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Full Length Research Paper

Evaluating integrals of the form $\int_0^\infty x^{s-1}e^{-ax^n}dx$ by Adomian decomposition

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This paper considers evaluation of the general class of integrals of the form $I[a;s;n]=\int_0^\infty x^{s-1}e^{-ax^n}dx$, using the Adomian decomposition method. This integral is solved in its general form and it is shown that particular values of s, a and n of this class of integrals correspond to Mellin and Gaussian integrals whose solutions are easily obtained.

Key words: General class of integrals, Adomian decomposition, Mellin integrals, Gaussian integrals.

INTRODUCTION

The Adomian decomposition method is an ingenious method for solving nonlinear functional equations of various kinds. It has been used to solve a wide class of stochastic and deterministic problems involving differential, nonlinear and integral equations. The method consists in calculating the solutions of nonlinear functional equations as infinite series in which each term can be easily determined. The series converges rapidly towards the true solution. El-Sayed (2002), Jafari et al. (2006), Kaya et al. (2004) and Mavoungou et al. (1992) have applied the method to solve systems of nonlinear equations; Haldar et al. (1996) used the method to solve integral equations; Abbaoui et al. (1999) and Babolian et al. (2005) employed the method to solve differential equations and Biazar (2006), Biazar et al. (2005a); Biazar and Montazeri (2005b) and Mavoungou et al. (1992) used the method to solve interaction equations. The convergence of the method has been proved by Adomian (1989, 1994), Cherruault (1989) and Paris et al. (2001); and Babolian et al., (2004) has used it to find Laplace transforms of given functions. In this paper, the Adomian method is used to obtain the solution of the general class of integrals of the form

$$I[a;s;n]=\int_0^\infty x^{s-1}e^{-ax^n}dx. \tag{1}$$

It is shown that:

$$I[a;s;n]=\frac{1}{na^{\frac{s}{n}}}\Gamma\left(\frac{s}{n}\right), \qquad n\neq 0. \tag{2}$$

From Equation 2, for various values of a, s and n, we can deduce many special cases of this integral.

THE ADOMIAN DECOMPOSITION METHOD

Following the analysis by Adomian (1989, 1994), consider the simple first-order differential equation:

$$\frac{dy}{dx}+P(x)y=Q(x),\ \ y(0)=0, \tag{3}$$

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which can be written in operator form as:

$$Ly + P(x)y = Q(x), \qquad y(0) = 0, \tag{4}$$

Where

$$L = \frac{d}{dx}.$$

The solution of Equation 3 is given by:

$$y(x)f(x) = \int Q(x)f(x)dx, \tag{5}$$

where

$$f(x) = e^{\int P(x)dx}. \tag{6}$$

Following Adomian (1989), we consider the asymptotic decomposition technique for the solution of Equation 4 by assuming that the solution takes the form:

$$y(x) = \sum_{m=0}^{\infty} y_m = y_0 + y_1 + y_2 + \cdots, \tag{7}$$

Where the terms $y_m(x),\ m = 0, 1, 2, \cdots,$ are given by the relations

$$y_0 = \frac{Q}{P}, \quad y_1 = \frac{y_0{}'}{P}, \quad y_2 = \frac{y_1{}'}{P}, \cdots, y_{m+1} = \frac{y_m{}'}{P}. \tag{8}$$

The convergence of the series of Equation 7 has been established by Adomian (1989, 1994), Cherruault (1989) and Mavoungou et al. (1992). Then, the solution of Equation 3 is defined by Equations 5 and 7 and is given by:

$$\int Q(x)f(x)dx = f(x)\sum_{m=0}^{\infty} y_m. \tag{9}$$

APPLICATION TO EQUATION 1

In Equation 1, let $Q(x) = x^{s-1}$, and set $f(x) = e^{-ax^n}$ so that, using Equation 6, $P(x) = -nax^{n-1}$. Then, the series of Equation 7 becomes:

$$y_0 = \frac{-x^{(s-n)}}{na},$$

$$y_1 = \frac{-(s-n)x^{(s-2n)}}{n^2a^2},$$

$$y_2 = \frac{-(s-n)(s-2n)x^{(s-3n)}}{n^3a^3},$$

$$y_3 = \frac{-(s-n)(s-2n)(s-3n)x^{(s-4n)}}{n^4a^4},$$

$$\vdots$$

$$y_{k-1} = \frac{-(s-n)(s-2n)(s-3n)\cdots(s-(k-1)n)x^{(s-kn)}}{n^ka^k}.$$

The series terminates when $s = kn$, that is, when

$$y_{\frac{s}{n}-1} = \frac{-n^{k-1}}{n^k}\frac{(k-1)(k-2)(k-3)\cdots 2\cdot 1}{a^k} = \frac{-1}{na^k}(k-1)!$$

$$= \frac{-1}{na^k}\Gamma(k) = \frac{-1}{na^{\frac{s}{n}}}\Gamma\left(\frac{s}{n}\right).$$

Thus, Equation 9 becomes:

$$\int_0^{\infty} x^{s-1}e^{-ax^n}dx = e^{-ax^n}\left[y_0 + y_1 + y_2 + \cdots + y_{k-1} + \cdots\right]\Big|_0^{\infty}, \tag{10}$$

so that Equation 1 gives:

$$I[a;s;n] = \int_0^{\infty} x^{s-1}e^{-ax^n}\,dx = \frac{1}{na^{\frac{s}{n}}}\Gamma\left(\frac{s}{n}\right), \tag{11}$$

as required.

EXAMPLES

In the following examples, it is shown that the general class of integrals of Equation 1 whose solution has been obtained by Adomain decomposition and is given by Equation 11, corresponds to Mellin and Gaussian integrals for particular values of a, s and n. Thus, solutions of these integrals can therefore be easily obtained from Equation 11 by substituting the appropriate values of a, s and n.

Example 1: Mellin transform of e^{-x}

When $a = 1$ and $n = 1$, Equation 11 becomes the Mellin transform of e^{-x}, which is also the definition of the Gamma function of s as given by Paris et al. (2001).

Thus, substituting these values in Equation 11 gives:

$$\int_0^\infty x^{s-1}e^{-x}dx = I[1;s;1] = \Gamma(s). \qquad (12)$$

Example 2: Mellin transform of e^{-ax}

Setting $n = 1$ in Equation 11 gives

$$\int_0^\infty x^{s-1}e^{-ax}dx = I[a;s;1] = a^{-s}\Gamma(s), \qquad (13)$$

which is the Mellin transform of e^{-ax} as defined by Paris et al. (2001).

Example 3: Mellin transform of e^{-x^2}

Set $a = 1$ and $n = 2$ in Equation 11 to obtain the Mellin integral of e^{-x^2}. Then:

$$\int_0^\infty x^{s-1}e^{-x^2}dx = I[1;s;2] = \tfrac{1}{2}\Gamma\left(\tfrac{s}{2}\right), \qquad (14)$$

which is in conformity with Paris et al. (2001).

In Examples 2 and 3, instead of substituting the values of a and n in Equation 11, the Mellin transforms of e^{-ax} and e^{-x^2} can be obtained by direct application of the Adomian decomposition method. The details of the procedure are shown in Appendix 1 and 2, respectively.

Example 4: The error function

Let $a = 1$, $s = 1$ and $n = 2$. Then, Equation 11 gives the error function, a special case of the Mellin integral. In this case

$$\int_0^\infty e^{-x^2}dx = I[1;1;2] = \tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2}. \qquad (15)$$

Example 5: The Gaussian integral

Let $n = 2$ in Equation 11 to obtain the Gaussian integral, which is also called the probability integral. In this case

$$\int_0^\infty x^{s-1}e^{-ax^2}dx = I[a;s;2] = \frac{1}{2a^{\frac{s}{2}}}\Gamma\left(\tfrac{s}{2}\right). \qquad (16)$$

Substituting, in turn, $s = 2$, $s = 2t+1$ and $s = 2t+2$, in Equation 14, we obtain the following well known results of the Gaussian integral (Paris et al., 2001).

Case (1)

When $s = 2$ we obtain

$$\int_0^\infty xe^{-ax^2}dx = I[a;2;2] = \frac{1}{2a}\Gamma(1) = \frac{1}{2a}, \qquad (17)$$

An expected elementary result.

Case (2)

When $s = 2t+1$ we have:

$\int_0^\infty x^{2t}e^{-ax^2}dx = I[a;2t+1;2] = \frac{1}{2a^{t+\frac{1}{2}}}\Gamma\left(t+\frac{1}{2}\right) =$
t+12!2at+12.
(18)

Equation 18 is a well known result of the Gaussian integral.

Case (3)

Set $s = 2t+2$ to obtain

$\int_0^\infty x^{2t+1}e^{-ax^2}dx = I[a;2t+2;2] = \frac{1}{2a^{t+1}}\Gamma(t+1) =$
t!2at+1.
(19)

Equation 19 is another well known result of the Gaussian integral.

CONCLUSION

The Adomain decomposition method has been used to evaluate a general class of integrals of the form of Equation 1. The examples worked out have shown that the use of the technique avoids the complexities involved when other methods of evaluation are used.

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APPENDIX 1: The Mellin transform of e^{-ax}

Let Q $(x) = x^{s-1}$, and $f(x) = e^{\int P(x)\,dx}$ so that $P(x) = -a$. Then the recurrence relations of Equation 8 become:

$$y_0 = \frac{-x^{s-1}}{a},$$

$$y_1 = \frac{-(s-1)x^{s-2}}{a^2},$$

$$y_2 = \frac{-(s-1)(s-2)x^{s-3}}{a^3},$$

$$y_3 = \frac{-(s-1)(s-2)(s-3)x^{s-4}}{a^4},$$

$$\vdots$$

$$y_{m-1} = \frac{-(s-1)(s-2)(s-3)\cdots(s-m)x^{s-m}}{a^m}.$$

$$\vdots$$

$$y_{s-1} = -\frac{(s-1)!}{a^s}$$

Thus, Equation (10) becomes:

$$\int_0^\infty x^{s-1}e^{-ax}\,dx = e^{-ax}\,[y_0 + y_1 + y_2 + \cdots + y_{k-1} + \cdot$$

Hence, $g(s) = a^{-s}\Gamma(s)$ as required.

APPENDIX 2: The Mellin transform of e^{-x^2}

Let Q $(x) = x^{s-1}$ and $f(x) = e^{-x^2} = e^{\int P(x)\,dx}$ so that $P(x) = -2x$. Then, the recurrence relations of Equation 8 become:

$$y_0 = \frac{-x^{s-2}}{2},$$

$$y_1 = \frac{-(s-2)x^{s-4}}{4},$$

$$y_2 = \frac{-(s-2)(s-4)x^{s-6}}{8},$$

$$y_3 = \frac{-(s-2)(s-4)(s-6)x^{s-8}}{16},$$

$$\vdots$$

$$y_{m-1} = \frac{-(s-2)(s-4)(s-6)\cdots(s-2m+2)x^{s-2m}}{2^m}.$$

The series terminates when $s = 2m$ and we then obtain

$$\vdots$$

$$y_{m-1} = -\frac{2^{m-1}}{2^m}(m-1)(m-2)\cdots 2\cdot 1 = -\frac{(m-1)!}{2}.$$

Thus, Equation 10 becomes:

$$\int_0^\infty x^{s-1}e^{-x^2}dx = e^{-x^2}[y_0 + y_1 + y_2 + \cdots + y_{k-1} + \cdots]\Big|_0^\infty = \frac{1}{2}\Gamma(m).$$

Hence, $g(s) = \frac{1}{2}\Gamma\left(\frac{s}{2}\right)$ as required.