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Full Length Research Paper

Determination of energy levels of the Klein–Gordon equation, with pseudo harmonic potential plus the ring shaped potential

M. R. Shojaei* and A. A. Rajabi

Physics Department, Shahrood University of Technology, P. O. Box 36155-316 Shahrood, Islamic Republic of Iran.

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In the present paper we have found the exact solution of the bound states of Klein–Gordon equation with pseudo harmonic potential plus the ring-shaped potential via Nikiforov–Uvarov method. We have supposed scalar and vector potentials are equal. By special selection the potential is reduced to pseudo harmonic and harmonic oscillator potentials. Finally, we found and plotted the ground state of energy as a function of the coefficients potential.

Key words: Klein–Gordon equation, Nikiforo Uvarov method, rring shaped potential.

INTRODUCTION

The exact solution of the Schrödinger equation and Klein–Gordon equation for central and non-central potentials has been of considerable interest in recent years (Aktas, 2009). The Klein–Gordon and Dirac equations are used to describe the particle dynamics in relativistic quantum mechanics (Gerry, 1986). The Klein–Gordon equation has been used for the motion of a spin-zero particles in large class of potentials (Sheng et al. 2003). The spin-zero particles, like pions or kaons interact strongly with other particle and fields (Berkdemir, 2007). In recent years, many authors have worked on solving the Klein–Gordon equation with physical potentials including Hulthen potential (Simsek and Egrifes, 2004; Ikhdair and Sever, 2007; Haouat and Chetouani, 2008) Posch-Teller potential, ring-shaped harmonic oscillator potential, etc (Ikhdair and Sever, 2007; Xu et al., 2010). The ring-shaped potential consists of radial and angular dependent potentials and useful in studying ring-shaped molecules (Chen et al., 2004). In this work we proposed ring-shaped potential combination of harmonics potential plus a non-central angular part. Its potential is composed of the superposition between the harmonic oscillator potential and negative second power function potentials, plus non-central potentials. We proposed the potential in spherical coordinates as follows:

$$v(r,q) = ar^2 + \frac{b}{r^2} + \frac{c}{r^2 \sin^2 q} \quad (1)$$

Where a, b and c are real constants. We replaced the coulomb term of the Hartmann's potential with inverse squared term plus the harmonics oscillator potential (Chen et al., 2004). This potential can be used in quantum chemistry and nuclear physics to describe the ring-shaped molecules like benzene and the interactions between the deformed pairs of the nuclei (Barnea et al., 2001). For solving the exact solution of the Schrödinger equation with non-central potential the most popular approximation schemes are the shifted 1/N expansion (Ikhdair, 2006) perturbation theory, path integral solution (Mandel, 2000) and supersymmetric method (Khare and Bhaduri, 1994; Alhaidari, 2002) and numerical methods (Bano et al., 2011, Ibijola et al., 2008). In this work we first discuss the Klein–Gordon equation with central and non-central potentials. Then we solve exact solution of the radial and angular parts using NU method (Alhaidari, 2006, 2004). We found the eigenvalue with this potential, and then we determined the eigenvalue as a function of coefficient potential.

*Corresponding author. E-mail: M.R.Shojaei@shahroodut.ac.ir.

The Klein–Gordon equation with non-central potentials in spherical coordinates

The Klein–Gordon equation describing a particle with scalar potential $u(r,\theta)$ and vector potential $v(r,\theta)$ is given by (Alhaiduri et al., 2006, Simsek and Egrifes, 2004):

$$\left[p^2 + (M+u(r,\theta))^2 + (E - v(r,\theta))\right]\psi(\vec{r}) = 0 \qquad (2)$$

We suppose the scalar potential is equal with vector potential $u(r,\theta) = v(r,\theta)$. Considering the Equation 1 (Oyewumi and Akoshile, 2010). In the spherical coordinates we have:

$$\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]\right.$$

$$\left.-(M+E)\left(ar^2+\frac{b}{r^2}+\frac{c}{r^2\sin^2\theta}\right)+\left(E^2-M^2\right)\right\}\psi(\vec{r}) = 0 \qquad (3)$$

For the wave function $\psi(r,\theta,\varphi)$ we make the following separation ansatz:

$$\psi(r,\theta,\varphi) = R(r)H(\theta)\Phi(\varphi) \qquad (4)$$

This leads to

$$\frac{d^2R(r)}{dr^2}+\frac{2}{r}\frac{dR(r)}{dr}+\left[\varepsilon' - Ar^2 - \frac{(\gamma'+B)}{r^2}\right]R(r) = 0 \qquad (5)$$

Where

$$\varepsilon' = -(M^2-E^2) \qquad A = (M+E)a$$
$$B = (M+E)b \qquad \gamma' = \gamma(\gamma+1) \qquad (6)$$

The angular part equation is the following form:

$$\frac{d^2H(\theta)}{d\theta^2}+\cot\theta\frac{dH(\theta)}{d\theta}+\left[\gamma' - \frac{m^2}{\sin^2\theta}-\frac{(E+M)c}{\sin^2\theta}\right]H(\theta) = 0 \qquad (7)$$

$$\frac{d^2\Phi(\varphi)}{d\varphi^2}+m^2\Phi(\varphi) = 0 \qquad (8)$$

Where m^2 separation constant, solution of the Equation (8) is as following:

$$\Phi(\varphi) = \frac{1}{\sqrt{2\pi}}e^{\pm im\varphi}$$

$$m = 0,\pm1,\pm2,\ldots\ldots,\pm l \qquad (9)$$

Equation 5 and 7 are radial and the polar angle equations and we will solve then using Nikiforov-Uvarov method.

EXACT SOLUTION OF THE RADIAL KLEIN–GORDON EQUATION VIA NU METHOD

In Equation 5 by using the new variable $R(r) = r^{-1}u(r)$ we have

$$\frac{d^2u(r)}{dr^2}+\left[\varepsilon' - Ar^2 - \frac{(\gamma'+B)}{r^2}\right]u(r) = 0 \qquad (10)$$

Supposing the change of variable as $x = r^2$ we get

$$\frac{d^2u(x)}{dx^2}+\frac{1}{2x}\frac{du(x)}{dx}+\frac{1}{4x^2}\left[\varepsilon' x - Ax^2 - (\gamma'+B)\right]u(x) = 0 \qquad (11)$$

We have solved the Equation 11 by using Nikiforov–Uvarov method, (Nikiforov, 1998; Aktas, 2000). The energy eigenvalue of radial equation is obtained as following:

$$E^2 = M^2 + 2A^{\frac{1}{2}}\left[(2n+1) - \frac{1}{2}(4\gamma'+4B+1)^{\frac{1}{2}}\right] \qquad (12)$$

Finally By using the NU method, the redial wave function is as follow

$$R(x) = N_n 2^n n! x^{\frac{\nu}{4}-1} e^{-\frac{\mu x}{2}} L_n^{\frac{(\nu-1)}{2}}(\mu x) \qquad (13)$$

Where

$$\nu = 1 - \delta_\ell^{\frac{1}{2}} \quad , \quad \mu = A^{\frac{1}{2}}.$$
$$\delta_\ell = 4(\gamma'+B)+1 \qquad (14)$$

$L_n^{\frac{(\nu-1)}{2}}(\mu x)$ are the generalized Laguerre polynomials. Therefore, we have found the eigenvalue and redial eigenfunction of Klein–Gordon equation.

Solution of angular part Klein - Gordon equation

After separating the Klein–Gordon equation, the polar angel part equation is as follow:

$$\frac{d^2H(\theta)}{d\theta^2}+\cot\theta\frac{dH(\theta)}{d\theta}+\left[\gamma' - \frac{m^2}{\sin^2\theta}-\frac{(E+M)c}{\sin^2\theta}\right]H(\theta) = 0 \qquad (15)$$

Using the change of variable $x = \cos\theta$ we have:

$$\frac{d^2H(x)}{dx^2}-\frac{2x}{1-x^2}\frac{dH(x)}{dx}+\frac{1}{(1-x^2)^2}\left[\gamma'(1-x^2)-m^2-(E+M)c\right]H(x) = 0 \qquad (16)$$

By using Nikiforov-Uvarov method, (Nikiforov, 1998; Aktas, 2000), we have:

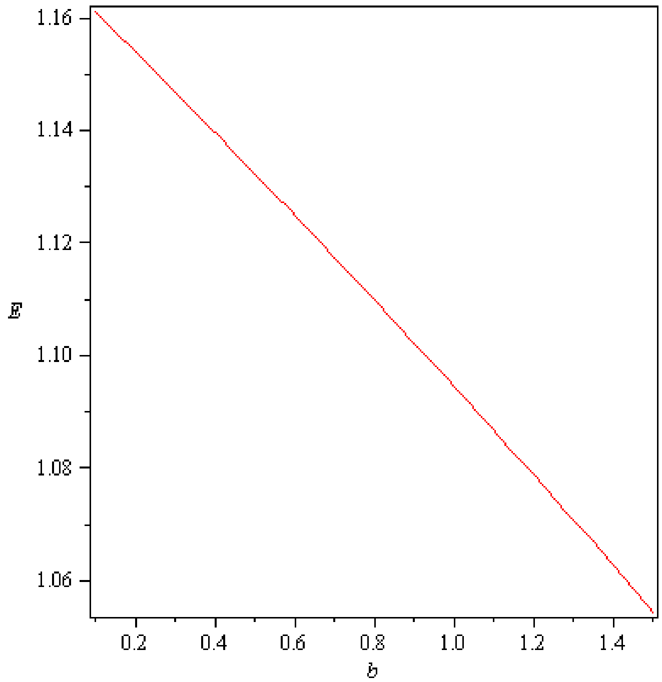


Figure 1. Energy of the ground state of the pseudoharmonic potential for $a = .1fm^{-3}$, $.1fm \le b \le 1.5fm$ and $M = .5fm^{-1}$.

$$g' = n(3-n) + (2n+1)(m^2 + c')^{\frac{1}{2}} + m^2 \tag{17}$$

Where $c' = (E + M)c$, Substituting Equation 17 in Equation 12, we find the final energy eigenvalue for angular part as follow:

$$E^2 = M^2 + 2A^{\frac{1}{2}}\left[(2n+1) - \frac{1}{2}\left[4n(3-n) + 4(2n+1)(m^2 + c')^{\frac{1}{2}} + 4m^2\right] + 4B + 1\right]^{\frac{1}{2}} \tag{18}$$

Finally By using NU method, the angular part wave function is as follow:

$$H_n(x) = \chi_n(x)P_n(x) = \left[N_n 2^n n!(-1)^n\right](x-1)^{-2n}\left(1-x^2\right)^{\frac{n}{2}} P_n^{(n,n)}(x) \tag{19}$$

Where the above equation from Rodriguez polynomials notation the Jacobi polynomials is as follow

$$P_n^{(a',b')}(x) = \frac{1}{2^n n!}(x-1)^{a'}(x+1)^{-b'}\frac{d^n}{dx^n}\left[(x-1)^{(n+a')}(x+1)^{(n+b')}\right] \tag{20}$$

Therefore, we derived the redial and angular part of the wave-function and energy eigenvalues of the Klein–Gordon equation analytically.

RESULTS AND DISCUSSION

Having various choices of potential parameters, our results can also be converted to the solution of some quantum mechanical systems. The results are as follows:

$i)$ In Equation 1 when the potential coefficient c is zero ($c = 0$), the potential reduced the pseudoharmonic potential. The standard form of this potential is as following:

$$v(r) = D_e\left(\frac{r}{r_e} - \frac{r_e}{r}\right)^2 \tag{22}$$

Where D_e is the dissociation energy between two atoms in a solid and r_e is the equilibrium inter molecular separation, which can be simply rewritten in the form of an isotropic harmonic oscillator plus an inverse quadratic potential. By comparing Equation 1 with Equation 22 we have $a = D_e r_e^{-2}$, $b = D_e r_e^2$. Substituting these parameters in Equation 18, the ground state energy is as follow:

$$E^2 = M^2 + 2\sqrt{(E+M)D_e r_e^{-2}}\left[1 - \frac{1}{2}\left(4(E+M)D_e r_e^2 + 1\right)^{\frac{1}{2}}\right] \tag{23}$$

Figure 1 shows energy of the ground state for $.1fm \le b \le 1.5fm$ and $a = .1fm^{-3}$. The ground state energy is a function of coefficient potential b.

According to the Figure 1, we can conclude that the ground state energy is the reverse of the coefficient potential. In this state, the other energy levels are given by

$$E^2 = M^2 + 2\sqrt{(E+M)D_e r_e^{-2}}\left[(2n+1) - \frac{1}{2}\left[4n(3-n) + 4(2n+1)m + 4m^2\right] + 4(M+E)D_e r_e^2 + 1\right]^{\frac{1}{2}} \tag{24}$$

Where $n = 1, 2, 3, ..$ and m depend on n. Also, M is mass of particle.

$ii)$ When the potential coefficients are $c = 0$, $b = 0$ the potential converts to the harmonic potential and the ground state eigenvalue energy is as follow:

$$E^2 = M^2 + \sqrt{(E+M)D_e r_e^{-2}} \tag{25}$$

In Figure 2, we show that the energy ground state as a

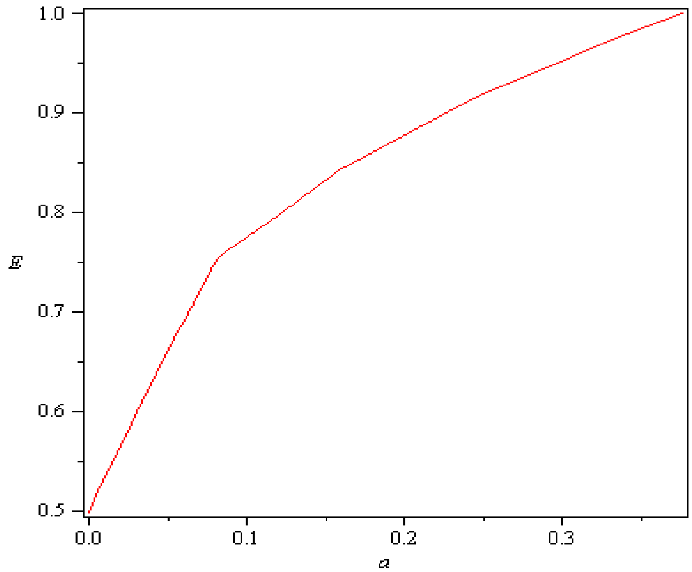


Figure 2. Energy of the ground state of the harmonic potential for $.1fm^{-3} \le a \le 1.5fm^{-3}$ and $M = .5fm^{-1}$.

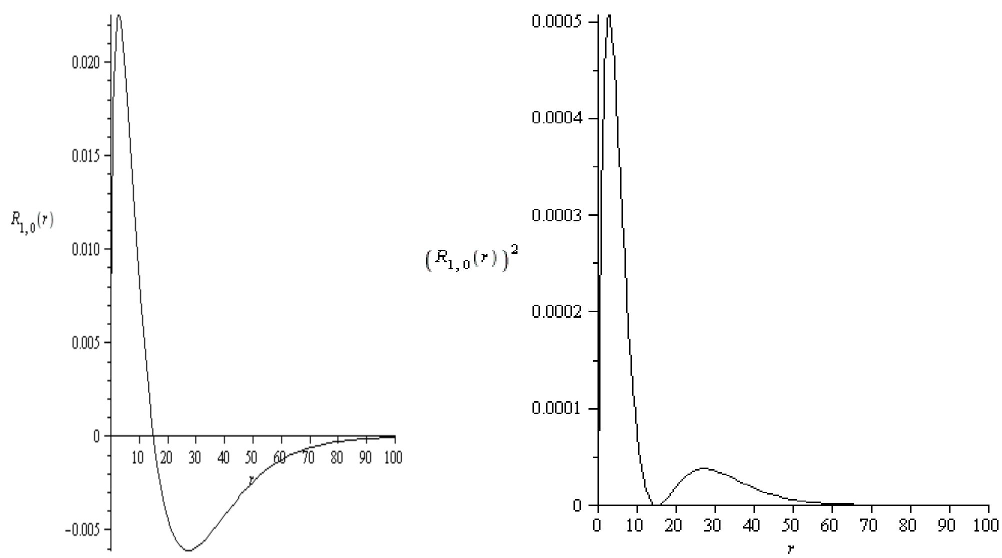


Figure 3. Radial wavefunction and probability density for $n = 1, l = 0$.

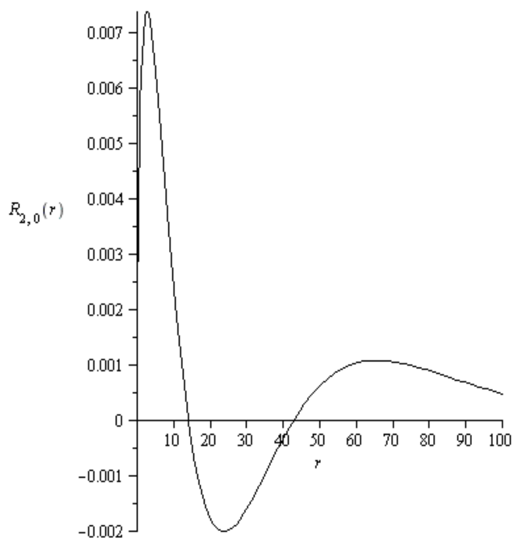
function of the coefficient potential for $.1(fm^{-3}) \le a \le 1.5(fm^{-3})$.

From Figure 2, we can conclude that the ground state energy has an increasing state. The other energy levels are given by

$$E^2 = M^2 + 2\sqrt{(E+M)a}\left[(2n+1) - \frac{1}{2}\left[n(3-n) + 4(2n+1)m + 4m^2 + 1\right]^{\frac{1}{2}}\right] \quad (26)$$

When the potential coefficients $a, b, c \neq 0$ the potential consist of central and ring-shaped potential and the eigenvalue energy is follow:

$$E^2 = M^2 + 2\sqrt{(E+M)a}\left[(2n+1) - \frac{1}{2}\left[n(3-n) + 4(2n+1)(m^2 + (E+M)C)^{\frac{1}{2}} + 4m^2 + 4(E+M)b + 1\right]^{\frac{1}{2}}\right] \quad (27)$$



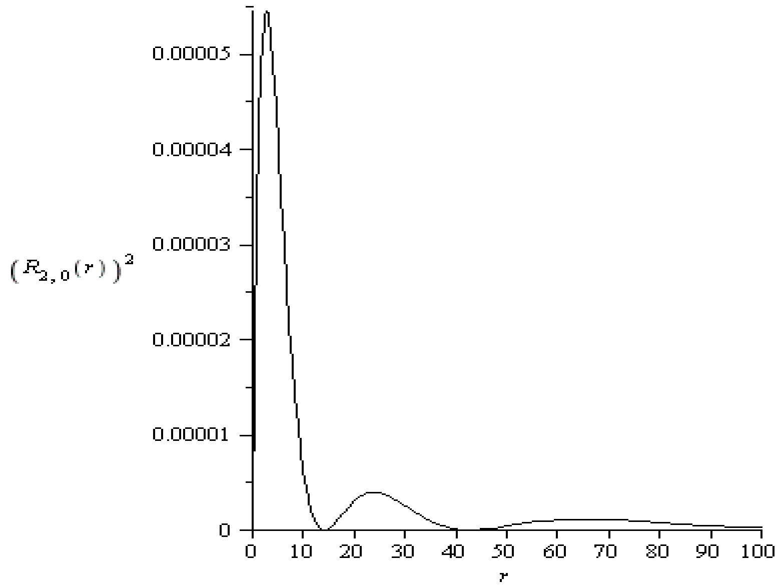


Figure 4. Radial wavefunction and probability density for $n = 2, l = 0$.

Where, $n = 0,1,2,3,..$ and m is depend on n also M is mass of particle. Thus, in this paper we found solved the Klein–Gordon equation analytically and exactly and found the energy levels of Klein–Gordon with pseudoharmonic potential plus ring-shaped potential. Then, in special physical condition, we analyzed these eigenvalues with the solved problems. Finally, we have plotted some the first few radial wave functions $R_n(x)$ and the probability density, $|R_n(x)|^2$ in Figures 3 and 4. This method of solving quantum mechanical problems may be useful in solving other complicated systems analytically. Given the above considerations, the authors believe that quantum chemistry and nuclear physics is field where the concepts and techniques and results can be put to good use.

CONCLUSION

We have studied the Klein–Gordon equation with special class of non-central potentials. Those potential can used to study the relativistic effect of the complex vibration-rotation energy structure of multi-electron atom and multi-atom molecules. Although we consider only the bound state problem here, this kind of soluble type of non-central potentials may have applications in scattering problems. This method of solving quantum mechanical problems may be useful in solving other complicated systems analytically.

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