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On computational methods for solving systems of fourth-order nonlinear boundary value problems

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In this paper, we use a modified variation of parameters method, which is an elegant coupling of variation of parameters method and Adomian's decomposition method, for finding the analytical solution of systems of fourth-order nonlinear boundary value problems. Examples are given to illustrate the implementation and efficiency of the modified variation of parameters method. Results obtained in this paper represent a refinement and improvement of the previous known results.

Key words: Adomian's polynomials, variation of parameters method, systems of nonlinear boundary value problems.

INTRODUCTION

In recent years much attention has been given to solve system of fourth-order boundary value problems, which arise in several branches of pure and applied sciences including oceanography, transportation, fluid flow through porous media, optimization, medical sciences (Al-Said et al., 1995, 2002, 2004), Noor et al. (2010) and references therein. In this paper, we consider the following systems of fourth-order nonlinear boundary value problems:

$$u^{(iv)} = \begin{cases} f\left(x,u\left(x\right)\right), & a \le x < c, \\ f\left(x,u\left(x\right)\right)+u\left(x\right)g\left(x\right)+r, & c \le x < d, \\ f\left(x,u\left(x\right)\right), & d \le x \le b, \end{cases} \tag{1}$$

and

$$u^{(iv)} = \begin{cases} f\left(x,u\left(x\right)\right)+u\left(x\right)g\left(x\right)+r, & a \le x < c, \\ f\left(x,u\left(x\right)\right), & c \le x < d, \\ f\left(x,u\left(x\right)\right)+u\left(x\right)g\left(x\right)+r, & d \le x \le b, \end{cases} \tag{2}$$

with boundary conditions

$u(a)=\alpha_1,\ u(c)=\alpha_2,\ u(c)=\alpha_3,\ u(b)=\alpha_4,$ and continuity conditions of $u(x), u'(x), u''(x)$ and $u'''(x)$ at internal points c and d of the interval $[a,b]$. Here r and $\alpha_i,\ i=1\ldots4$ are real and finite constants, $g(x)$ is a continuous function on $[a,b]$ and $f(x,u(x)) = f(u)$ is a nonlinear continuous function. If $f(u)=f,$ is a function of space variable only, then these problems have been studied extensively. We would like to mention that such type of problems arise in the study of obstacle, contact, unilateral and equilibrium problems arising in economics, transportation, nonlinear optimization, oceanography, ocean wave engineering, fluid flow through porous media and some other branches of pure and applied sciences (Al-Said et al., 1995, 2002, 2004) and references therein.

Several techniques have been developed for solving a system of fourth-order linear boundary value problems associated with obstacle problems. Al-Said et al. (2004) used finite difference method for fourth-order obstacle problems, Khalifa and Noor (1990) applied quintic spline method for contact problems, Noor and Khalifa (1994) applied quartic spline method for obstacle problems, Noor and Al-Said (1990) used some numerical methods

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for system of fourth-order boundary value problems, Al-Said and Noor (2002) used quartic spline method for fourth-order obstacle problems, Momani et al. (2006) used decomposition method for solving system of fourth-order obstacle problems. Most of these methods involve numerical and huge computational work. In this paper, we use the modified variation of parameters method to solve system of fourth-order nonlinear boundary value problems associated with obstacle problems. Noor et al. (2008) have used the variation of parameters method for solving a wide classes of higher orders initial and boundary value problems. Ma et al. (2004, 2008) have applied the variation of parameters method for solving some non homogenous partial differential equations. Ramos (2008) has used this technique to find the frequency of some nonlinear oscillators. Ramos (2008) has also shown that this technique is equivalent to the variational iteration method. It is further investigated by Noor et al. (2010) that proposed technique is totally different from the variational iteration method in many aspects. The multiplier used in the variation of parameters method is obtained by Wronskian technique, which is totally different from Lagrange multiplier of the variational iteration method. It is remarked that the variation of parameters method removes the higher order derivative term from its iterative scheme which is clear advantage over the variational iteration method. Thus we conclude the variation of parameters method has reduced lot of computational work involved due to this term as compared to some other existing techniques using this term. This shows that this method has clear advantage over other techniques such as variational iteration method and decomposition methods.

In this paper, we use the modified variation of parameters method, which is obtained by combing the classical variational of parameters method and the Adomian decomposition method. This technique is used by Noor et al. (2010) for solving a system of second-order nonlinear boundary value problem. We would like to point out that the modified variation of a parameter method is very flexible and efficient as compared with other methods. The use of multiplier and Adomian's polynomial together in the modified variation of parameters method increases the rate of convergence by reducing the number of iterations and successive application of integral operators. This technique makes the solution procedure simple while still maintaining the higher level of accuracy. In this paper, we implement this technique for solving systems of forth-order nonlinear boundary value problems associated with obstacle, unilateral and contact problems. Some examples are given to illustrate the implementation and efficiency of the modified variation of a parameter method. It is well-known that the obstacle problems can be studied through the variational inequalities, Noor (1994, 2004, 2009, 2009a), Noor et al. (1993) and the references therein. It is an interesting and open problem to extend this technique for solving the variational inequalities. This may lead to further research in this field and related optimization problems.

MODIFIED VARIATION OF PARAMETERS METHOD

To convey the basic concept of the variation of parameter method for solving the differential equations, we consider the general form of the nonlinear equation of this type.

$$Lu(x) + Ru(x) + Nu(x) = g(x), \tag{3}$$

where L is a higher order linear operator, R is a linear operator of order less than L, N is a nonlinear operator and g is a source term. Using variation of parameters method in Mohyud-Din et al. (2009, 2011), we have the following general solution of Equation (3)

$$u(x) = \sum_{i=0}^{n-1} \frac{B_i x^i}{i!} + \int_0^x \lambda(x,s)\big(-Nu(s) - Ru(s) + g(s)\big)ds, \tag{4}$$

where n is order of given differential equation and $B_i's$ are unknowns which can be determined by initial/boundary conditions. Here $\lambda(x,s)$ is multiplier which can be obtained with the help of Wronskian technique. This multiplier removes the successive application of integrals in iterative scheme and it depends upon the order of equation. Mohyud-Din et al. (2009, 2011), have obtained the following for finding the multiplier $\lambda(x,s)$ as

$$\lambda(x,s) = \sum_{i=1}^{n} \frac{s^{i-1} x^{n-i} (-1)^{i-1}}{(i-1)!(n-i)!}. \tag{5}$$

Hence, we have the following iterative scheme from Equation (4)

$$u_{k+1}(x) = u_k(x) + \int_0^x \lambda(x,s)\big(-Nu_k(s) - Ru_k(s) + g(s)\big)ds \qquad k = 0,1,2,\ldots \tag{6}$$

It is observed that the fixed value of initial guess in each iteration provides the better approximation, that is, $u_k(x) = u_0(x)$, for $k = 1,2,\cdots$. One can modify the initial guess by dividing $u_0(x)$ in two parts and using one of them as initial guess. It is more convenient way in case of more than two terms in $u_0(x)$.

In the Adomian decomposition method, one defines the solution $u(x)$ by the following series $u(x) = \sum_{k=0}^{\infty} u_k(x),$ and the nonlinear terms are decomposed by infinite number of polynomials as follows $N(u) = \sum_{k=0}^{\infty} A_k(u_0, u_1, u_2, ..., u_i),$ where u is a function of x and A_k are the so-called Adomian's polynomials. These polynomials can be generated for various classes of nonlinearities by specific algorithm developed in Wazwaz (2000) as follows:

$$A_k=\left(\frac{1}{k!}\right)\left(\frac{d^k}{d\lambda^k}\right)N\left(\sum_{i=0}^{n}\left(\lambda^i u_i\right)\right)_{\lambda=0},\quad k=0,1,2,\cdots.$$

Hence, we have the following iterative scheme for finding the approximate solution of Equation (3) as

$$u_{k+1}(x)=u_k(x)+\int_0^x \lambda(x,s)\left(-A_k-Ru_k(s)+g(s)\right)ds. \qquad (7)$$

Modified variation of parameters method for solving the system of fourth-order nonlinear boundary value problems may be viewed as an important and significant improvement as compared with other similar method. This method has been used by Noor et al. (2010) for solving a system of second-order nonlinear boundary value problems. The results are very encouraging as compared with other techniques. The main motivation of this paper is to use this modified variation of parameters method for solving the fourth-order system of nonlinear boundary value problems.

NUMERICAL RESULTS

We consider some examples to illustrate the efficiency and the implementation of the modified variation of parameter method.

Example 1.

Consider following system of fourth-order nonlinear boundary value problems of the type:

$$u^{(iv)}=\begin{cases}\frac{u^3}{3!}+\frac{u^2}{2!}+u+1, & \text{for } -1\le x<-\frac{1}{2},\\ \frac{u^3}{3!}+\frac{u^2}{2!}-3u+2, & \text{for } -\frac{1}{2}\le x<\frac{1}{2},\\ \frac{u^3}{3!}+\frac{u^2}{2!}+u+1, & \text{for } \frac{1}{2}\le x\le 1,\end{cases} \qquad (8)$$

with boundary conditions

$$u(-1)=u(1)=u\left(-\frac{1}{2}\right)=u\left(\frac{1}{2}\right)=0.$$

We will use modified variation of parameters method for solving system of fourth-order nonlinear boundary value problems (8). Using the modified variation of parameters method, we have the following iterative scheme to solve nonlinear system (8):

$$u_{k+1}(x)=\begin{cases}u_k(x)+\int_0^x\lambda(x,s)\left(A_k+u_k+1\right)ds, & \text{for } -1\le x<-\frac{1}{2},\\ u_k(x)+\int_0^x\lambda(x,s)\left(A_k-3u_k+2\right)ds, & \text{for } -\frac{1}{2}\le x<\frac{1}{2},\\ u_k(x)+\int_0^x\lambda(x,s)\left(A_k+u_k+1\right)ds, & \text{for } \frac{1}{2}\le x\le 1.\end{cases}$$

Since the governing equation is of fourth-order order, using $\lambda(x,s)=\frac{x^3}{3!}-\frac{x^2s}{2!}+\frac{xs^2}{2!}-\frac{s^3}{3!}$, we have

$$u_{k+1}(x)=\begin{cases}u_k(x)+\int_0^x\left(\frac{x^3}{3!}-\frac{x^2s}{2!}+\frac{xs^2}{2!}-\frac{s^3}{3!}\right)\left(A_k+u_k+1\right)ds, & \text{for} -1\le x<-\frac{1}{2},\\ u_k(x)+\int_0^x\left(\frac{x^3}{3!}-\frac{x^2s}{2!}+\frac{xs^2}{2!}-\frac{s^3}{3!}\right)\left(A_k-3u_k+2\right)ds, & \text{for } -\frac{1}{2}\le x<\frac{1}{2},\\ u_k(x)+\int_0^x\left(\frac{x^3}{3!}-\frac{x^2s}{2!}+\frac{xs^2}{2!}-\frac{s^3}{3!}\right)\left(A_k+u_k+1\right)ds, & \text{for } \frac{1}{2}\le x\le 1,\end{cases}$$

Case 1: $-1\le x<-\frac{1}{2}$.

In this case, we choose the initial value as: $u_0=c_3x,$ and obtain:

$$u_1(x)=c_1\frac{x^3}{3!}+c_2\frac{x^2}{2!}+c_4+\int_0^x\left(\frac{x^3}{3!}-\frac{x^2s}{2!}+\frac{xs^2}{2!}-\frac{s^3}{3!}\right)\left(A_0+u_0+1\right)ds,$$

$$u_1(x)=c_4+\frac{1}{2}c_2x^2+\frac{1}{6}x^3c_1+\frac{1}{24}x^4+\frac{1}{120}x^5c_3+\frac{1}{720}x^6c_3^{\,2}+\frac{1}{5040}x^7c_3^{\,3},$$

$$u_{k+2}(x)=\int_0^x\left(\frac{x^3}{3!}-\frac{x^2s}{2!}+\frac{xs^2}{2!}-\frac{s^3}{3!}\right)\left(A_{k+1}+u_{k+1}\right)ds,\quad \text{for}$$

$k=0,1,2,\ldots,$

$$\begin{aligned}u_2(x)=&\frac{1}{24}c_4x^4+\frac{1}{120}c_3c_4x^5+\frac{1}{720}c_3^{\,2}c_4x^6+\left(\frac{1}{5040}c_1+\frac{1}{1680}c_3c_2\right)x^7\\&+\left(\frac{1}{40320}+\frac{1}{6720}c_3c_2+\frac{1}{10080}c_1\right)c_3x^8+\left(\frac{1}{36288}c_3c_1+\frac{1}{60480}\right)c_3x^9\\&+\frac{11}{1814400}c_3^{\,2}x^{10}+\frac{29}{39916800}c_3^{\,3}x^{11}+\frac{1}{13305600}c_3^{\,4}x^{12}\\&+\frac{1}{172972800}c_3^{\,5}x^{13},\end{aligned}$$

$\vdots$

Case 2: $-\frac{1}{2}\le x<\frac{1}{2}$. In this case, we have

$$u_0=c_7x,$$

$$u_1(x)=c_5\frac{x^3}{3!}+c_6\frac{x^2}{2!}+c_8+\int_0^x\left(\frac{x^3}{3!}-\frac{x^2s}{2!}+\frac{xs^2}{2!}-\frac{s^3}{3!}\right)\left(A_0-3u_0+2\right)ds,$$

$$u_1(x)=c_8+\frac{1}{2}c_6x^2+\frac{1}{6}x^3c_5+\frac{1}{12}x^4-\frac{1}{40}x^5c_7+\frac{1}{720}x^6c_7^{\,2}+\frac{1}{5040}x^7c_7^{\,3},$$

$$u_{k+2}(x)=\int_0^x\left(\frac{x^3}{3!}-\frac{x^2s}{2!}+\frac{xs^2}{2!}-\frac{s^3}{3!}\right)\left(A_{k+1}-3u_{k+1}\right)ds,$$

for $k=0,1,2,\ldots,$

$$u_2(x) = -\frac{1}{8}c_8 x^4 + \frac{1}{120}c_7 c_8 x^5 + \left(\frac{1}{720}c_7^{\,2}c_8 - \frac{1}{240}\right)x^6 + \left(\frac{1}{1680}c_6 c_7 - \frac{1}{1680}c_5\right)x^7$$
$$+\left(-\frac{1}{6720} + \frac{1}{6720}c_7^{\,2}c_6 + \frac{1}{10080}c_5 c_7\right)c_3 x^8 + \left(\frac{19}{362880}c_7 + \frac{1}{36288}c_7 c_5\right)c_7 x^9$$
$$+\frac{1}{403200}c_7^{\,2}x^{10} - \frac{59}{39916800}c_7^{\,3}x^{11} + \frac{1}{13305600}c_7^{\,4}x^{12} + \frac{1}{172972800}c_7^{\,5}x^{13},$$
$\vdots$

Case 3: $\frac{1}{2} \le x \le 1$. In this case, we proceed as follows by taking $u_0 = c_{11}x$ and obtain

$$u_1(x) = c_9\frac{x^3}{3!} + c_{10}\frac{x^2}{2!} + c_{12} + \int_0^x \left(\frac{x^3}{3!} - \frac{x^2 s}{2!} + \frac{xs^2}{2!} - \frac{s^3}{3!}\right)(A_0 + u_0 + 1)\,ds,$$

$$u_1(x) = c_{12} + \frac{1}{2}c_{10}x^2 + \frac{1}{6}x^3 c_9 + \frac{1}{24}x^4 + \frac{1}{120}x^5 c_{11} + \frac{1}{720}x^6 c_{11}^{\,2} + \frac{1}{5040}x^7 c_{11}^{\,3},$$

$$u_{k+2}(x) = \int_0^x \left(\frac{x^3}{3!} - \frac{x^2 s}{2!} + \frac{xs^2}{2!} - \frac{s^3}{3!}\right)(A_{k+1} + u_{k+1})\,ds,$$

for $k = 0, 1, 2, \ldots,$

$$u_2(x) = \frac{1}{24}c_{12}x^4 + \frac{1}{120}c_{11}c_{12}x^5 + \frac{1}{720}c_{11}^{\,2}c_{12}x^6 + \left(\frac{1}{5040}c_9 + \frac{1}{1680}c_{11}c_{10}\right)x^7$$
$$+\left(\frac{1}{40320} + \frac{1}{6720}c_{11}c_{10} + \frac{1}{10080}c_9\right)c_{11}x^8 + \left(\frac{1}{36288}c_{11}c_9 + \frac{1}{60480}\right)c_{11}x^9$$
$$+\frac{11}{1814400}c_{11}^{\,2}x^{10} + \frac{29}{39916800}c_{11}^{\,3}x^{11} + \frac{1}{13305600}c_{11}^{\,4}x^{12}$$
$$+\frac{1}{172972800}c_{11}^{\,5}x^{13},$$
$\vdots$

Using the modified variation of parameters method, we have following formula for getting series solution in the whole domain from the above cases

$$u(x) = \begin{cases} \sum_{k=0}^{\infty} u_k(x), & \text{for } -1 \le x \le -\frac{1}{2}, \\ \sum_{k=0}^{\infty} u_k(x), & \text{for } -\frac{1}{2} \le x \le \frac{1}{2}, \\ \sum_{k=0}^{\infty} u_k(x), & \text{for } \frac{1}{2} \le x \le 1. \end{cases}$$

Hence, we have the following series solution after two Iterations

$$u(x) = \begin{cases} c_4 + c_3 x + \frac{1}{2}c_2 x^2 + \frac{1}{6}x^3 c_1 + \left(\frac{1}{24} + \frac{1}{24}c_4\right)x^4 + (c_4+1)\frac{1}{120}x^5 c_3 + (c_4+1)\frac{1}{720}x^6 c_3^{\,2} \\ +\left(\frac{1}{5040}c_1 + \frac{1}{5040}c_3^{\,3} + \frac{1}{1680}c_3 c_2\right)x^7 + \left(\frac{1}{40320} + \frac{1}{6720}c_3 c_2 + \frac{1}{10080}c_1\right)c_3 x^8 + \left(\frac{1}{36288}c_3 c_1 + \frac{1}{60480}\right)c_3 x^9 \\ +\frac{11}{1814400}c_3^{\,2}x^{10} + \frac{29}{39916800}c_3^{\,3}x^{11} + \frac{1}{13305600}c_3^{\,4}x^{12} + \frac{1}{172972800}c_3^{\,5}x^{13}, \quad \text{for } -1 \le x < -\frac{1}{2}, \\ c_8 + c_7 x + \frac{1}{2}c_6 x^2 + \frac{1}{6}x^3 c_5 x^4 + \left(\frac{1}{12} - \frac{1}{8}c_8\right)x^4 + \left(\frac{1}{120}c_8 - \frac{1}{40}\right)x^5 c_7 + \left(\frac{1}{720}c_7^{\,2}c_8 - \frac{1}{240} + \frac{1}{720}c_7^{\,2}\right)x^6 \\ +\left(\frac{1}{1680}c_6 c_7 - \frac{1}{1680}c_5 + \frac{1}{5040}c_7^{\,3}\right)x^7 + \left(-\frac{1}{6720} + \frac{1}{6720}c_7^{\,2}c_6 + \frac{1}{10080}c_5 c_7\right)c_3 x^8 \\ +\left(\frac{19}{362880}c_7 + \frac{1}{36288}c_7 c_5\right)c_7 x^9 + \frac{1}{403200}c_7^{\,2}x^{10} - \frac{59}{39916800}c_7^{\,3}x^{11} + \frac{1}{13305600}c_7^{\,4}x^{12} \\ +\frac{1}{172972800}c_7^{\,5}x^{13}, \quad \text{for } -\frac{1}{2} \le x < \frac{1}{2}, \\ c_{12} + c_{11}x + \frac{1}{2}c_{10}x^2 + \frac{1}{6}x^3 c_9 + \left(\frac{1}{24} + \frac{1}{24}c_{12}\right)x^4 + (c_{12}+1)\frac{1}{120}x^5 c_{11} + (c_{12}+1)\frac{1}{720}x^6 c_{11}^{\,2} \\ +\left(\frac{1}{5040}c_9 + \frac{1}{5040}c_{11}^{\,3} + \frac{1}{1680}c_{11}c_{10}\right)x^7 + \left(\frac{1}{40320} + \frac{1}{6720}c_{11}c_{10} + \frac{1}{10080}c_9\right)c_{11}x^8 + \left(\frac{1}{36288}c_{11}c_9 + \frac{1}{60480}\right)c_{11}x^9 \\ +\frac{11}{1814400}c_{11}^{\,2}x^{10} + \frac{29}{39916800}c_{11}^{\,3}x^{11} + \frac{1}{13305600}c_{11}^{\,4}x^{12} + \frac{1}{172972800}c_{11}^{\,5}x^{13}, \quad \text{for } \frac{1}{2} \le x \le 1. \end{cases} \tag{9}$$

By using boundary conditions and continuity conditions at $x = -\frac{1}{2}$ and $x = \frac{1}{2}$, we have a system of nonlinear equations. By using Newton's method for system of nonlinear equations, we have the following values of unknown constants:

$$\begin{aligned} &c_1 = -.4754938514,\ c_2 = -.3174510299,\ c_3 = -.0202520873,\ c_4 = .0170014799, \\ &c_5 = -.0000000003,\ c_6 = -.1969122745,\ c_7 = 0,\ c_8 = .0195461669, \\ &c_9 = .4754938516,\ c_{10} = -.3174510303,\ c_{11} = .0202520873,\ c_{12} = .0170014799. \end{aligned} \tag{10}$$

By using values of unknowns from Equation (10) into (9), we have the following analytic solution of system of forth-order nonlinear boundary value problem associated with obstacle problem (8)

$$u(x) = \begin{cases} .01700147990 - .02025208730\,x + .1587255150\,x^2 - .07924897523\,x^3 + .04237506167\,x^4 \\ -.0001716366897\,x^5 - .0004403248747\,x^6 - .00009051885313\,x^7 + 2.573754371\times10^{-5}\,x^8 \\ -3.402302352\times10^{-7}x^9 + 2.486561640\times10^{-9}x^{10} - 6.034643963\times10^{-12}\,x^{11} \\ +1.264284169\times10^{-14}\,x^{12} - 1969568721\times10^{-17}\,x^{13}, \quad \text{for } -1 \le x < -\frac{1}{2}, \\ .0195461669 - .09845613725x^2 - .50000000x^3 + .08089006247x^4 + .0008204678104x^6 \\ +1.785714286\times10^{-13}x^7 - .0001488095238x^8, \quad \text{for } -\frac{1}{2} \le x < \frac{1}{2}, \\ .01700147990 + .02025208730\,x - .1587255150\,x^2 + .07924897523\,x^3 + .04237506167\,x^4 \\ +.0001716366897\,x^5 - .0004403248747\,x^6 + .00009051885313\,x^7 + 2.573754371\times10^{-5}\,x^8 \\ +3.402302352\times10^{-7}x^9 + 2.486561640\times10^{-9}x^{10} + 6.034643963\times10^{-12}\,x^{11} \\ +1.264284169\times10^{-14}\,x^{12} + 1969568721\times10^{-17}\,x^{13}, \quad \text{for } \frac{1}{2} \le x \le 1. \end{cases}$$

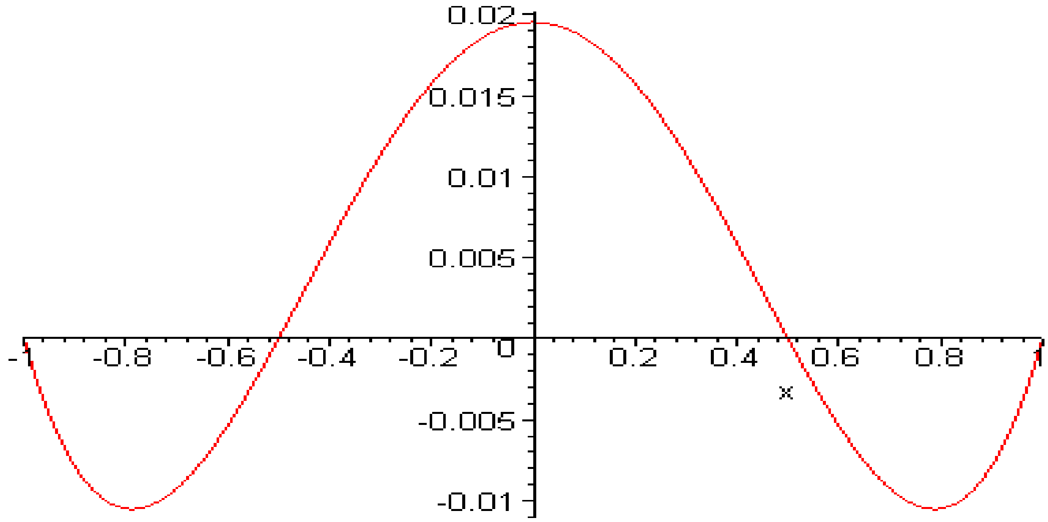

Figure 1. Graphical representation of analytical solution.

Figure 1 shows the graphical representation of analytical solution of system of forth-order nonlinear boundary value for problem (8).

Example 2

The value problem of the fourth-order nonlinear boundary, relevant to system (2), is considered as follows:

$$u^{(iv)} = \begin{cases} 1-4u+2u^3, & \text{for } -1 \le x < -\frac{1}{2}, \\ 2u^3, & \text{for } -\frac{1}{2} \le x < \frac{1}{2}, \\ 1-4u+2u^3, & \text{for } \frac{1}{2} \le x \le 1, \end{cases} \qquad (11)$$

with boundary conditions $u(-1)=u(1)=u\left(-\frac{1}{2}\right)=u\left(\frac{1}{2}\right)=0.$ Proceeding as before, we have the following iterative scheme to solve nonlinear system (11) by using the modified variation of parameters method:

$$u_{k+1}(x)=\begin{cases} u_k(x)+\int\limits_0^x\left(\frac{x^3}{3!}-\frac{x^2s}{2!}+\frac{xs^2}{2!}-\frac{s^3}{3!}\right)(1-4u+A_k)\,ds, & \text{for } -1\le x<-\frac{1}{2}, \\ u_k(x)+\int\limits_0^x\left(\frac{x^3}{3!}-\frac{x^2s}{2!}+\frac{xs^2}{2!}-\frac{s^3}{3!}\right)(A_k)\,ds, & \text{for } -\frac{1}{2}\le x<\frac{1}{2}, \\ u_k(x)+\int\limits_0^x\left(\frac{x^3}{3!}-\frac{x^2s}{2!}+\frac{xs^2}{2!}-\frac{s^3}{3!}\right)(1-4u+A_k)\,ds, & \text{for } \frac{1}{2}\le x\le 1, \end{cases}$$

Case 1: $-1 \le x < -\frac{1}{2}$. In this case, we consider the initial value as: $u_0 = c_3 x,$ and obtain further iterations as follows:

$$u_1(x)=c_1\frac{x^3}{3!}+c_2\frac{x^2}{2!}+c_4+\int\limits_0^x\left(\frac{x^3}{3!}-\frac{x^2s}{2!}+\frac{xs^2}{2!}-\frac{s^3}{3!}\right)(A_0-4u_0+1)\,ds,$$

$$u_1(x)=c_4+\frac{1}{2}c_2x^2+\frac{1}{6}x^3c_1+\frac{1}{24}x^4-\frac{1}{30}x^5c_3+\frac{1}{420}x^7c_3^3,$$

$$u_{k+2}(x)=\int\limits_0^x\left(\frac{x^3}{3!}-\frac{x^2s}{2!}+\frac{xs^2}{2!}-\frac{s^3}{3!}\right)(A_{k+1}-4u_{k+1})\,ds,$$

for $k=0,1,2,\ldots,$

$$u_2(x)=-\frac{1}{6}c_4x^4+\left(-\frac{1}{80}c_2+\frac{1}{60}c_3^2c_4\right)x^6-\frac{1}{1260}c_1x^7+\left(\frac{1}{560}c_3^2c_2-\frac{1}{10080}\right)x^8$$
$$+\left(\frac{1}{3024}c_3c_1+\frac{1}{22680}\right)c_3x^9+\frac{1}{20160}c_3^2x^{10}-\frac{1}{37800}c_3^3x^{11}+\frac{1}{1201200}c_3^5x^{13},$$

$\vdots$

Case 2: $-\frac{1}{2} \le x < \frac{1}{2}$. In this case, we have:

$$u_0 = c_7 x,$$

$$u_1(x)=c_5\frac{x^3}{3!}+c_6\frac{x^2}{2!}+c_8+\int\limits_0^x\left(\frac{x^3}{3!}-\frac{x^2s}{2!}+\frac{xs^2}{2!}-\frac{s^3}{3!}\right)(A_0)\,ds,$$

$$u_1(x) = c_8 + \frac{1}{2}c_6 x^2 + \frac{1}{6}c_5 x^3 + \frac{1}{420}c_7^{\,3} x^7,$$

$$u_{k+2}(x) = \int_0^x \left(\frac{x^3}{3!} - \frac{x^2 s}{2!} + \frac{x s^2}{2!} - \frac{s^3}{3!}\right)(A_{k+1})ds,$$

for $k = 0,1,2,\ldots,$

$$u_2(x) = \frac{1}{60}c_7^{\,2}c_8 x^6 + \frac{1}{560}c_7^{\,2}c_6 x^8 + \frac{1}{3024}c_5 c_7^{\,2} x^9 + \frac{1}{1201200}c_7^{\,2} x^{13},$$

$\vdots$

Case 3: $\frac{1}{2} \le x \le 1$. In this case, we obtain

$$u_0 = c_{11}x,$$

$$u_1(x) = c_9\frac{x^3}{3!} + c_{10}\frac{x^2}{2!} + c_{12} + \int_0^x \left(\frac{x^3}{3!} - \frac{x^2 s}{2!} + \frac{x s^2}{2!} - \frac{s^3}{3!}\right)(A_0 - 4u_0 + 1)\,ds,$$

$$u_1(x) = c_{12} + \frac{1}{2}c_{10}x^2 + \frac{1}{6}x^3 c_9 + \frac{1}{24}x^4 - \frac{1}{30}x^5 c_{11} + \frac{1}{420}x^7 c_{11}^{\,3},$$

$$u_{k+2}(x) = \int_0^x \left(\frac{x^3}{3!} - \frac{x^2 s}{2!} + \frac{x s^2}{2!} - \frac{s^3}{3!}\right)(A_{k+1} - 4u_{k+1})\,ds,$$

for $k = 0,1,2,\ldots,$

$$\begin{aligned} u_2(x) = &-\frac{1}{6}c_{12}x^4 + \left(-\frac{1}{80}c_{10} + \frac{1}{60}c_{11}^{\,2}c_{12}\right)x^6 - \frac{1}{1260}c_9 x^7 + \left(\frac{1}{560}c_{11}^{\,2}c_{10} - \frac{1}{10080}\right)x^8 \\ &+ \left(\frac{1}{3024}c_{11}c_9 + \frac{1}{22680}\right)c_{11}x^9 + \frac{1}{20160}c_{11}^{\,2}x^{10} - \frac{1}{37800}c_{11}^{\,3}x^{11} \\ &+ \frac{1}{1201200}c_{11}^{\,5}x^{13}, \end{aligned}$$

$\vdots$

Hence, we have the following series solution after two iterations

$$u(x) = \begin{cases} c_4 + c_3 x + \frac{1}{2}c_2 x^2 + \frac{1}{6}x^3 c_1 + \left(\frac{1}{24} - \frac{1}{6}c_4\right)x^4 - \frac{1}{30}x^5 c_3 + \left(-\frac{1}{80}c_2 + \frac{1}{60}c_3^{\,2}c_4\right)x^6 \\ + \left(\frac{1}{420}c_3^{\,3} - \frac{1}{1260}c_1\right)x^7 + \left(\frac{1}{560}c_3^{\,2}c_2 - \frac{1}{10080}\right)x^8 + \left(\frac{1}{3024}c_3 c_1 + \frac{1}{22680}\right)c_3 x^9 \\ + \frac{1}{20160}c_3^{\,2}x^{10} - \frac{1}{37800}c_3^{\,3}x^{11} + \frac{1}{1201200}c_3^{\,5}x^{13}, & \text{for } -1 \le x < -\frac{1}{2}, \\ c_8 + c_7 x + \frac{1}{2}c_6 x^2 + \frac{1}{6}c_5 x^3 + \frac{1}{60}c_7^{\,2}c_8 x^6 + \frac{1}{420}c_7^{\,3}x^7 + \frac{1}{560}c_7^{\,2}c_6 x^8 + \frac{1}{3024}c_5 c_7^{\,2}x^9 \\ + \frac{1}{1201200}c_7^{\,2}x^{13}, & \text{for } -\frac{1}{2} \le x < \frac{1}{2}, \\ c_{12} + c_{11}x + \frac{1}{2}c_{10}x^2 + \frac{1}{6}x^3 c_9 + \left(\frac{1}{24} - \frac{1}{6}c_{12}\right)x^4 - \frac{1}{30}c_{11}x^5 + \left(-\frac{1}{80}c_{10} + \frac{1}{60}c_{11}^{\,2}c_{12}\right)x^6 \\ + \left(\frac{1}{420}c_{11}^{\,3} - \frac{1}{1260}c_9\right)x^7 + \left(\frac{1}{560}c_{11}^{\,2}c_{10} - \frac{1}{10080}\right)x^8 + \left(\frac{1}{3024}c_{11}c_9 + \frac{1}{22680}\right)c_{11}x^9 \\ + \frac{1}{20160}c_{11}^{\,2}x^{10} - \frac{1}{37800}c_{11}^{\,3}x^{11} + \frac{1}{1201200}c_{11}^{\,5}x^{13}, & \text{for } \frac{1}{2} \le x \le 1. \end{cases} \tag{12}$$

By using boundary conditions and continuity conditions at $x = -\frac{1}{2}$ and $x = \frac{1}{2}$ and we have a system of nonlinear equations. By using Newton's method for system of nonlinear equations, we have the following values of unknown constants:

$$\begin{aligned} &c_1 = .4977848610,\ c_2 = .1177376514,\ c_3 = .0207967163,\ c_4 = .0034695031, \\ &c_5 = -.0000000002,\ c_6 = -.0069516345,\ c_7 = 0,\ \ c_8 = .0008689543, \\ &c_9 = -.4977848609,\ c_{10} = .1177376510,\ c_{11} = -.0207967161,\ c_{12} = .0034695030 \end{aligned} \tag{13}$$

By using values of unknowns from Equation (13) into (12), we have following analytic solution of system of forth-order nonlinear boundary value problem associated with obstacle problem (11)

$$u(x) = \begin{cases} .0034695031 + .0207967163x + .05886882570x^2 + .08296414350x^3 + .04108841615x^4 \\ -.0006932238767x^5 - .0006540730538x^6 - .0003950459342x^7 + 9.911541718\times10^{-5}x^8 \\ +9.881577896\times10^{-7}x^9 + 2.145354211\times10^{-9}x^{10} - 2.379537220\times10^{-10}x^{11} \\ +3.238608963\times10^{-15}x^{13}, & \text{for } -1 \le x < -\frac{1}{2}, \\ .0008689543 - .003475817250x^2 - 3.333333333\times10^{-11}x^3, & \text{for } -\frac{1}{2} \le x < \frac{1}{2}, \\ .0034695030 - .0207967161x - .05886882550x^2 - .08296414348x^3 + .04108841617x^4 \\ +.00069322387x^5 - .0006540730516x^6 + .0003950459341x^7 - 9.911541718\times10^{-5}x^8 \\ -9.881577795\times10^{-7}x^9 + 2.145354169\times10^{-8}x^{10} + 2.379537152\times10^{-10}x^{11} \\ +1.264284169\times10^{-14}x^{12} + 1969568721\times10^{-17}x^{13}, & \text{for } \frac{1}{2} \le x \le 1. \end{cases}$$

Figure 2 shows the graphical representation of the solution of problem (11).

CONCLUSION

In this paper, we have used the modified variation of parameters method, which is a combination of variation of parameters method and Adomian's decomposition method for solving system of fourth-order nonlinear boundary value problem. It is worth mentioning that we have solved nonlinear systems of boundary value problem by our proposed technique while most of the methods in the literature are proposed to solve linear systems of boundary value problems associated with obstacle problems. We took two examples for both the systems which are highly nonlinear in their nature. After applying our proposed technique we obtained series solutions as well as their graphical representation over the whole domain. We have analyzed that our proposed method is well suited for such physical problems as it

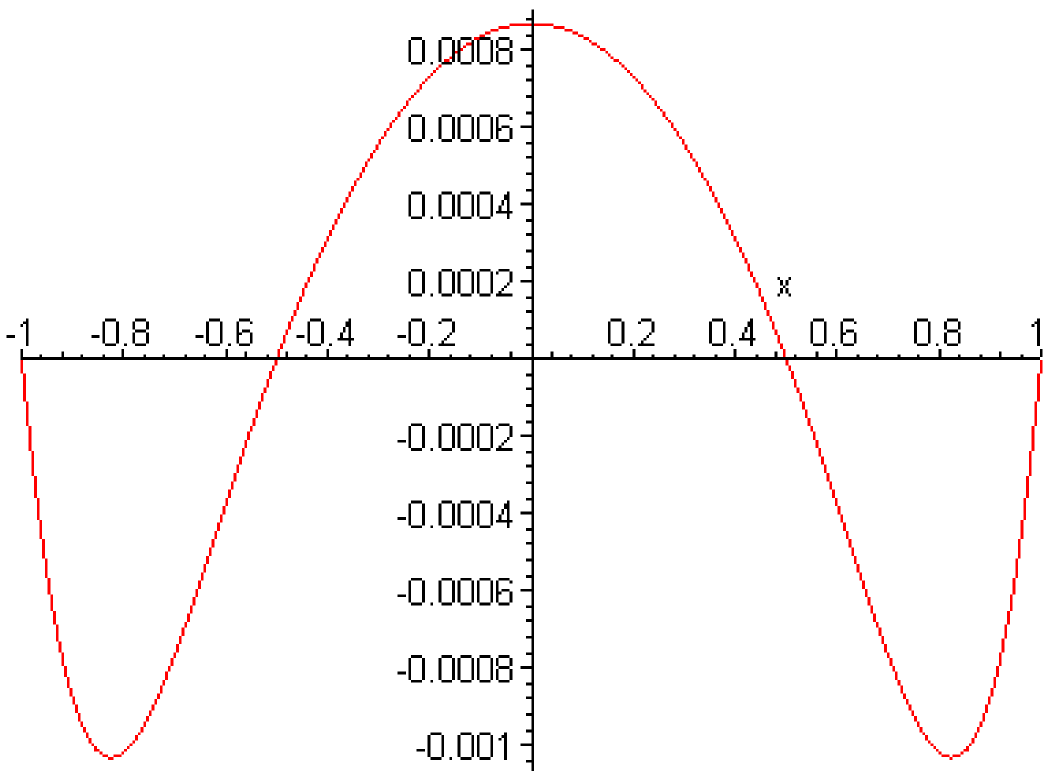

Figure 2. Graphical representation of the solution of problem.

provides best solution in less number of iterations. It is worth mentioning that the method is capable of reducing the volume of the computational work as compared to the existing classical methods. The use of multiplier gives this technique a clear edge over the decomposition method by removing successive application of integrals. We conclude that the modified variation of parameter method is very powerful and efficient technique for finding the analytical solutions for a wide class of systems of nonlinear boundary value problems and other related problems.

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