International Journal of the Physical Sciences Vol. 6(35), pp. 7921 - 7925, 23 December, 2011
Available online at http://www.academicjournals.org/IJPS
DOI: 10.5897/IJPS11.1325
ISSN 1992 - 1950 ©2011 Academic Journals

Full Length Research Paper

Symmetry reductions and exact solutions of a variable coefficient (2+1)-Zakharov-Kuznetsov equation

L. D. Moleleki, A. G. Johnpillai and C. M. Khalique*

Department of Mathematical Sciences, International Institute for Symmetry Analysis and Mathematical Modelling, North-West University, Mafikeng Campus, Private Bag X 2046, Mmabatho 2735, Republic of South Africa.

Accepted 16 November, 2011

We study the generalized (2+1)-Zakharov-Kuznetsov (ZK) equation of time dependent variable coefficients from the Lie group-theoretic point of view. The Lie point symmetry generators of a special form of the class of equations are derived. We classify the Lie point symmetry generators to obtain the optimal system of one-dimensional subalgebras of the Lie symmetry algebras. These subalgebras are then used to construct a number of symmetry reductions and exact group-invariant solutions to the underlying equation.

Key words: Generalized ZK equation, solitons, Lie symmetries, optimal system, symmetry reduction, group-invariant solutions.

INTRODUCTION

The study of the exact solutions of a nonlinear evolution equation plays an important role to understand the nonlinear physical phenomena which are described by these equations. The importance of deriving such exact solutions to these nonlinear equations facilitate the verification of numerical methods and helps in the stability analysis of solutions.

This paper studies, the exact solutions of one nonlinear evolution equation, the generalized (2+1)-Zakharov-Kuznetsov equation of the form

$$u_t + f(t)uu_x + g(t)u_{xxx} + h(t)u_{xyy} = 0 \tag{1}$$

of time dependent variable coefficients. Here *f(t), g(t)* and *h(t)* are arbitrary smooth functions of the variable *t* and $fgh \neq 0$. Equation 1, models the nonlinear development of ion-acoustic waves in a magnetized plasma under the restrictions of small wave amplitude, weak dispersion, and strong magnetic fields (Zakharov and Kuznetsov, 1974). Equation 1 also appears in different forms in many areas of Physics, Applied Mathematics and Engineering (Biswas and Zerrad, 2010; Biswas, 2009).

The transformation

$$\tilde{t} = \int f(t)dt, \quad \tilde{x} = x, \quad \tilde{y} = y, \quad \tilde{u} = u \tag{2}$$

maps Equation 1 to

$$\tilde{u}_t + \tilde{f}(t)\tilde{u}\tilde{u}_x + \tilde{g}(t)\tilde{u}_{xxx} + \tilde{h}(t)\tilde{u}_{xyy} = 0, \tag{3}$$

where $\tilde{f}(t) = 1, \quad \tilde{g}(t) = g(t)/f(t)$ and $\tilde{h}(t) = h(t)/f(t)$. Therefore, without loss of generality, we can consider the equations of the general form

$$u_t + uu_x + a(t)u_{xxx} + b(t)u_{xyy} = 0 \tag{4}$$

in our analysis as all the results of the class 4 can be extended to the class 1 by the transformation 2.

In Peng et al. (2008), travelling wave-like solutions for the Equation 1 were obtained. In Moussa (2001) and Changzheng (1995), similarity reductions and some exact

*Corresponding author. E-mail: Masood.Khalique@nwu.ac.za. Tel: +27 18 389 2009. Fax: +27 18 389 2052.

solutions were obtained for the special cases of the class of Equation 4, (where *a(t)* and *b(t)* are constants) using symmetry group method. For the theory and application of the Lie symmetry methods see Bluman and Kumei, (1989), Olver (1993), Ovsiannikov (1982), Ibragimov (1994; 1996). Recently, Kraenkel and Senthilvelan (2011), utilized the method of Lie groups to derive solutions to an integrable equation governing short waves in a long-wave model.

In this study, we present the Lie point symmetries of a special case of Equation 4 and we also construct the optimal system of one-dimensional subalgebras of the Lie symmetry algebra of the special form of the equation. Moreover, using the optimal system of subalgebras symmetry reductions and exact group-invariant solutions of the underlying equation are obtained.

LIE POINT SYMMETRIES

We consider a special case of the class of Equation 4. That is, for the time dependent coefficients $a(t)=a_0/t$ and $b(t)=b_0/t,$ where a_0 and b_0 are arbitrary constants, we utilize the Lie symmetry group method to obtain symmetry reductions and group-invariant solutions of the underlying equation. Therefore, the equation that is going to be studied in this paper takes the following form:

$$u_t + uu_x + \frac{a_0}{t}u_{xxx} + \frac{b_0}{t}u_{xyy} = 0. \qquad (5)$$

A vector field

$$X = \tau(t,x,y,u)\frac{\partial}{\partial t} + \xi(t,x,y,u)\frac{\partial}{\partial x} + \psi(t,x,y,u)\frac{\partial}{\partial y} + \eta(t,x,y,u)\frac{\partial}{\partial u}, \qquad (6)$$

is a generator of point symmetry of the equation (5) if

$$X^{[3]}\left(u_t + uu_x + \frac{a_0}{t}u_{xxx} + \frac{b_0}{t}u_{xyy} \right)\Big|_{(5)} = 0, \qquad (7)$$

Where the operator $X^{[3]}$ is the third prolongation of the operator X defined by

$$X^{[3]} = X + \zeta_t\frac{\partial}{\partial u_t} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{xxx}\frac{\partial}{\partial u_{xxx}} + \zeta_{xyy}\frac{\partial}{\partial u_{xyy}},$$

the coefficients $\zeta_t, \zeta_x, \zeta_{xxx}$ and ζ_{xyy} are given by

$$\zeta_t = D_t(\eta) - u_tD_t(\tau) - u_xD_t(\xi) - u_yD_t(\psi),$$

$$\zeta_x = D_x(\eta) - u_tD_x(\tau) - u_xD_x(\xi) - u_yD_x(\psi),$$

$$\zeta_{xx} = D_x(\zeta_x) - u_{xt}D_x(\tau) - u_{xx}D_x(\xi) - u_{xy}D_x(\psi),$$

$$\zeta_{xy} = D_y(\zeta_x) - u_{xt}D_y(\tau) - u_{xx}D_y(\xi) - u_{xy}D_y(\psi),$$

$$\zeta_{xxx} = D_x(\zeta_{xx}) - u_{xxt}D_x(\tau) - u_{xxx}D_x(\xi) - u_{xxy}D_x(\psi),$$

$$\zeta_{xyy} = D_y(\zeta_{xy}) - u_{xyt}D_y(\tau) - u_{xxy}D_y(\xi) - u_{xyy}D_y(\psi).$$

Here D_i denotes the total derivative operator and is defined by

$$D_i = \frac{\partial}{\partial x^i} + u_i\frac{\partial}{\partial u} + u_{ij}\frac{\partial}{\partial u_j} + \ldots, i = 1,2,3, \quad \text{and}$$

$$(x^1, x^2, x^3) = (t, x, y).$$

The coefficient functions τ, ξ, ψ and η are calculated by solving the determining Equation 7. Since τ, ξ, ψ and η are independent of the derivatives of u, the coefficients of like derivatives of u in Equation 7 can be equated to yield an over determined system of linear partial differential equations (PDEs). Therefore, the determining equation for symmetries after lengthy calculations yielded:

$$\tau = \tau(t), \xi = \xi(t,x), \psi = \psi(y), \xi_{xx} = 0, \eta_{xu} = 0, \eta_{uu} = 0, \qquad (8)$$

$$(-1/t^2)\tau + (1/t)\tau_t - (3/t)\xi_x = 0, \qquad (9)$$

$$(-1/t^2)\tau + (1/t)\tau_t - (1/t)\xi_x - (2/t)\psi_y = 0, \qquad (10)$$

$$2\eta_{yu} - \psi_{yy} = 0, \qquad (11)$$

$$\eta + \tau_t u - \xi_t - \xi_x u + (b_0/t)\eta_{yyu} = 0, \qquad (12)$$

$$\eta_t + \eta_x u + (a_0/t)\eta_{xxx} + (b_0/t)\eta_{xyy} = 0. \qquad (13)$$

Solving the determining Equations 8, 9 ,10, 11, 12 and 13 for τ, ξ, ψ and η, we obtained the following symmetry group generators given by

$$X_1 = \partial_x, X_2 = \partial_y, X_3 = t\partial_x + \partial_u, X_4 = t\partial_t - u\partial_u.$$

SYMMETRY REDUCTIONS AND EXACT GROUP-INVARIANT SOLUTIONS OF EQUATION 5

Here, we first construct the optimal system of one-

Table 1. Commutator table of the Lie algebra of Equation 5.

Ad	X_1	X_2	X_3	X_4
X_1	0	0	0	0
X_2	0	0	0	0
X_3	0	0	0	$-X_3$
X_4	0	0	X_3	0

Table 2. Adjoint table of the Lie algebra of Equation 5.

Ad	X_1	X_2	X_3	X_4
X_1	X_1	X_2	X_3	X_4
X_2	X_1	X_2	X_3	X_4
X_3	X_1	X_2	X_3	$X_4 + \varepsilon X_3$
X_4	X_1	X_2	$e^{-\varepsilon} X_3$	X_4

dimensional subalgebras of the Lie algebra admitted by the Equation 5. The classification of the one-dimensional subalgebras are then used to reduce the Equation 5 into a partial differential equation (PDE) having two independent variables. Then, we also studied the symmetry properties of the reduced PDE to derive further symmetry reductions and exact group-invariant solutions for the underlying equation.

The results on the classification of the Lie point symmetries of the Equation 5 are summarized (Tables 1, 2 and 3). The commutator table of the Lie point symmetries of the Equation 5 and the adjoint representations of the symmetry group of Equation 5 on its Lie algebra are given in Tables 1 and 2, respectively. Tables 1 and 2 are used to construct the optimal system of one-dimensional subalgebras for Equation 5 which is given in Table 3 (Olver, 1993).

Case 1: In this case, the group-invariant solution corresponding to the symmetry generator $X_4 + \lambda X_1 + \mu X_2$ reduces the Equation 5 to the PDE:

$$\lambda h_\alpha + \mu h_\beta - h h_\alpha + h - a_0 h_{\alpha\alpha\alpha} - b_0 h_{\alpha\beta\beta} = 0. \quad (14)$$

Now the Equation 14 admits the following symmetry generators given by

$$X_1 = \partial_\alpha, \quad X_2 = \partial_\beta.$$

(a) X_1

The group-invariant solution corresponding to X_1 is $h = H(\gamma)$, where $\gamma = \beta$ is the group invariant of X_1, the substitution of this solution into Equation 14 and solving we obtain a solution $u(t,x,y) = Ce^{-y/\mu}$ for Equation 5, where C is a constant.

(b) $X_1 + \rho X_2$, where ρ is a constant.

$X_1 + \rho X_2$ leads to the group-invariant solution $h = H(\gamma)$, where $\gamma = \beta - \rho\alpha$ is the group invariant.

Substitution of this solution into Equation 14 gives rise to the ordinary differential equation (ODE).

$$(\rho^3 a_0 + \rho b_0)H'' + \rho H H' + (\mu - \lambda\rho)H' + H = 0, \quad (15)$$

where `prime' denotes differentiation with respect to γ.

Case 2: The group-invariant solution arising from $X_2 + \upsilon X_1$ reduces the Equation 5 to the PDE:

$$h_\alpha + h h_\beta + \frac{(a_0 + b_0\upsilon^2)}{\alpha} h_{\beta\beta\beta} = 0. \quad (16)$$

The Equation 16 admits the following three Lie point symmetry generators:

$$X_1 = \partial_\beta, \quad X_2 = \alpha\partial_\beta + \partial_h, \quad X_3 = -\alpha\partial_\alpha + h\partial_h.$$

The optimal system of one-dimensional subalgebras are $X_3 + cX_1, X_2 + dX_1, X_1,$ where c is an arbitrary real constant and $d = 0, \pm 1$.

(a) $X_3 + cX_1$

The group-invariant solution corresponding to $X_3 + cX_1$ is $h = \frac{1}{\alpha} H(\gamma)$, where $\gamma = \beta + c\ln\alpha$ is the group invariant of $X_3 + cX_1$, the substitution of this solution into the Equation 16 results in the following ODE

$$(a_0 + b_0\upsilon^2)H'' + HH' + a_0 H' - H = 0, \quad (17)$$

where `prime' denotes differentiation with respect to γ.

(b) $X_2 + dX_1$.

$X_2 + dX_1$ leads to the group - invariant solution

Table 3. Subalgebra, group invariants, group-invariant solutions of Equation 5.

S/N	X	α	β	Group – invariant solution
1	$X_4 + \lambda X_1 + \mu X_2$	$x - \lambda \ln t$	$y - \mu \ln t$	$u = \frac{1}{t} h(\alpha, \beta)$
2	$X_2 + \upsilon X_1$	t	$x - \upsilon y$	$u = h(\alpha, \beta)$
3	$X_3 + \varepsilon X_1$	t	y	$u = \frac{x}{(t+\varepsilon)} + h(\alpha, \beta)$
4	$X_3 + \delta X_2 + \varepsilon X_1$	t	$\delta x - (t+\varepsilon) y$	$u = \frac{x}{(t+\varepsilon)} + h(\alpha, \beta)$
5	X_1	t	y	$u = h(\alpha, \beta)$

Here $\varepsilon = 0, \pm 1, \delta = \pm 1$ and λ, μ and υ are arbitrary constants.

$h = \frac{\beta}{(\alpha + d)} + H(\gamma)$, where $\gamma = \alpha$ is the group invariant. Substitution of this solution into the Equation 16 gives the solution

$$u(t, x, y) = \frac{x - \upsilon y + C}{(t + d)},$$

Where C is a constant.

(c) X_1

The symmetry generator X_1 gives the trivial solution $u(t, x, y) = C$, where C is a constant.

Case 3: The group-invariant solution that corresponds to $X_3 + \varepsilon X_1$ reduces Equation 5 to the PDE:

$$h_\alpha + \frac{h}{\alpha + \varepsilon} = 0. \tag{18}$$

Hence, the solution of Equation 5 is given by

$$u(t,x,y) = \frac{x + H(y)}{(t + \varepsilon)},$$

where $H(y)$ is an arbitrary function of its argument.

Case 4: The $X_3 + \delta X_2 + \varepsilon X_1$-invariant solution reduces the Equation 1 to the PDE:

$$h_\alpha + \frac{\beta}{(\alpha+\varepsilon)} h_\beta + \delta h h_\beta + \frac{h}{(\alpha+\varepsilon)} + \left[\frac{a_0 \delta^3}{\alpha} + \frac{b_0 \delta (\alpha+\varepsilon)^2}{\alpha}\right] h_{\beta\beta\beta} = 0. \tag{19}$$

(a) $\varepsilon = 0$

In this case, the PDE (Equation 19) becomes

$$h_\alpha + \frac{\beta}{\alpha} h_\beta + \delta h h_\beta + \frac{h}{\alpha} + \left[\frac{a_0 \delta^3}{\alpha} + b_0 \delta \alpha\right] h_{\beta\beta\beta} = 0. \tag{20}$$

The Equation 20 admits the Lie algebra spanned by the following symmetry generators

$$X_1 = \alpha \partial_\beta, \quad X_2 = \delta \partial_\beta - 1/\alpha \partial_h.$$

(i) X_1

The group-invariant solution corresponding to X_1 is $h = H(\gamma)$, where $\gamma = \alpha$ is the group invariant of X_1, the substitution of this solution into Equation 20 and solving, we obtain the solution $u(t, x, y) = (x + C)/t$, where C is a constant.

(ii) $X_1 + \omega X_2$, where ω is a constant.

The group-invariant solution corresponding to $X_1 + \omega X_2$ is $h = -\beta / \alpha(\omega\alpha + \delta) + H(\gamma)$, where $\gamma = \alpha$ is the group invariant of $X_1 + \omega X_2$, the substitution of this solution into Equation 20 and solving, we obtain the solution

$$u(t,x,y)=\frac{\omega x+y+C}{\omega t+\delta},$$ where C is a constant.

(b) $\varepsilon \neq 0$.

In this instance, the PDE (Equation 19) admits the following symmetry generators

$$X_1=\delta\partial_\beta-1/(\alpha+\varepsilon)\partial_h,\quad X_2=\delta\alpha\partial_\beta+\varepsilon/(\alpha+\varepsilon)\partial_h.$$

(i) X_1

The group-invariant solution corresponding to X_1 is $h=-\beta/\delta(\alpha+\varepsilon)+H(\gamma)$, where $\gamma=\alpha$ is the group invariant of X_1, the substitution of this solution into the equation (19) and solving we obtain the solution $u(t,x,y)=\frac{y}{\delta}+C$, where C is a constant.

(ii) $X_2+\omega X_1$, where ω is a constant.

The $X_2+\omega X_1$-invariant solution is given by $h=(\varepsilon-\omega)\beta/\delta(\alpha+\omega)(\alpha+\varepsilon)+H(\gamma)$, where $\gamma=\alpha$ is the group invariant of $X_2+\omega X_1$, the substitution of this solution into Equation 19 and solving we obtain the solution

$$u(t,x,y)=\frac{\delta x-(\varepsilon-\omega)y+C}{\delta(t+\omega)},$$ where C is a constant.

Case 5: The X_1-invariant solution reduces Equation 5 to $h_\alpha=0$. Hence, the solution of the Equation 5 is given by $u(t,x,y)=H(y)$, where $H(y)$ is an arbitrary function of its argument.

Conclusion

In this paper, we have studied the generalized (2+1)-ZK Equation 4 with time dependent variable coefficients using the Lie symmetry group method. The special case of this equation, when *a(t)* and *b(t)* are constants, was studied in Moussa (2001) and Changzheng (1995), in which, the similarity reductions and some exact solutions were obtained using symmetry group method. we derived the Lie point symmetry generators of a special form of the underlying class of equations. The Lie symmetry classification with respect to the special form of the time dependent variable coefficients equation was presented. We used this classification of optimal system of one-dimensional subalgebras of the Lie symmetry algebras to construct symmetry reductions and exact group-invariant solutions for the special form of the equation.

REFERENCES

Biswas A, Zerrad E (2010). Solitary wave solution of the Zakharov-Kuznetsov equation in plasmas with power law nonlinearity. Nonlinear Analysis: Real World Appl., 11: 3272-3274.

Biswas A (2009). 1-Soliton solution of the generalized Zakharov-Kuznetsov equation with nonlinear dispersion and time-dependent coefficients. Physics Letters A, 373: 2931-2934.

Bluman GW, Kumei S (1989). Symmetries and Differential Equations, Springer, New York.

Changzheng Q (1995). Nonclassical analysis for the Zakharov-Kuznetsov equation. Int. J. Theor. Phys., 34: 99-108.

Ibragimov NH (1994-1996). CRC Handbook of Lie Group Analysis of Differential Equations, Vols.1-3, CRC Press, Boca Raton, FL.

Kraenkel RA, Senthilvelan M (2011). On the particular solutions of an integrable equation governing short waves in a long-wave model, Nonlinear Analysis: Real World Appl., 12: 446-449.

Moussa MHM (2001). Similarity solutions to non-linear partial differential equation of physical phenomena represented by the Zakharov-Kuznetsov equation. Int. J. Eng. Sci., 39: 1565-1575.

Olver PJ (1993). Applications of Lie Groups to Differential Equations. Second Edition, Springer, New York.

Ovsiannikov LV (1982). Group Analysis of Differential Equations. Academic Press, New York.

Peng YZ, Krishnan EV, Feng H (2008). Travelling wave-like solutions of the Zakharov-Kuznetsov equation with variable coeffcients. Pramana. J. Phys., 71: 49-55.

Zakharov VE, Kuznetsov EM (1974). On three-dimensional solitons. Sov. Phys. JETP, 39: 285-288.