, x^j is the j^{th} spatial co-ordinate, D_{ij} is an element of second order self-diffusion tensor and F_i is the i^{th} component of the flux vector F .

Thus, the Fick’s law assumes a linear relationship between the flux and the concentration gradient for a skewed-isotropic diffusion tensor.

Most probably, many mathematical quantities could be derived from the higher order tensors. For instance, the complete characteristics of the higher order tensors can be reduced to offer single numbers (scalar and vector) for a description of the features of essential diffusion process. From this stand point, tensor contraction is a crucial operation. Full contraction of an even order tensor of rank n is defined like;

$$I^{(n)} = g_{i_1 i_2} g_{i_3 i_4} \dots g_{i_{n-1} i_n} D_{i_1 i_2 i_3 \dots i_n} \tag{10}$$

where g_{ij} is a real manifold symmetric metric tensor which for a Cartesian co-ordinate is an identity matrix and $I^{(n)}$ is a scalar invariant, also called the trace of a tensor

of rank n .

Steady-state concentration

3-D tensor field as conductivity *tensor* (Donnell et al., 2002) used the Fick's first law to estimate white matter connectivity in diffusion tensor MRI. In such estimation, Fick's first law relates a concentration difference to a flux (a flow across a unit area), which states that the flux \vec{j} , in any direction is proportional to the concentration gradient ∇u in the opposite direction. The constant of proportionality \underline{d} is the diffusivity along the direction under consideration;

$$\vec{j} = -\underline{d}\nabla u \quad (11)$$

Since, the flow filed of water molecules does not follow the concentration gradient directly, as the material property of axon fiber also affect diffusion and hence the diffusion tensor \underline{D} is introduces, that is,

$$\vec{j} = -\underline{D}\nabla u \quad (12)$$

The standard model of diffusion explores that over time, the concentration gradient of the solute will change as the divergence of the flux, that is,

$$u_t = \nabla \cdot (\underline{D}\nabla u). \quad (13)$$

This is due to the conservation of mass.

SPATIAL TRANSFORMATION OF DT-MRI USING CERTAIN GEOMETRIC OPERATIONS

A natural elucidation of the 'degree of connectivity' between two adjacent or neighboring points is the measure or distance between the points in some metric manifold. However, in our proposed study, the distance between two points of a fiber or fibers or two anatomical positions would depend on the nature of diffusion tensor filed.

The diffusion tensor under consideration could likely to be connected with a Riemannian metric tensor $G \equiv g_{ij}$ via the relation $G = D^{-1}$. Obviously, this relation can allow us computing various geometric quantities, like distance between two points in the brain and other anatomies of body, length of the axon fibers, inclination of fibers and geodesic paths etc. Also, once we have the fundamental metric tensor, we can calculate different spatial transformations of DT-MRI. Here, we shall first like to briefly discuss the manifold of diffusion tensors as described by (Fletcher et al., 2004).

Manifold generated by diffusion tensor

Diffusion tensor magnetic resonance imaging produces a 3D diffusion tensor of second order, that is, a 3×3 symmetric positive definite matrix, at each voxel of an imaging volume. This tensor represents the covariance in a Brownian motion model of water diffusion at that individual voxel. Also such a 2-tensor has been commonly used to approximate the diffusion profile at each lattice point of the image lattice (Basser et al., 1994). The approximation yields a diffusion tensor data set which is a matrix valued image. These tensors are the elements of the manifold of 3×3 positive definite matrix denoted by $PD(3)$. Mathematically, the positive definite diffusion tensor belongs to a curved symmetric manifold somewhat like Riemannian symmetric manifold, that is, the space of diffusion tensor can be thought to be a Riemannian symmetric manifold rather than a linear manifold. Moreover, Riemannian symmetric manifolds can thought to be connected Riemannian manifolds M such that for each $x \in M$, there is an isometry σ_x which (i) is involute, that is, $\sigma_x^2 = id$, and (ii) has x as an isolated fixed point, that is, there is a neighborhood U of x where σ_x leaves only x fixed. However, while applying the geometric operations to the diffusion tensor, we shall make use of the concepts of symmetric spaces as studied by (Kumar's Thesis, 2008):

A curved manifold (that is, Riemannian manifold) of class C^∞ of any dimension is said to be Riemannian symmetric in the sense of Cartan (1925), if it satisfies the following identity:

$$\nabla_a R_{ijk}^h = 0, \text{ or } \nabla_a R_{hijk} = 0, \quad (14)$$

where R_{ijk}^h , or R_{hijk} is the well known Riemannian curvature tensor defined as

$$R_{ijk}^h = -\partial_k \Gamma_{ij}^h + \partial_j \Gamma_{ik}^h + \Gamma_{ia}^h \Gamma_{jk}^a - \Gamma_{ja}^h \Gamma_{ik}^a \quad (15)$$

and $\text{del} (\text{i. e., } \nabla)$ followed by a subscripted index stands for the operator of covariant differentiation with respect to the symmetric connection Γ_{jk}^i .

Also, if we contract the Equation (14) with respect to the indices h and k , we get the following expression:

$$\nabla_a R_{ij} = 0, \quad (16)$$

where R_{ij} is called the Ricci tensor or contracted curvature tensor and if the expression (16) holds good in the given curved manifold, we shall say that the manifold under consideration is Ricci symmetric or Riemannian semi-symmetric.

Thus, it is evident that if the manifold of diffusion tensor

is *Riemannian symmetric*, it is naturally a semi-symmetric manifold.

Let us now proceed to discuss the spatial transformations for DT-MRI with the following mathematical analysis: Firstly, in order to calculate the inverse of diffusion tensor (2), consider the co-ordinate transformation of flux vector (9), as:

$$F'_i = D_{ij} F_j, \tag{17}$$

where the prime is used to distinguish co-ordinate systems.

This transformation law is what defines a tensor of rank first. If one wanted to transform the co-ordinates in other direction, from prime co-ordinate system to unprimed, one would use the inverse transformation matrix

$$D_{ij}^{-1} = \hat{x}'_i \cdot \hat{x}_j = D_{ij}^T = D_{ji}, \tag{18}$$

where $D_{ij}^T = D_{ji}$ is the transpose of D_{ij} .

This means that to obtain the inverse of the transformation in Equation (17), one only need to switch the rows and columns in the 3×3 transformation matrix. Also a transformation of co-ordinates from one system to another and then back again should produce the original co-ordinates and hence:

$$D_{ij} D_{jl}^{-1} = D_{im}^{-1} D_{mi} = \delta_{il},$$

where δ_{il} , the Kronecker delta, is essentially equivalent to the identity matrix. Evidently the conjugate or reciprocal of the diffusion tensor D_{ij} , written as D^{ij} would be the inverse tensor of D_{ij} . Now since the Riemannian metric tensor and diffusion tensor are connected by the relation $G = D^{-1}$, therefore in view of the aforementioned detail, $g_{ij} = D^{ij}$.

The study now considers a continuous curve C in the sense of axon fiber of brain tissue lying in the manifold of diffusion tensor (that is, a Riemannian symmetric manifold). Let the parametric equation of C be $x^i = x^i(t)$, where t is some parameter, then the length ds of the element of individual axon fiber joining the two adjacent points x^i and $x^i + dx^i$ would be given by the measure function.

$$ds = \sqrt{\sum_{a=1}^3 \frac{\partial X^a}{\partial x^i} \frac{\partial X^a}{\partial x^j} \delta x^i \delta x^j} \tag{19}$$

So far, it's only defined for Euclidean manifold, but once it is defined in this way, it does transform as a T^2_0 tensor as follows:

$$ds = \sqrt{D^{ij} \delta x^i \delta x^j}, \tag{20}$$

where we have defined the conjugate diffusion tensor

$$D^{ij} \text{ as } D^{ij} = \sum_{a=1}^3 \frac{\partial X^a}{\partial x^i} \frac{\partial X^a}{\partial x^j}.$$

Thus, the arc length of entire axon fiber corresponding to the values t_0 & t_1 of the parameter is defined as

$$s = \int_{t_0}^{t_1} \sqrt{\left(D^{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} \right)} dt \tag{21}$$

If we replace the upper limit t_1 by t , we will have s as a function of t . Hence, if x^i be the co-ordinates of any diffused water molecule along any axon fiber, x^i must be functions of the arc length s of fiber measured from a fixed point. It then follows from Equation (21) that:

$$D^{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} = \mathbf{1}, \tag{22}$$

which shows that $\frac{dx^i}{ds}$ is a unit tangent vector.

The study now proceeds to calculate the inclination between any two axon fibers and then try to calculate the same between any two fiber tracts or voxels of an imaging volume, whenever due to some kind of neuropsychiatric disorder, these axon fibers lose their original parallel orientations.

Suppose that we are give two individual fibers of the brain tissue, which we denote by l_t & m_r passing through any point P_0 and defined by

$$\omega^{\dagger} r = \left(\dot{r} dt, \omega^{\dagger} r = \mu^{\dagger} r d\tau \right),$$

where t & τ are nothing but the arbitrary parameters, then if the diffusion of water molecules along each fiber tract is isotropic, the angle φ between these two fibers will be defined as the angle between the hyperspheres $d_t P_0$ & $d_\tau P_0$, which are orthogonal to the fibers l_t & m_r respectively. Here dt & $d\tau$ are the operators of differentiation with respect to the parameters t & τ respectively. Then by the usual formula;

$$\cos^2 \varphi = \left[\frac{r_1^2 + r_2^2 - |a - b|^2}{2r_1 r_2} \right]^2,$$

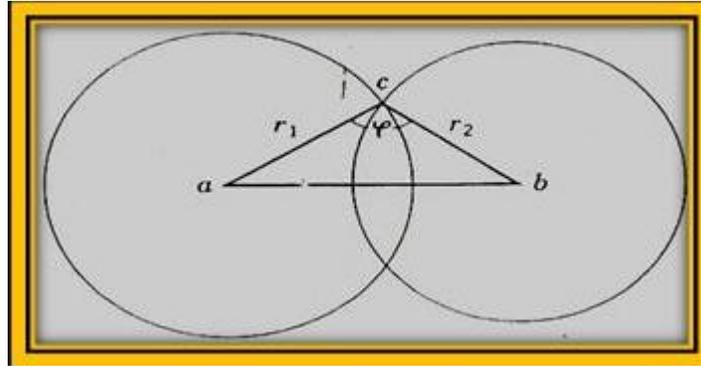


Figure 5. Angle between two hyperspheres as an angle between two intersecting circles.

where $|a - b| = r_1^2 + r_2^2 - 2r_1r_2 \cos\varphi$,

we get

$$\cos^2\varphi = \frac{\langle d_t P_0, d_\tau P_0 \rangle^2}{\langle d_t P_0, d_t P_0 \rangle \cdot \langle d_\tau P_0, d_\tau P_0 \rangle},$$

where the symbol $\langle d_t P_0, d_\tau P_0 \rangle$ etc. is used to denote the scalar product of the enclosed quantities.

If we now set $\langle dP_0, dP_0 \rangle = D^{rs}\omega^r\omega^s$, a conformal structure would be defined in the neighborhood of point P_0 by the quadratic form $D^{rs}\omega^r\omega^s$, which is positive definite in the Riemannian symmetric manifold $PD(3)$. Thus $\cos^2\varphi$ would then be written as;

$$\cos^2\varphi = \frac{[(D^{rs}(r^i s^j))]^2}{(D^{rs}(r^i s^j))^2} = \frac{D^{pq} \mu^p \mu^q}{D^{pq} \mu^p \mu^q}. \tag{23}$$

The following Figure 5 depicts angle between two hyperspheres (orthogonal to considered fibers) which is an angle between two intersecting circles having centers at a and b. The equation $D^{rs}\omega^r\omega^s = 0$ may define an isotropic cone and since $\langle dP_0, dP_0 \rangle = D^{rs}\omega^r\omega^s$ is the positive definite; this cone will be purely imaginary. However, the same angle could be calculated by assuming two tangent vectors \hat{u}_t^i and \hat{v}_τ^i to the fibers l_t and m_τ , such that:

$$\cos^2\varphi = \frac{(D^{ij}\hat{u}_t^i\hat{v}_\tau^j)^2}{(D^{ij}\hat{u}_t^i\hat{u}_t^j) \cdot (D^{ij}\hat{v}_\tau^i\hat{v}_\tau^j)}, \tag{24}$$

where $\hat{u}_t^i \equiv \frac{du_t^i}{dt}$ and $\hat{v}_\tau^i \equiv \frac{dv_\tau^i}{d\tau}$, as l_t and m_τ are fibers depending upon the arbitrary parameters t and τ , respectively.

Now, both the formulae given by equations (23) and (24) are quite similar, but using formula (24), wherein the tangent vectors are involved the isotropic behavior of diffusion tensor could not be directly introduced by means of hyperspheres generated by Brownian motion of water molecules.

There is another interesting case which we seem necessary to introduce, that is, instead of considering any two individual fibers of brain tissues, we consider the whole fiber tract. In this case, one can think the fiber tract as a hypersurface which is a variety of our $PD(3)$ manifold. Let η and ζ be the scalar functions of coordinates x^1, x^2, \dots, x^n . Then $\eta(x^i) = \text{constant}$ & $\zeta(x^i) = \text{constant}$ will be the two families of fiber tracts. From these fiber

tracts, it is evident that $d\eta = \frac{\partial\eta}{\partial x^i} dx^i = 0$ & $d\zeta = \frac{\partial\zeta}{\partial x^i} dx^i = 0$, which simply imply dx^i is orthogonal to $\frac{\partial\eta}{\partial x^i}$ & $\frac{\partial\zeta}{\partial x^i}$ respectively and hence the gradient vector to the fiber

tract, that is, $\frac{\partial\eta}{\partial x^i}$ & $\frac{\partial\zeta}{\partial x^i}$ are normal to both $\eta(x^i)$ and $\zeta(x^i)$. Now, if the aforementioned fiber tracts are mutually inclined at an angle φ , then φ will also be an angle between their respective normals, that is,

$$\cos^2\varphi = \frac{(D^{ij} \frac{\partial\eta}{\partial x^i} \frac{\partial\zeta}{\partial x^j})^2}{D^{ij} \frac{\partial\eta}{\partial x^i} \frac{\partial\eta}{\partial x^j} \cdot D^{ij} \frac{\partial\zeta}{\partial x^i} \frac{\partial\zeta}{\partial x^j}}. \tag{25}$$

In case if we take the fiber tracts as the co-ordinate fiber tracts of parameters x^i and x^m , then

$$\cos^2\varphi = \frac{(D^{ij} \delta_i^l \delta_j^m)^2}{(D^{ij} \delta_i^l \delta_j^l) \cdot (D^{ij} \delta_i^m \delta_j^m)} = \frac{(D^{lm})^2}{D^{ll} \cdot D^{mm}}. \tag{26}$$

Thus if θ_{ij} be the angle between co-ordinate fiber tracts, we have

$$\cos^2 \theta_{ij} = \frac{(D^{ij})^2}{D^{ii} \cdot D^{jj}} \tag{27}$$

It follows from equation (27) that if the co-ordinate fiber tracts become orthogonal due to some neuropsychiatric disorder, then $D^{ij} = 0$ as $\cos \theta_{ij} = \cos 90 \Rightarrow \theta_{ij} = 90$. Mathematically, we can say that in case if fiber tracts lose their original orientations and become orthogonal, the inverse diffusion process inside them will vanish and thereby diffusion process may run indefinitely and infinitely.

Comparison of two fibers by means of involute-evolute geometry

Here, we discuss a comparison theory of two fibers of an individual fiber tract based on their geometric configurations. In this case, we let that there is a one-to-one correspondence between the water molecules (at some specific time) diffused along the fibers l_t and m_r and that the 1-1 correspondence is such that the tangent vector at any water molecule of l_t at that specific instant is normal to the corresponding water molecule of m_r , then by definition, m_r is called an involute of l_t and l_t is called an evolute of m_r . Thus by adopting this theory, one can completely characterize the geometric property of brain tissue. Let $r = r(s)$, (where s being the arc length parameter) be the vectorial equation of the fiber l_t , then the position vector of any water molecule P_1 at some specific instant is given by;

$$r_1 = r + (\lambda t) \tag{28}$$

where λ is to be determined. Differentiating Equation (28) with respect to s_1 , we obtain

$$\begin{aligned} \frac{dr_1}{ds_1} &= \frac{dr}{ds} \frac{ds}{ds_1} + \lambda' t \frac{ds}{ds_1} + \lambda \frac{dt}{ds} \frac{ds}{ds_1} \frac{dr_1}{ds_1} = \frac{dr}{ds} \frac{ds}{ds_1} + \lambda' t \frac{ds}{ds_1} + \lambda \frac{dt}{ds} \frac{ds}{ds_1} \frac{dr_1}{ds_1} \\ &= \left(\frac{dr}{ds} + \lambda' t + \lambda \frac{dt}{ds} \right) \frac{ds}{ds_1} \frac{dr_1}{ds_1} \\ &= \left(\frac{dr}{ds} + \lambda' t + \lambda \frac{dt}{ds} \right) \frac{ds}{ds_1} \frac{dr_1}{ds_1} \\ &= \frac{dr}{ds} \frac{ds}{ds_1} + \lambda' t \frac{ds}{ds_1} + \lambda \frac{dt}{ds} \frac{ds}{ds_1} = \left(\frac{dr}{ds} + \lambda' t + \lambda \frac{dt}{ds} \right) \frac{dr_1}{ds_1} \\ &= \frac{dr}{ds} \frac{ds}{ds_1} + \lambda' t \frac{ds}{ds_1} + \lambda \frac{dt}{ds} \frac{ds}{ds_1} \\ &= \left(\frac{dr}{ds} + \lambda' t \right. \\ &\left. + \lambda \frac{dt}{ds} \right) \frac{ds}{ds_1} \frac{dr_1}{ds_1} = \frac{dr}{ds} \frac{ds}{ds_1} + \lambda' t \frac{ds}{ds_1} + \lambda \frac{dt}{ds} \frac{ds}{ds_1} = \left(\frac{dr}{ds} + \lambda' t + \lambda \frac{dt}{ds} \right) \frac{ds}{ds_1} \cdot \frac{ds}{ds_1} \end{aligned}$$

But $\frac{dr_1}{ds_1} = t_1, \frac{dr}{ds} = t$ and $\frac{dt}{ds} = kn$, where k is curvature and n is the normal vector. Therefore, we have

$$t_1 = (t + (t' t + (kn) ds) / [ds])_1 \tag{29}$$

By definition, t is perpendicular to t_1 (Figure 6), so taking dot product on both side of Equation (29) with t and using $t \cdot t_1 = 0$, we have $(1 + (t' t) ds / [ds])_1 = 0$ or, $1 + (t' t) = 0$, which on integration implies $s + (c \text{ or } (s - c))$, c being some constant of integration. Therefore, $r_1 = r + (s - c)t$, which is the equation of involute m_r of the evolute l_t . On substituting the values of $(t$ and t' in Equation (29), we get $t_1 = (c - s)k \frac{ds}{ds_1} n$, which implies t_1 is parallel to n . We now take the positive sense along involute fiber such that $t_1 = n$, so that last expression yields $\frac{ds_1}{ds} = k(c - s)$. Thus once we have the equation of either of the fiber (l_t or m_r), we can analyze much more geometric features of others.

Fiber geodesics measurement using affine connection

As it is well known that the manifold of diffusion tensor is a curved manifold which is really a Riemannian symmetric manifold. Hence such a hypothesis may lead to the very desirable geodesics equations which are essential to define statistical methods for diffusion tensors. Moreover the white matter seen in DT-MRI as a 3D manifold $PD(3)$ and thereby the fibers become geodesics of this manifold. The approach of calculating geodesic distance on the manifold of diffusion tensor involves identification of such a manifold. Figure 7 shows the diffusion tensors on their underlying manifold (a non-linear manifold) in which the green dotted line represents the Euclidean distance $ds = \sqrt{\sum_{i=1}^3 (\delta X^i)^2} = \sqrt{\sum_{i=1}^3 \frac{\partial X^i}{\partial x^j} \delta x^j}$ between two nearby points which are the elements of $PD(3)$ and this line does not lie along the manifold while the red line represents the geodesic distance along the manifold. Thus on $PD(3)$ any manifold there are special intrinsic curves, called geodesics which are analogous to straight lines in Euclidean space $[(e.g., R)^3]$. Because such intrinsic curves are the curves of shortest distance, the problem is given any two points A and B on the manifold to find out all the curves joining two points A and B those which give the least arc length? To treat this problem properly is very typical as the lengths of various arcs AB certainly have a non-zero greatest lower and least upper bounds. However, the problem leads to a

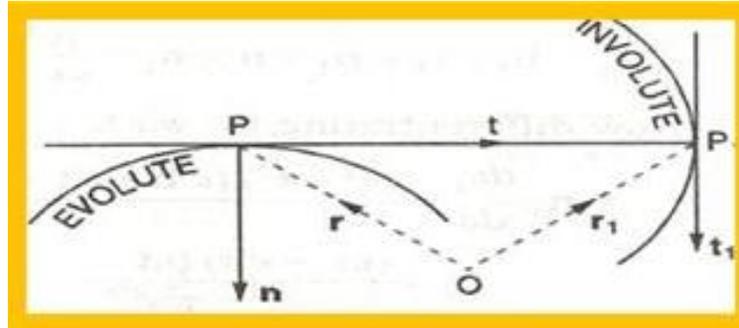


Figure 6. Two fibers which behave like involute and evolute in the brain cortical manifold.

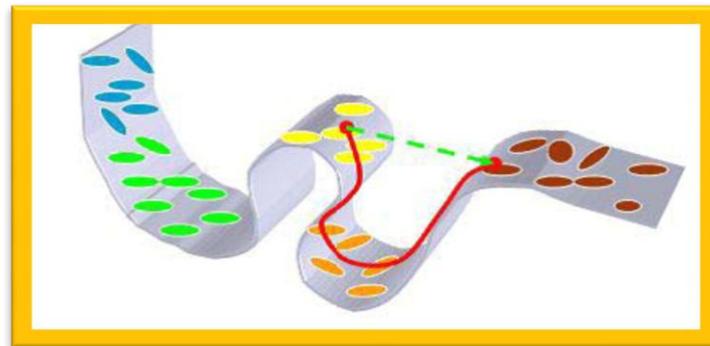


Figure 7. Manifold structure of tensors. The black surface represents the non-linear manifold fitted through the tensors represented as ellipses. The green line represents the Euclidean distance between tensors treated as elements of $PD(3)$ and the red line represents the geodesic distance along the manifold that will be used for all tensor manipulations (Khurd et al., 1985).

very surprising possibility in DT-MRI study, that if we have an approximate measure of least and greatest fiber in each voxel of brain depending on the size of individual brain (that is, a fixed limit of least and greatest fiber in smallest and largest brain), we can probably predict about the ailments of individual’s brain.

We, now, define the geodesic fibers as “*straightest possible*” fibers which are not strictly shortest but the filaments of stationary lengths in $PD(3)$, where the straightest means the tangent vector drawn to any water molecule diffused along fiber propagates parallel along the fiber is still parallel to its initial position, *that is*., the tangent vectors are auto parallel.

Let us now consider a fiber γ with an arbitrary parameterization τ , which is defined by $\gamma: \omega^i = \mu^i d\tau$. Let the tangent vector to the water molecule (at some specific instant say t_0) is

$$T^i \equiv \dot{\mu}_{\tau}^i = \frac{d\mu^i}{d\tau} \tag{30}$$

Then the fiber $\gamma: \omega^i = \mu^i d\tau$ is a geodesic if we have a transport function such that;

$$\llbracket [Transport] \rrbracket_1((\gamma(\tau_2) \rightarrow \tau(\tau_1); \gamma)) \rrbracket T^i(\tau_2) (T^i(\tau_1)) \tag{31}$$

That is

$$\llbracket [Transport] \rrbracket_1((\gamma(\tau_2) \rightarrow \tau(\tau_1); \gamma)) \rrbracket \mu^i(\tau_2) (\mu^i(\tau_1)) \tag{32}$$

where the “*Transport*” function has the property that $\llbracket [Transport] \rrbracket_{y \rightarrow x; \gamma}: T_y \rightarrow T_x$ should be invertible mapping but is might well depend on the specific fiber path chosen to go water molecule from $y \rightarrow x: \gamma_{fiber}$. Equation (32) describes a geodesic fiber denoted by γ_{fiber} of a fiber γ , such that its tangent vector is auto parallel to itself when transported parallel along the fiber. The Equation(32) may be written in infinitesimal form by differentiating with respect to τ and letting $y \rightarrow x, i.e., \tau_2 \rightarrow \tau_1$ as follows:

$$\left[\frac{dT}{d\tau} \right]^i + d/d\tau \left[\left[\text{Transport} \right]_i((\gamma(\tau_1, 2) \rightarrow \tau(\tau_1, 1); \gamma)) \right] T^i(T^i),$$

which in local co-ordinates takes the form:

$$\left[\frac{dT}{d\tau} \right]^i + \left[\frac{dx}{d\tau} \right]^i / d\tau \partial / (\partial x^i) \left[\left[\text{Transport} \right]_i((\gamma(\tau_1, 2) \rightarrow \tau(\tau_1, 1); \gamma)) \right] T^i(T^i),$$

that is

$$\left[\frac{dT}{d\tau} \right]^i + \{ T^k \Gamma_{jk}^i \} T^j(T^i),$$

or alternatively

$$\left[\frac{dT}{d\tau} \right]^i + \Gamma_{jk}^i T^j T^k(T^i). \tag{33}$$

Substituting for T^i from Equation (30), Equation (31) implies

$$(d^2 \mu^i) / (d\tau)^2 + \Gamma_{jk}^i [d\mu]^j / d\tau [d\mu]^k / d\tau \propto [d\mu]^i / d\tau, \tag{34}$$

which is the fiber geodesic of a fiber $\gamma: \omega^i = \mu^i d\tau$ and the symbol Γ_{jk}^i is a well known affine connection which in case of torsion free metric connection becomes Christoffel's second kind bracket $\left(\begin{smallmatrix} i \\ j \ k \end{smallmatrix} \right) = D^{hi} [h, jk]$, where D^{hi} are the components of reciprocal diffusion tensor.

Fiber geodesics in case of torsion full affine connection

However, due to the above reason, there arises a particular case where the affine connection is not a torsion free one, and then what would be the equation of fiber geodesics? That is when the diffusion of water molecules along particular fiber geodesics is hindered and propagates along the binormal direction of geodesics, and hence diffusion becomes anisotropic, then to calculate geodesic fiber, one must introduce the following geometry:

Let Γ_{jk}^i be an affine connection defining the fiber geodesic γ_{fiber} , then a tensor of type T_2^1 defined by:

$$T_{jk}^i = \frac{1}{2} [\Gamma_{jk}^i - \Gamma_{kj}^i] = \Gamma_{[jk]}^i$$

where the square index bracket stands for the anti-symmetry of affine connection is called the torsion of the connection which in general is the arc rate of rotation of binormal vector drawn to any water molecule to show the presence of anisotropic diffusion in the fiber geodesics under consideration. This torsion may arise from the part of parallel transport operation that is independent of geodesics but may play a vital role in the study of DT-MRI. Also, it is well known that affine connection Γ_{jk}^i is

called symmetric if $\Gamma_{jk}^i = \frac{1}{2} [\Gamma_{jk}^i + \Gamma_{kj}^i] \equiv \Gamma_{(jk)}^i$, where the round index bracket stands for the symmetry of connection. With the help of above relations, we start to calculate γ_{fiber} with non-metricity of D^{ij} as follows:

$$\nabla_k D_{ij} = \partial_k D_{ij} - D_{mj} \Gamma_{ik}^m - D_{im} \Gamma_{jk}^m, \tag{35}$$

where del followed by subscripted index stands for the operator of covariant differentiation with respect to symmetric affine connection. And now, rewrite Equation (35) as follows;

$$q_{ijk} = \partial_k D_{ij} - \Gamma_{jik} - \Gamma_{ijk} = \nabla_k D_{ij} - \Gamma_{jik} - \Gamma_{ijk} \tag{36}$$

Consider a combination of partial derivatives of a symmetric diffusion tensor D_{ij} as

$$\{ij; k\} = \frac{1}{2 [\nabla_j D_{ik} + \nabla_i D_{jk} - \nabla_k D_{ij}]}, \tag{37}$$

we then compute

$$\{ij; k\} = \frac{1}{2} \{q_{ikj} + q_{jki} - q_{ijk}\} + \frac{1}{2 [\Gamma_{ikj} + \Gamma_{kij} + \Gamma_{jki} + \Gamma_{kji} - \Gamma_{jik} - \Gamma_{ijk}]} \tag{38}$$

that is,

$$\{ij; k\} = \frac{1}{2} \{q_{ikj} + q_{jki} - q_{ijk}\} - \{T_{ijk} + T_{jik}\} + \frac{1}{2 [\Gamma_{kij} + \Gamma_{kji}]}. \tag{39}$$

$$\otimes \{ij; k\} = \frac{1}{2} \{q_{ikj} + q_{jki} - q_{ijk}\} - \{T_{ijk} + T_{jik} - T_{kji}\} + \Gamma_{kji}. \tag{40}$$

We now re-arrange Equation (40) as

$$\Gamma_{kij} = \{ij; k\} - \frac{1}{2} \{q_{ikj} + q_{jki} - q_{ijk}\} - \{T_{ijk} + T_{jik} - T_{kji}\}. \tag{41}$$

Let us now transvect Equation (38) throughout with D^{km} and arrange the indices arbitrarily, we get

$$\Gamma_{ij}^k = \left[\left(\begin{smallmatrix} k \\ i \ j \end{smallmatrix} \right) + D^{km} \left\{ -\frac{1}{2} \{q_{ikj} + q_{jki} - q_{ijk}\} + \{T_{ijk} + T_{jik} - T_{kji}\} \right\} \right], \tag{42}$$

which is the expression for affine connection in terms of non-metricity of diffusion tensor D_{ij} , its reciprocal diffusion tensor D^{ij} and the torsion tensor T_{ijk} .

If we now substitute Equation (42) in Equation (34), we get a fiber geodesic in case of anisotropic diffusion along the fiber as follows:

$$(d^2 \mu^i) / (d\tau)^2 + \{ (k | i \ j) \} + D^{im} \{ -1/2 \{q_{ikj} + q_{jki} - q_{ijk}\} + \{T_{ijk} + T_{jik} - T_{kji}\} \} [d\mu]^j / d\tau [d\mu]^k / d\tau \propto [d\mu]^i / d\tau \tag{43}$$

In Equation (43), the combination $\{T_{ijk} + T_{jik} - T_{kji}\}$ is sometimes called co-torsion of the geodesic fiber.

Here are few more things about our affine connection that if we define two parallel transports by means of two affine connections namely;

$$\Gamma_{jk}^i \equiv \frac{\partial}{\partial y^k} [Transport_{y \rightarrow x; \gamma_{geodesic}}]_{y \rightarrow x}^i, \text{ where } \textit{Limit } y \rightarrow x \text{ is}$$

taken by moving along the fiber γ so that the limit approaches to some trivial fiber γ_0 and $\tilde{\Gamma}_{jk}^i$ on the same diffusion manifold. Then their difference is also a tensor of type T_2^1 defined as:

$$\tilde{\Gamma}_{jk}^i = \Gamma_{jk}^i + P_{jk}^i. \tag{44}$$

By this technique, we can easily calculate the Riemannian curvature tensor for $\tilde{\Gamma}_{jk}^i$ and Γ_{jk}^i as

$$[\tilde{R}]_{jkl}^i = [R]_{jkl}^i + P_{jkl}^i. \tag{45}$$

Thus, if we consider an individual voxel or fiber tract of brain white matter, we can compare the Riemannian curvatures of any two nearby fiber geodesics or fibers by means of defining the affine connections. This comparison may lead to a crucial analysis of fiber tracking and fiber orientations.

Furthermore, once we have a data set of inverse diffusion tensor D^{ij} for any voxel, we can also calculate the mean diffusivity of diffusion tensor without seeking its latent roots, but by means of affine connection defined to that voxel as follows:

$\Gamma_i = \Gamma_{ki}^k \equiv Trace [\Gamma_{ki}^k]$ and hence mean diffusivity may be given as

$$\langle D \rangle = \frac{Trace [\Gamma_{ki}^k]}{3}. \tag{46}$$

Analysis of spatial transformation using lie derivatives

The analysis of spatial transformations of DT-MRI using certain geometric operations is much more concerned with the differential geometry of Lie derivatives in which one may model the brain cortical surfaces as manifolds, (that is, Riemannian symmetric manifold). The brain cortical surface can thought to be a topological manifold V with a set of local co-ordinate charts $\{(U)_i, \phi_i\}$, where U_i are open sets on V and the union of U_i envelopes the whole manifold V . Also, $\phi_i: U_i \rightarrow R^2$ is a homeomorphism that transforms U_i to the planar parameter domain. Moreover, a Riemannian metric on the individual brain cortical surface V is a quadratic form

given by:

$$ds^2(u, v) = E(u, v)du^2 + 2F(u, v)dudv + G(u, v)dv^2,$$

where (u, v) are local co-ordinates of any point on such a cortical surface. Also, all transition functions defined over brain cortical surface are analytic (that is., Cauchy-Riemann partial differential equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ are satisfied) and on each chart the Riemannian metric will have a special form:

This kind of local co-ordinates are called isothermal co-ordinates. Now, we define the Lie derivative denoted by \mathcal{L}_v of a tensor with respect to a contravariant vector field v^i as a linear mapping $\mathcal{L}_v: T_s^r \rightarrow T_s^r$ that is also linear in the argument v^i and satisfy the Leibnitz's rule on cartesian product of tensors;

$$\mathcal{L}_v(X_1 \otimes X_2) = (\mathcal{L}_v X_1) \otimes X_2 + X_1 \otimes (\mathcal{L}_v X_2).$$

Most probably the differential geometry of Lie derivatives can lucidly illustrate the surface based modeling which is valuable in brain imaging to help analyzing anatomical shape, to detect abnormalities of cortical surface folding and to statistically compare 3D anatomical models.

We, now, consider a point transformation in the diffusion tensor manifold $PD(3)$ as

$$\pi: P \rightarrow P', \tag{47a}$$

or, in local co-ordinate system

$$\pi: x^h \rightarrow x'^h. \tag{47b}$$

Suppose that we have an arbitrary anatomical shape (or geometric object) $\Omega(P)$ and we bring back the anatomical shape $\Omega(P')$ at P' to P by (the differential of) the transformation inverse to π , then we would have an anatomical shape $\Omega'(P)$ at P and we call the difference $\Omega'(P) - \Omega(P)$ the Lie difference between two anatomical shapes of brain cortical manifold. Let us introduce an open parameter group of transformation $x'^h \rightarrow f^h(x, t)$ over the $PD(3)$ manifold generated by a vector field v^i then $\mathcal{L}_v \Omega = \lim_{t \rightarrow 0} \frac{1}{t} \{(\Omega'(P) - \Omega(P))\}$ is called Lie derivative of anatomical shape Ω with respect to v^i and t is a canonical parameter.

Isometric collineation

If the point transformation given by Equation (47a, b) in the brain cortical manifold (which are $PD(3)$ manifolds) does not change the arc length of any individual fiber in

the manifold, we say that the transformation does not change the metric $g_{ij} \equiv D^{ij}$ and hence the reciprocal diffusion tensor D^{ij} . Then such a transformation is called isometric collineation or simply a motion. Hence in order that a one parameter group of transformation $x^h \rightarrow f^h(x, t)$ generated by a contravariant vector field $v^i = v^i \partial_i$ admits a group of isometric collineation, it is necessary and sufficient that

$$\mathcal{L}_v g_{ij} \equiv \mathcal{L}_v D^{ij} = 0, \text{ or}$$

$$\nabla_i v_j + \nabla_j v_i = v^m \partial_m D^{ij} - D^{mj} \partial_m v^i + D^{im} \partial_m v^j = 0. \quad (48)$$

Here v_i & v_j are the covariant components of a vector field v . Thus from Equation (48), we conclude that if the Lie derivative of reciprocal diffusion tensor of the brain cortical manifold has same values everywhere, the anatomical shape of the brain cortical manifold will be ideal.

Affine collineation or transformation

If the point transformation given by Equation(47a, b) defined in the brain cortical manifold does not change the torsion free affine connection, *that is,*, a symmetric affine connection $\Gamma^h_{(ij)}$ defined on some arbitrary fiber (or fiber geodesics) of cortical manifold, we call such a spatial transformation an affine collineation or simply affine motion. Thus to exist affine motion, we must have a relation of the type;

$$\mathcal{L}_v \Gamma^h_{(ij)} = \nabla_j \nabla_i v^h + R^h_{ijk} v^k = 0, \quad (49)$$

where R^h_{ijk} is the Riemannian curvature tensor as defined by Equation(15). Moreover, if two affine connections $\tilde{\Gamma}^i_{jk}$ and Γ^i_{jk} defined over the fibers γ_2 & γ_1 respectively of $PD(3)$ manifold emphasize the same notion of auto-parallelism of tangent vectors drawn to corresponding water molecules diffused along these fibers, then there must exist a co-vector f_k such that:

$$\tilde{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j f_k. \quad (50)$$

These two connections related in this way are said to differ by a projective transformation and we can also call them protectively equivalent and the co-vector f_k may assume to be local components of π .

Conformal collineation or transformation

We define a conformal map as an angle preserving transformation both in the sense of magnitude and direction. Suppose γ_1 & γ_2 be two fibers or fiber geodesics intersecting at the point P with angle φ . If the point transformation given by equation (47a, b) maps any two anatomical shapes P & P' of $PD(3)$ manifold, such that $[\varphi(\gamma_1)]$ & $[\varphi(\gamma_2)]$ be two fibers on P' intersecting at the point $\varphi(P)$ with an angle α then the point transformation will be called a conformal transformation iff

$$\mathcal{L}_v D^{ij} = v^m \partial_m D^{ij} - D^{mj} \partial_m v^i + D^{im} \partial_m v^j = 2\rho \quad (51)$$

where ρ being some function. Moreover, if in place of the point transformation (Equation 47a, b), we have a Gaussian mapping $\pi: S \rightarrow S^2$ from the brain cortical surface S to the unit sphere S^2 which spatially transforms each point P of S to its normal $n(P)$. Then this map will conformally transform infinitesimal circles to infinitesimal circles and will preserve the intersecting angle among circles. The following Figures (8 and 9) illustrate how a conformal transformation spatially maps the brain cortical surfaces.

CONCLUSION

Here is the brief discussion over some crucial results obtained from our article written in favor of spatial transformation of DT-MRI:

- (1) In order to justify the results of Riemannian geometry in the study of DT-MRI, we have calculated the inverse diffusion tensor whose value is defined as $D^{ij} = \sum_{a=1}^3 \frac{\partial X^a}{\partial x^i} \frac{\partial X^a}{\partial x^j}$. Also, with its help we have defined a metric between any two nearby water molecules diffused along any axon fiber given by Equation (20) and measured the length of that arbitrary axon fiber given by Equation (21).
- (2) As it has been verified that diffusion tensor manifolds are generally Riemannian symmetric and hence we have introduced a condition given by Equation (14) for Riemannian symmetry, from which we have deduced [Equation (16)] that the brain cortical manifold, *that is,*, the diffusion manifold is not merely a Riemannian symmetric but also a semi-symmetric manifold. Thereby, not only the results of Riemannian geometry can be made valid for DT-MRI studies, but the results of semi-symmetric geometry can also be justified therein.
- (3) We have setup a relation given by Equation (23) for

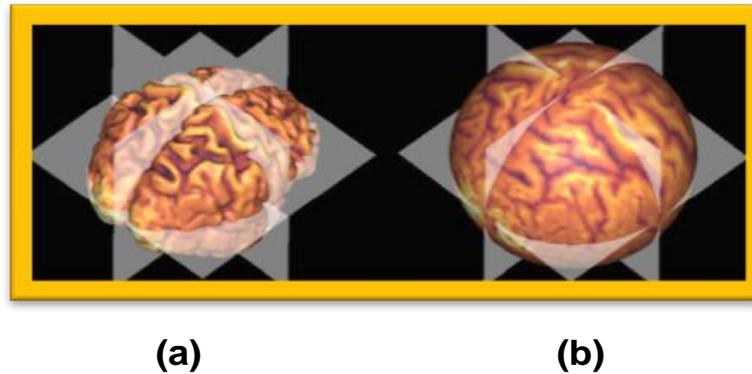


Figure 8. (a) shows the brain cortical surface with its mass center positioned at the coordinate origin, the yz plane aligned to the hemisphere plane and the originating points of the central sulcus near hemisphere clefts defines the xy plane. (b) Shows the conformal mapping of the brain cortical surface to a unit sphere (Zou et al., 2006).

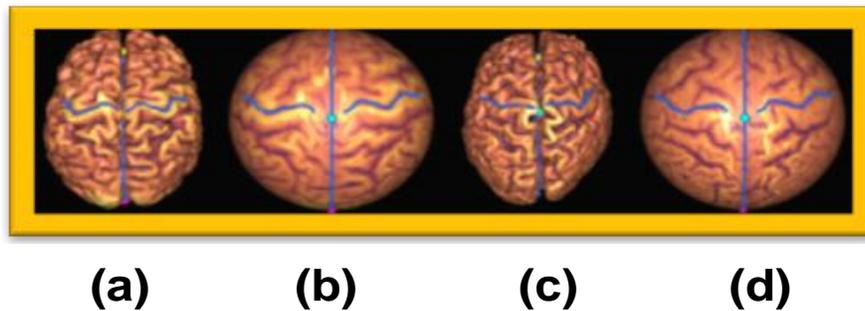


Figure 9. (a) Shows the template brain cortical surface with hemisphere cleft, central sulky, etc. as the landmarks. (b) Shows the result of mapping the cortical surface of a subject, with the same anatomical landmark specified. (c) Onto the unit sphere by the normal conformal mapping. (d) Shows the cortical surface of a subject, with the same anatomical landmark specified. (e) Shows the result of the cortical surface registration for the subject at the spherical domain (Zou et al., 2006).

the inclination of two axon fibers in terms of the angle between two hyper spheres which are orthogonal to the considered fibers. Also, the same (Equation 24) has been calculated in terms of the inclination of two tangent vectors drawn to any two water molecules of corresponding axon fibers at some specific instant. By comparing Equations (23) and (24), we have concluded that the isotropic behavior of diffusion tensor could not be directly introduced in case of Equation (24) as in this equation we cannot introduced an isotropic cone generated in Equation (23). The inclination is also calculated between two fiber tracts of brain white matter in terms of reciprocal diffusion tensor and its leading diagonal components.

(4) Hopefully, the third conclusion of this article can express a bit mechanism for the brain injury. The MRI study of popular neural disorders, the “seizure disorders” mainly involves detection of malformations of cortical

brain surface. Such malformations disturb the parallel orientations of axon fibers and fiber tracts and are a common cause of epilepsy. Thus by using third conclusion, one can geometrically detect the status of seizure disorder by simply calculating the angular disorders among fibers or fiber tracts.

(5) We have discussed a comparison theory of two axon fibers by means of involute-evolute geometry. By this study, one is able to check out the geometric configuration of fibers in the brain white matter as if we establish a 1-1 correspondence between the diffused water molecules along considered axon fibers, then by having information about one of the fiber, the same information can be assessed for others.

(6) The fiber geodesics for the axon fibers lying in the brain cortical manifold have also been calculated which can be used to identify the underlying manifold. Also, it is emphasized that by making use of Equation (34), if one

calculates the least and greatest fiber geodesics depending upon the size of healthy brain, the calculated data can be used to find out the symptoms of neuropsychiatric disorders in unhealthy brains by means of comparative data studies. Fiber geodesics given by Equation (43), in case of torsion full affine connection is also manipulated.

(7) Finally, spatial transformations, like affine collineation, isometric collineation, projective motion and conformal transformation of DT-MRI by making use of geometry of Lie derivatives have been studied. The strong reason behind the use of Lie geometry for such a study is that whenever spatial transformations are applied to DT-MR images, they could be distorted or deformed (distortion or deformation of DT-MR images may depend upon the nature of spatial transformation applied to them) and hence in order to preserve the consistency and validity of DT-MRI data, it is mandatory to re-orient the transformed image and the Lie operator is the only operator which can bring back the transformed image to its original position.

(8) The above spatial transformations, specially the conformal transformations could be used to examine the symptoms arise due to brain tumors, schizophrenia and other psychiatric disorders as in case of brain tumors the axon fiber tracts may get disruption (angular or parallel) or displacement and hence conformal transformation are able to map the orientation of anatomical shapes of brain both in terms of sense and magnitude. Also, the case of schizophrenia can be geometrically examined using the aforementioned spatial transformations, as in this disease the fiber tracts lose their orientation.

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