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Review

Solution of Einstein's equations, for a fluid and homogeneous star

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Using the mathematical aspects of the Schwarzschild metric, we present different variables transformation which proves that the central singularity hypothesis does not exist. The geometric interpretation of the black hole from the Schwarzschild metric is not mathematically convincing. The different mathematical approaches taken by many authors were viewed, such as the Painleve metric and its complex variant, and the Schwarzschild metric, to deduce a metric with a throat sphere which leads to a mirror space-time. Subsequently, the possibility of a bi-metric tangent to the Schwarzschild metric's throat sphere was deduced. It was also shown that a false interpretation of the variables of the Schwarzschild metric can lead to false physical deductions and, in particular, to the concept of singularity. We computed the general solution of Einstein's equations in the presence of a non-zero energy tensor, that is, for a homogeneous fluid ball with energy conditions. This study method of resolution involves a reformulation of the Einstein equation and integration of the differential system. The metrics found are asymptotic to the Schwarzschild metric outside the fluid ball. Assumptions were presented for the pressure inside the fluid ball and the corresponding metrics were derived. Then, by solving the continuity equation of the energy-impulse tensor, we deduce an expression for the pressure inside the star that permits the express on of the interior and exterior metrics.

Key words: Schwarzs child's metric, black hole, singularity, gravity, bi-metric, internal metric.

INTRODUCTION

The Schwarzschild metric has become popular since the beginning of the 20th century for two main reasons. The first reason is that the first explicit solutions of Einstein's equations in vacuum, and demonstrated significant advancement in general relativity. The enthusiasm of cosmologists about the Schwarzschild metric was later undermined by 20th century mathematicians and physicists Eddington (1921), and then in the 21st century by Vankov (2011) and Mizony (2015), who have addressed the subject. The physical interpretation of variables in the metric cannot be subject to untested theories of time reversal singularity, or even spacetime rupture. The popular science press, eager for sensationalism, the science fiction movies have contributed to keeping its popularity alive the basis of Einstein's equations and, consequently, the role of

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variables in a metric requires a serious reassessment.

The second reason for its popularity is that the gravitational field deduced from the metric solves the problems of the anomalous advance of Mercury's perihelion and gravitational lensing effects. However, the scientific basis of these discoveries has been poorly disseminated and misunderstood. Cinema and the popular press, as well as fallacious computational simulations, under the pretext of educating and entertaining the general public, have made erroneous representations of the phenomena. Until very recently, radiophotography of the M87* black hole in the center of a galaxy, the same name had confused both the public and some dedicated physicists. This image, reconstructed from the contrast of radio radiation, is certainly of great interest, but cannot confirm the presence of a black hole. It could be a very massive neutron star, or strange star, with a redshift magnitude of 5 to 6, surrounded by a disk of a hot material.

Moreover, many mathematicians and physicists have warned about the interpretation of the Schwarzschild metric. Even if it seems to meet some physical assumption such as stability, asymptotic convergence at infinity to the Minkowski metric, and spherical symmetry, the questions raised by this metric, and especially by the description of the black hole it describes, have three types.

First, this metric is deduced from the Einstein tensor expressed in the vacuum. The equations deduced from it express both the vacuum and presence of a central mass. This contradiction remains unresolved.

Second, the Schwarzschild radius, the limit beyond which space-time is no longer real, does not have a coherent physical interpretation. The various papers on the black hole problem often omit the reality of a four-dimensional (4D) topology and continue to explain a three-dimensional (3D) time-dependent phenomenon, which is not the same physical phenomenon.

Third, Birkhoff's theorem is often misused by some physicists who, instead of considering a 4D spherical symmetry, continue to solve Einstein's equations with the assumption of a 3D central symmetry, which leads to misinterpretations of the theorems of Hawking and Penrose(1965 - 1970) on the emergence of a singularity by gravitational collapse.

From mathematical calculations, we present alternative physical interpretations of the Schwarzschild metric and the probable nature of the singularity. The general inner and outer solution of the Einstein equations for a homogeneous fluid star was also studied.

REVIEW OF THE SCHWARZSCHILD METRIC IN A VACUUM

Let us consider a metric of the following form:

$$ds^2 = Ad\xi^2 - Bd\mathcal{R}^2 - Edw^2 \tag{1}$$

where $dw^2 = (d\theta^2 + sin^2\,\theta\; d\varphi^2)$ and $A, B, and\ E$ are functions of the variable $\mathcal{R}$ satisfying the equations of Einstein in a vacuum.

$$G_{ik} = R_{ik} - \frac{1}{2} R g_{ik} = 0 \tag{2}$$

From a physical perspective, the problem is less evident, but many questions have been raised (Painlevé, 1921; Chazy, 1930; Eddington 1960; Mizony, 2015; Crothers, 2015).

It was recalled that Einstein, was the first to raise the Schwarzschild problem and gave an approximate solution before Schwarzschild gave an exact solution in 1916 in two remarkable publications.

One can, quickly solve the mathematical problem in a vacuum, then interpret the constants by assuming the presence of a central mass and by prejudging the interpretation of the variables of the metric. The latter is adapted to the physical problem of the black hole posed a posteriori. This method, which is used in many articles and is also taught, has often been contested.

First, we define the variable $(\xi, \mathcal{R}, \theta, \varphi)$ of the space-time where the metric will be calculated.

Variable «$\mathcal{R}$» is not the radial distance, but is a monotonic function of the radial distance $OM = r$. When r becomes very large, we can assimilate r into $\mathcal{R}$. Then, $\mathcal{R}(r) \approx r$ very far from the star. After the mathematical resolution of Equations 2 in vacuum, we obtain the following metric:

$$ds^2 = \left(1 + \frac{C}{\mathcal{R}}\right) d\xi^2 - \left(1 + \frac{C}{\mathcal{R}}\right)^{-1} d\mathcal{R}^2 - \mathcal{R}^2 dw^2 \tag{3}$$

whose determinant is equal to:

$$det(g) = -\mathcal{R}^4 sin^2(\theta)$$

Constant C is determined according to the laws of physics, and the systems limit conditions.

With the correct constant, this metric is spherically symmetric, static, and asymptotically equivalent to the Minkowski metric at infinity. It can be written as the Schwarzschild metric, with C given by the following:

$$C = -R_s = -\frac{2MG}{c^2}$$

where *M* being the mass of the star, G the gravitational constant, c the speed of light that is taken to be equal to 1 in the following, and R_s the Schwarzschild radius or the black hole horizon. In many scientific papers, variables ξ and $\mathcal{R}$ are arbitrarily and respectively assimilated to time *t* and radial distance r.

Thus we have:

$$ds^2 = \left(1 - \frac{R_s}{r}\right)dt^2 - \frac{1}{\left(1 - \frac{R_s}{r}\right)}dr^2 - r^2dw^2 \quad (4)$$

We note that Equation 4 shows the presence of two singularities, at $r = 0$, and at $r = R_s$, the horizon of the black hole.

In this paper, we consider that far from a star of mass M and radius a, that is, $r \gg a$, variables ξ and $\mathcal{R}$ are, respectively assimilated into time t, and radial distance r. Close to the star and inside the star, these variables cannot be considered as time and radial distance.

CHANGE OF VARIABLES IN THE METRIC

In the following, we investigate what impacts is observed on the metric and curvature if variables, that is is, are changed without changing the physical cause. Let us posit:

$$\mathcal{R} = \mathcal{R}(r) \implies d\mathcal{R} = \frac{\partial \mathcal{R}}{\partial r}dr$$

The Schwarzschild metric in Equation 3 can be written as:

$$ds^2 = \left(1 + \frac{C}{\mathcal{R}(r)}\right)d\xi^2 - \left(\frac{\partial \mathcal{R}}{\partial r}\right)^2\left(1 + \frac{C}{\mathcal{R}(r)}\right)^{-1}dr^2 - \mathcal{R}^2(r)dw^2 \quad (5)$$

Which satisfies the physical compatibility conditions if and only if the space is asymptotically flat for large r; that is,

$$\begin{cases} \left(\frac{\partial \mathcal{R}}{\partial r}\right)^2 \longrightarrow 1 \text{ when } r \longrightarrow \infty \\ \mathcal{R}(r) \longrightarrow r \text{ when } r \longrightarrow \infty \end{cases}$$

For example:

$$\mathcal{R}(r) = (r^n + a^n)^{\frac{1}{n}}, \quad (5.1)$$
$$\mathcal{R}(r) = r + a$$
$$\mathcal{R}(r) = r + f(r)$$

where $f(r)$ is a function derivable and monotone on $[0\,;+\infty]$, and asymptotically flat at infinity. There are infinitely many such examples.

All these metrics in Equation (5.1) verify the Einstein equation in vacuum $G_{ik} = 0$, and are spherically symmetric, static, and asymptotically flat. By a change of variables, infinitely many metrics equivalent to the Schwarzschild metric could be found.

Let us consider the following metric:

Changing variable ξ as $\xi = t - \int \psi(\mathcal{R})$, which translates to
$d\xi = dt - \psi d\mathcal{R}$

where ψ is a continuous and differentiable function on $[0\,;+\infty]$, the Schwarzschild metric can be written as:

$$ds^2 = \left(1 + \frac{2C}{\mathcal{R}}\right)dt^2$$
$$-\left(\left(1 + \frac{2C}{\mathcal{R}}\right)^{-1} - \psi^2\left(1 + \frac{2C}{\mathcal{R}}\right)\right)d\mathcal{R}^2$$
$$-2\psi(\mathcal{R})\left(1 + \frac{2C}{\mathcal{R}}\right)dtd\mathcal{R} - \mathcal{R}^2dw^2$$

with $lim_{\mathcal{R}\to\infty}\, \psi(\mathcal{R}) = 0$

Discussion on sign of the constant:

If c ≤0, with c= -R_s, we choose $\psi(\mathcal{R}) = \left(\frac{R_s}{\mathcal{R}}\right)^{\frac{1}{2}}\left(1 - \frac{R_s}{\mathcal{R}}\right)^{-1}$,

and

$$d\xi = dt - \sqrt{\frac{R_s}{\mathcal{R}}}\left(1 - \frac{R_s}{\mathcal{R}}\right)^{-1}d\mathcal{R}$$

With this change of variable, the metric is written:

$$ds^2 = \left(1 - \frac{R_s}{\mathcal{R}}\right)dt^2 - d\mathcal{R}^2 - 2\sqrt{\frac{R_s}{\mathcal{R}}}dtd\mathcal{R} - \mathcal{R}^2dw^2 \quad (6)$$

This formulation is called the Painleve-Gullstrand metric (Gullstrand, 1922; Fric, 2013; Crothers, 2015).

It can also be written as:

$$ds^2 = dt^2 - \left(d\mathcal{R} + \sqrt{\frac{R_s}{\mathcal{R}}}dt\right)^2 - \mathcal{R}^2 dw^2 \quad (7)$$

If c≥0,

we choose $\psi(\mathcal{R}) = i\left(\frac{R_s}{\mathcal{R}}\right)^{\frac{1}{2}}\left(1+\frac{2C}{\mathcal{R}}\right)^{-1}$

Then the metric can be written as:

$$ds^2 = \left(1+\frac{2C}{\mathcal{R}}\right)dt^2 - d\mathcal{R}^2 - 2i\sqrt{\frac{2C}{\mathcal{R}}}dt d\mathcal{R} - \mathcal{R}^2 dw^2 \quad (8)$$

This is a Riemannian metric on a holomorphic fibration tangent to the space $\mathbb{C}^2$, which is isomorphic to $\mathbb{R}^4$ (Dumitrescu, 2001).

It can also be written as:

$$ds^2 = dt^2 - \left(d\mathcal{R} + i\sqrt{\frac{R_s}{\mathcal{R}}}dt\right)^2 - \mathcal{R}^2 dw^2 \quad (9)$$

We do not wish to give a physical interpretation of the metric. Our approach is to find mathematical expressions of the metrics, by solving a tensor equation. The physical interpretation will be given as required.

Remark

For the case $\boldsymbol{C} \leq \boldsymbol{0}$ with $\boldsymbol{C} = -\boldsymbol{R_s}$, we can observe from Equation 6 that the singularity at $r = R_s$ does not exist and Equation 6 can be written as:

$$ds^2 = \left(1 - \frac{R_s}{\mathcal{R}(R_s)}\right)dt^2 - d\mathcal{R}^2 - 2\sqrt{\frac{R_s}{\mathcal{R}(R_s)}}dt d\mathcal{R} - \mathcal{R}^2(R_s)dw^2 \quad (10)$$

Minkowski's metric

The Minkowski metric, which represents an empty spacetime, is written as:

$$ds^2 = dt^2 - dr^2 - r^2 dw^2$$

The determinant of this metric is equal to:

$-r^4 sin^2(\theta)$

To the Minkowski metric, let us apply the following changes of variables:

$$dt = dT + \frac{\Phi}{1-\Phi^2}dR$$

$$dr = \Phi dT + \frac{1}{1-\Phi^2}dR$$

with $\Phi = \Phi(R)$ being a continuous function, except at localized points, and function $\Phi(R)$ tending asymptotically to 0 at infinity.

The change of variables can be written in matrix form:

$$(dt; dr) = \mathcal{M}\begin{pmatrix} dT \\ dR \end{pmatrix} = \begin{pmatrix} 1 & \frac{\Phi}{1-\Phi^2} \\ \Phi & \frac{1}{1-\Phi^2} \end{pmatrix}\begin{pmatrix} dT \\ dR \end{pmatrix}$$

where $\mathcal{M}$ the matrix of the change of variables. It is isometric as the determinant of this matrix is equal to 1.

The metric can be written as:

$$ds^2 = (1-\Phi^2)dT^2 - (1-\Phi^2)^{-1}dR^2 - r^2(\Phi(R))dw^2$$

Even if we posit $(R) = \sqrt{\frac{R_s}{R}}$, this metric cannot be interpreted as the Schwarzschild metric. Thus there is indeed a difference between the so-called Schwarzschild solution and empty space metric.

Also, we will assume that the Schwarzschild metric is a particular solution of the Einstein's equation.

To consider that the energy tensor is zero ($T_\mu^\nu = 0$) and that the solution represents a central mass does not have physical or mathematical basis. We demonstrate that the solution of Einstein's equation with second term allows us to find a general solution in a non-empty space-time, and that the particular solution is asymptotically the Schwarzschild solution (Chapter F).

Remarks on singularity

All Schwarzschild metrics have a physical singularity at the center of mass, that is, at $\mathcal{R}(r) = 0$. The Kretschmann scalar for these metrics is written using the Riemann tensor as:

$$R^{\mu\nu\sigma\sigma}R_{\mu\nu\sigma\sigma} = \frac{48R_s^{\ 2}}{\mathcal{R}^6(OM)}$$

This shows first that the singularity at $r = R_s$ is purely geometric and that the variable $\mathcal{R}$ is not the radial distance. Thus there is no singularity at OM=0, unless $\mathcal{R}(0) = 0$.

According to Equation 5.1, there always exists a metric solution of the Einstein equations such that $\mathcal{R}(0) \neq 0$.

$$\mathcal{R}(0) = (0^n + a^n)^{\frac{1}{n}}$$
$$\mathcal{R}(0) = 0 + a \qquad if\ a \neq 0$$
$$\mathcal{R}(0) = 0 + f(0), \qquad if\ \ f(0) \neq 0$$

According to current definitions, gravitational singularities in general relativity are locations in spacetime where the gravitational field becomes infinite. Some physicists, such as De Witt (1967), Dvali and Gomez (2014), Farnes (2018), Barrau et al. (2019), and philosophers, such as Saint-Ours (2011) propose that because the density of matter tends towards infinity in the singularity, the laws of behavior of spacetime are no longer compatible with classical physics. This has given rise to a multitude of theories, such as quantum gravity, loop quantum gravity, string theory applied to black holes, and space-time reversing.

Nevertheless, there still are debates and general disagreement between physicists, mathematicians, and philosophers regarding the definition of singularity (Saint-Ours, 2011; Fromholz et al., 2014).

Although it changes the local geometry, it is difficult to consider a singularity as a point that lies at a location in spacetime. Therefore, some physicists and philosophers cautiously refer singular space-time instead of singularities (Curiel and Bokulich, 2018). The most important definitions allude either to incomplete paths or to the idea of missing space in space-time. The concept is often called singular structure with pathological behavior.

Hawking and Penrose (1970) showed the existence of a singularitý during gravitational collapse. However, one should be careful about the meaning of this term. In their work, these authors do not prove the existence of a point where the geometry of space-time would become singular in the mathematical sense. What they have explained is the existence of incomplete-geodesics specific to time or light zone of space-time where the history of the objects that penetrate it stops after a finite time.

To illustrate this mathematical concept, let $t = Cte$ and $\theta = \frac{\pi}{2}$ in the Schwarzschild metric. We have:

$$ds^2 = -\left(1 - \frac{R_s}{\mathcal{R}}\right)^{-1} d\mathcal{R}^2 - \mathcal{R}^2 d\varphi^2$$

We seek a representation in a space such that:

$$ds^2 = -dz^2 - d\mathcal{R}^2 - \mathcal{R}^2 d\varphi^2$$

$$= -\left(1 + \left(\frac{dz}{d\mathcal{R}}\right)^2\right) d\mathcal{R}^2 - \mathcal{R}^2 d\varphi^2$$

By equating the coefficient in the two relations we have:

$$\left(1 + \left(\frac{dz}{d\mathcal{R}}\right)^2\right) = \frac{1}{1 - \frac{R_s}{\mathcal{R}}}$$

The visualization of the Schwarzschild space-time is obtained using of a 2D surface embedded in a space-time of dimension 3D. The Schwarzschild surface is thus visualized by the function $z(\mathcal{R})$, which can be written as follows for $t = Cte$ and $\theta = \frac{\pi}{2}$:

$$z(\mathcal{R}) = \int_0^{\mathcal{R}} du \sqrt{\frac{\frac{R_s}{u}}{1 - \frac{R_s}{u}}} = 2\sqrt{R_s(\mathcal{R} - R_s)} + z_0$$

That is,

$$z^2 = \left(4R_s(\mathcal{R} - R_s)\right) + 4z_0\sqrt{R_s(\mathcal{R} - R_s)} + z_0^{\ 2}$$

which is a Flamm paraboloid, with a throat circle at $\mathcal{R} = R_s$. This represents two 2D parabolic layers connected by a 1D throat circle with parameter $\mathcal{R} = R_s$ in 3D space.

For a very large, that is, far from the center of mass, we have:

$$\left(1 + \left(\frac{dz}{dr}\right)^2\right) \approx 1$$

$$z(r) = z_0$$

At infinity, this two dimensional surface embedded in a 3D dimensional space is a visualization of Schwarzschild's asymptotic space-time, that is, Minkowski's flat space-time (Figure 1).

The physical singularity at $r = 0$ does not exist as the throat circle at $r = R_s$ is the physical limit for any object plunging into the Schwarzschild metric. An object traveling from the upper sheet, which dives towards the center of mass following a parabolic geodesic, crosses the gorge, then slides toward the lower sheet, and disappears forever. The object cannot be trace any further. In this interpretation of the black hole, it is the zone of no return, where the history of the object stops.

The hypothesis, that for the object, the throat circle is impossible to cross because of the pulsation of the circle remains to be verified.

The aforementioned visualization of the Schwarzschild space is a sheet of dimension 2 with a hole, but the reality, with four dimensions, would rather be a hyperplane of dimension 4D with a throat sphere of dimension 3D.

An object falling from the upper sheet and having a slightly oblique trajectory will go around the throat one or more times before passing on the other sheet and disappearing forever. The throat sphere functions as a transition zone between our space-time and another space-time to be determined.

We call the space-time the upper sheet, the one in which we live E^+ and the lower space-time, the complementary space-time of the lower sheet E^-.

Then, a metric was defined on each space-time: ${g_{ik}}^+$ and ${g_{ik}}^-$ respectively and an Einstein tensor for each of the metrics, such that ${G_{ik}}^+$ and ${G_{ik}}^-$ satisfy the equations of relativity, respecting the continuity on the throat sphere between the two space-times. We have:

$${G_{ik}}^+ = {R_{ik}}^+ - \frac{1}{2}R^+ {g_{ik}}^+$$
$${G_{ik}}^- = {R_{ik}}^- - \frac{1}{2}R^- {g_{ik}}^-$$

This defines two metrics. In our space-time, with positive masses:

$$ds^{2+} = \left(1 - \frac{R_s}{\mathcal{R}}\right) d\xi^2 - \left(1 - \frac{R_s}{\mathcal{R}}\right)^{-1} d\mathcal{R}^2 - \mathcal{R}^2 dw^2 \qquad (11)$$

Meanwhile in the mirror space-time with negative masses:

$$ds^{2-} = \left(1 + \frac{R_s}{\mathcal{R}}\right) d\xi^2 - \left(1 + \frac{R_s}{\mathcal{R}}\right)^{-1} d\mathcal{R}^2 - \mathcal{R}^2 dw^2 \qquad (12)$$

This hypothesis of negative mass has been developed and is perfectly accepted in contemporary physics. The consequences on the concepts of energy, frequencies, and momentum in mechanics must be reviewed based on these new concepts.

Complementary space-times

Bondi (1957) and Rosen (1973) were the first to express a bi-metric representation of space-time. Subsequently, Sakharov (1980), Hossenfelder (2008), Hassan and Rosen (2012), Damour and Nikiforova (2019), and Petit et al. (2021) also developed their models on this basis. Boyle et al. (2018) published a cosmological model based on the existence of a mirror universe, populated by antimatter and going back in time, such in Sakharov's model. The scientific literature shows that the absence of negative mass matter in our known universe, supports the hypothesis of a bimetric of space-time that separates the known matter from this negative mass matter.

Many researchers, starting with Dirac, predicted intuitively that the mirror universe (at the antipodes of our universe), should be sought not in our space, but rather in a space where particles have masses and energies of opposite signs. Since the masses in our universe are positive, those in the mirror universe will be negative, according to Borissova and Rabounski (2009).

Both Newton's and Einstein's theories of gravitation predict non-intuitive behavior for negative masses. For two bodies of equal and opposite masses, the positive mass attracts the negative mass, but the latter repels the positive mass; the two masses pursue each other. The motion along the line joining the centers of mass of the considered bodies would thus be a motion with constant acceleration.

The throat sphere that separates our universe from the mirror universe prevents particles of negative and positive mass from coming into contact, thus prohibiting any particle annihilation, except in the case of quantum tunneling.

From the geometrical perspective, the throat sphere in 3D contains particles of null mass that move tangentially to the regions occupied by particles characterized by $(m > 0)$, or $(m < 0)$. The particles of zero mass, can interact both with the particles of our universe $(m > 0)$, and with those of the mirror universe, $(m < 0)$.

The throat sphere contains only energy in the form of

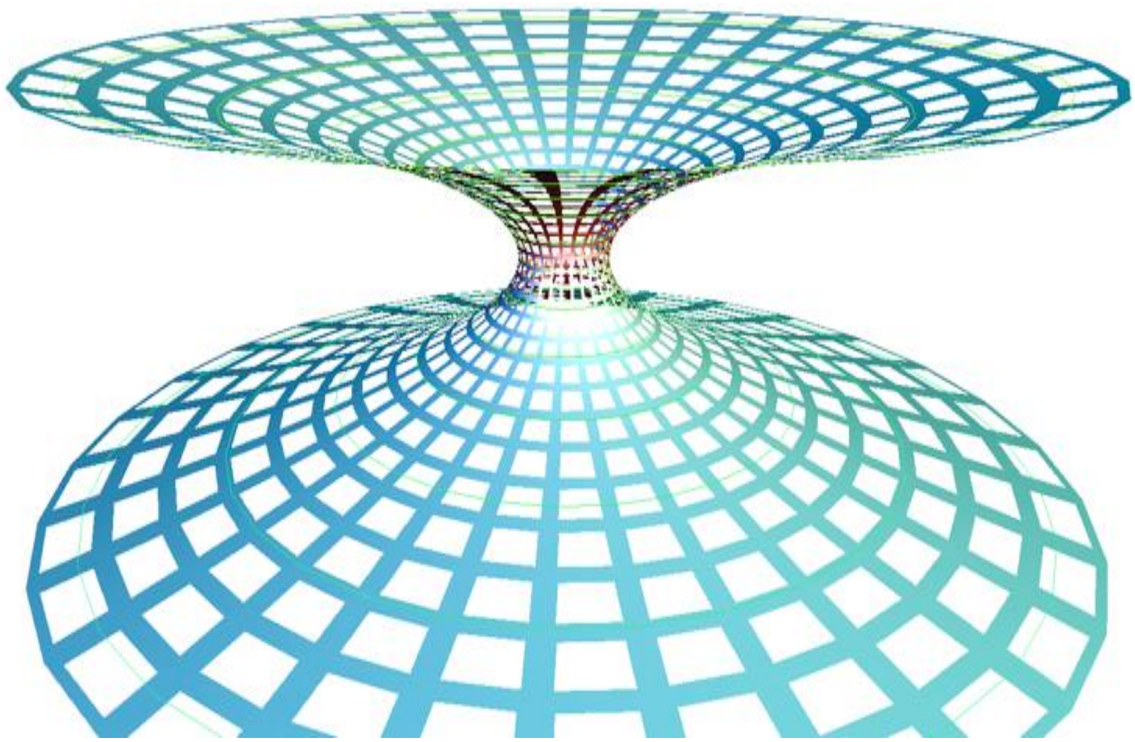

Figure 1. 3D visualization of the Schwarzschild hyper-surface.
Source: Grapher v 2.7 – Apple Inc.

elementary particles of zero mass, described by quantum fields. This energy contributes to generating the gravitational field.

Solution of Einstein's general equation in a nonempty space

Here, we present the mathematical solution of Einstein's equations in a non-empty space-time. We suppose that a massive, fluid, and homogeneous spherical object of radius a, generates a gravitational field inside and outside the object.

Let $M = \frac{4\pi}{3}(a^3 - R_s^3)$ be the mass of the homogeneous fluid contained in the sphere of radius a. We assume in that the Schwarzschild radius of the star is such that $R_s \ll a$.

The matter in the interior of the star is described by a fluid of energy-impulse tensor T_μ^ν as proposed by Schwarzschild (cf Haag, 1931; Brillouin, 1935), Antoci (1999), Barletta et al.,(2020).

The energy tensor is written as:

$$T_\mu^\nu = \rho U_\mu U^\nu - P_\mu^\nu$$

where $\rho(\mathcal{R})$ represents the proper density, P_μ^ν is the internal pressure tensor, and U^ν is the quadratic components of the generalized velocity.

Also, we assumed that the metric is of general form:

$$ds^2 = Ad\xi^2 - Bd\mathcal{R}^2 - \mathcal{R}^2 dw^2$$

with $A(\mathcal{R})$ and $B(\mathcal{R})$ two functions of variable $\mathcal{R}$.

We define the metric tensor as:

$$g_{\xi\xi} = A(\mathcal{R}), \quad g_{\mathcal{R}\mathcal{R}} = -B(\mathcal{R}), \quad g_{\theta\theta} = -\mathcal{R}^2,$$
$$g_{\varphi\varphi} = -\mathcal{R}^2 sin^2(\theta),$$

The pressure in the fluid is described by the equation of state $P_\mu^\nu = P_\mu^\nu(\rho, \mathcal{R})$, and because of homogeneity, we assume that:

$$P_\mu^0 = P_0^\nu = 0$$
$$P_\mu^\nu = 0 \quad (\mu \neq \nu)$$

$P_1^1 = P(\mathcal{R}); P_2^2 = P_3^3 = Q(\mathcal{R})$; the pressure tensor is then written as:

$$P_\mu^\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -P(\mathcal{R}) & 0 & 0 \\ 0 & 0 & -Q(\mathcal{R}) & 0 \\ 0 & 0 & 0 & -Q(\mathcal{R}) \end{pmatrix}$$

The pressure has a normal component $P(\mathcal{R})$ and a transverse component $Q(\mathcal{R})$.

The energy tensor in the star is then written:

$$T_\mu^\nu = \begin{pmatrix} \rho(\mathcal{R}) & 0 & 0 & 0 \\ 0 & -P(\mathcal{R}) & 0 & 0 \\ 0 & 0 & -Q(\mathcal{R}) & 0 \\ 0 & 0 & 0 & -Q(\mathcal{R}) \end{pmatrix}.$$

Considering the metric, we have:

$$T_{\mu\nu} = \begin{pmatrix} \rho A & 0 & 0 & 0 \\ 0 & PB & 0 & 0 \\ 0 & 0 & Q\mathcal{R}^2 & 0 \\ 0 & 0 & 0 & Q\mathcal{R}^2 sin^2(\theta) \end{pmatrix}$$

with $P(\mathcal{R}) = Q(\mathcal{R}) = \rho(\mathcal{R}) = 0$ located outside the fluid sphere; thus $T_\mu^\nu = 0$ located outside the star.

This hypothesis is compatible with the presence of a fluid with density $\rho(\mathcal{R})$. The structure of the gravitational field of the star is thus determined by four functions $A, B, P,$ and ρ which are functions of variable $\mathcal{R}$.

The general equations of Einstein are written as:

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = \aleph T^{\mu\nu}$$

where R is the curvature radius of the metric, $g^{\mu\nu}$ is the metric tensor, and $\aleph = 8\pi G$.

$g_{i\mu} g_{j\nu} G^{ij} = G_{\mu\nu}$, and $g_{\mu\nu}\left(R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu}\right) = -R$

$g_{\mu\nu} T^{\mu\nu} = T;\ \ then,\ \ R = -\aleph T$

Then Einstein equations are written as:

$$R_{\mu\nu} = \aleph\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right) \qquad (13)$$

After calculations, the Ricci tensor is:

$$\begin{cases} R_{\xi\xi} = \frac{A}{B}\left[\frac{A''}{2A} + \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{\mathcal{R}A}\right] \\ R_{\mathcal{R}\mathcal{R}} = \frac{A''}{2A} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{\mathcal{R}B} \\ R_{\theta\theta} = \frac{\mathcal{R}}{B}\left(\frac{A'}{A} - \frac{B'}{B} + \frac{2}{\mathcal{R}} - \frac{2B}{\mathcal{R}}\right) \\ R_{\varphi\varphi} = sin^2(\theta) R_{\theta\theta} \end{cases}$$

As the trace of the energy tensor is:
$T = T_i^i = \rho(\mathcal{R}) - P(\mathcal{R}) - 2Q(\mathcal{R})$ with $T_{\mathcal{R}\mathcal{R}} = P(\mathcal{R}) B(\mathcal{R})$;
$T_{\theta\theta} = Q(\mathcal{R})\mathcal{R}^2$

$T_{\varphi\varphi} = Q(\mathcal{R})\mathcal{R}^2 sin^2(\theta)$

$T_{\xi\xi} = \rho(\mathcal{R}) A(\mathcal{R})$.

Equation 13 can be expressed by the following system of four nonlinear differential equations:

$$\begin{cases} R_{\xi\xi} = \aleph A \frac{1}{2}(\rho + P + 2Q) \\ R_{\mathcal{R}\mathcal{R}} = \aleph B \frac{1}{2}(\rho + P - 2Q) \\ R_{\theta\theta} = \frac{1}{2}\aleph\mathcal{R}^2(P - \rho) \\ R_{\varphi\varphi} = sin^2(\theta)\aleph\mathcal{R}^2\frac{1}{2}(P - \rho) \end{cases}$$

That is

$$\begin{cases} \frac{A''}{2A} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{A'}{\mathcal{R}A} = \aleph B \frac{1}{2}(\rho + P + 2Q) & (a) \\ \frac{A''}{2A} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{\mathcal{R}B} = \aleph B \frac{1}{2}(2Q - P - \rho) & (b) \\ \left(\frac{A'}{A} - \frac{B'}{B}\right) - \frac{2B}{\mathcal{R}} + \frac{2}{\mathcal{R}} = \aleph\mathcal{R}B(P - \rho) & (c) \end{cases}$$

where $A(\mathcal{R}), B(\mathcal{R})$ are functions defining the desired interior and exterior metrics. These functions depend on the parameters of the energy tensor $P, Q, \rho, and\ \aleph$.

We calculate:

$$(a) - (b) \rightarrow \frac{1}{\mathcal{R}}\left(\frac{A'}{A} + \frac{B'}{B}\right) = \aleph B(\rho + P)$$

That is,

$$\left(\frac{A'}{A} + \frac{B'}{B}\right) = \aleph\mathcal{R}B(P + \rho) \qquad (14)$$

Equation (c) can be rewritten as:

$$\left(\frac{A'}{A} - \frac{B'}{B}\right) = \frac{2B - 2}{\mathcal{R}} + \aleph\mathcal{R}B(P - \rho) \qquad (15)$$

Differentiating and adding Equations 14 and 15, we obtain:

$$\frac{A'}{A} = \frac{B - 1}{\mathcal{R}} + \aleph\mathcal{R}BP(\mathcal{R}) \qquad (16)$$

$$\frac{B'}{B} = \frac{1 - B}{\mathcal{R}} + \aleph\mathcal{R}B\rho(\mathcal{R}) \qquad (17)$$

To solve these equations, in the first example, we assume some mathematical hypothesis such as the density is constant, or the pression is a simple explicit function of variable R. In the second example we show that the energy conditions in the fluid will determine the metric. The equations relating the pressure and density of the fluid determine these energy conditions.

Hypothesis F.1

We assume that the density $\rho = \rho_0$ is constant in throughout the fluid. This assumption is compatible with the physics of classical stars.

If the pressure in the fluid is a function of $\mathcal{R}$ (this is a mathematical hypothesis), then:

$$P(a) = P(R_s) = 0$$

It can be written:

$$P(\mathcal{R}) = P_0\left(1 - \frac{\mathcal{R}}{a}\right)\left(\frac{\mathcal{R}}{R_s} - 1\right), \qquad R_s \le \mathcal{R} \le a$$

The general solution of Equation 17 is:

$$B(\mathcal{R}) = \frac{3\mathcal{R}}{3C + 3\mathcal{R} - \aleph\rho_0\mathcal{R}^3} \quad (18)$$

For $\rho_0 = 0$, we find the particular solution of the metric outside the star:

$$B_0(\mathcal{R}) = \frac{\mathcal{R}}{\mathcal{R} + C} = \left(1 + \frac{C}{\mathcal{R}}\right)^{-1}$$

Since the density is constant in the star and given the general expression for $B(\mathcal{R})$, we write Equation 16:

$$\frac{A'}{A} = \frac{-1}{\mathcal{R}} + \frac{3(1 + \aleph\mathcal{R}^2 P(\mathcal{R}))}{3C + 3\mathcal{R} - \aleph\rho_0\mathcal{R}^3} \quad (19)$$

The solution of this equation is:

$$A(\mathcal{R}) = e^{\zeta(\mathcal{R})}\prod_{j=1}^{3}\frac{(\mathcal{R} - z_j)^{\beta_j}}{\mathcal{R}} \quad (20)$$

$$\zeta(\mathcal{R}) = \frac{3P_0}{2aR_s\rho_0}\mathcal{R}^2 - \frac{3P_0(R_s + a)}{\rho_0 a R_s}\mathcal{R}, \qquad R_s \le \mathcal{R} \le a$$

where $K(\mathcal{R}) = 3C + 3\mathcal{R} - \aleph\rho_0\mathcal{R}^3$ is a polynomial, with roots z_j, and β_j are coefficients depending on constants $C, a, R_s, and\ \aleph\rho_0$.

For $\rho_0 \to 0, P(\mathcal{R} \ge a) = 0$, the particular solution of the metric located outside the star is:

$$A_0(\mathcal{R}) = \frac{\mathcal{R} + C}{\mathcal{R}} = 1 + \frac{C}{\mathcal{R}}$$

Hypothesis F.2

If the pressure in the fluid is a function of $\mathcal{R}$ (this is a mathematical hypothesis) with $P(a) = P(R_s) = 0$. Then, it can be written:

$$P(\mathcal{R}) = P_0\left(\frac{a}{\mathcal{R}} - 1\right)\left(1 - \frac{R_s}{\mathcal{R}}\right)$$

The solutions for the Einstein equations are obtained in the form of Equations 18 and 20, assuming the continuity of pressure and limit conditions ($\mathcal{R} = a$) and ($\mathcal{R} = R_s$).

We have:

$$ds^2 = A(\mathcal{R})d\xi^2 - B(\mathcal{R})d\mathcal{R}^2 - \mathcal{R}^2 dw^2$$

with

$$A(\mathcal{R}) = \prod_{j=1}^{3}\frac{(\mathcal{R} - w_j)^{\gamma_j}}{\mathcal{R}}$$

$$B(\mathcal{R}) = \frac{3\mathcal{R}}{3C + 3\mathcal{R} - \aleph\rho_0\mathcal{R}^3}$$

Then

$$AB = \prod_{j}^{3}\frac{(\mathcal{R} - w_j)^{\gamma_j}}{(\mathcal{R} - z_j)^{\beta_j}}$$

For $\rho_0 \to 0, and\ P(\mathcal{R} \ge a) = 0$, the particular solution of the metric located outside the star is:

$$A_0B_0 = 1$$

$$B_0(\mathcal{R}) = \frac{\mathcal{R}}{\mathcal{R}+C} = \left(1+\frac{C}{\mathcal{R}}\right)^{-1}$$

PHYSICS OF PRESSURE AND DENSITY IN THE STAR

The divergence of the energy tensor verifies the following physical property because of Einstein's equations: The energy condition is:

$$\partial_i T^{ij} = 0$$

That is,

$$P' + \frac{(P+\rho)}{2}\frac{A'}{A} + \frac{(P-Q)}{\mathcal{R}} = 0$$

In a perfect fluid sphere, we have: $P = Q$

The previous equation reduces to the following expression:

$$P' + \frac{(P+\rho)}{2}\frac{A'}{A} = 0 \qquad (21)$$

$P(\mathcal{R})$ and $\rho(\mathcal{R})$, can be deduced from Equation 21,

$B(\mathcal{R})$ and $A(\mathcal{R})$ can be calculated.

Hypothesis G.1

For $P+\rho = Cte = K \neq 0$,

$$P' = \frac{-K}{2}\frac{A'}{A}$$

$$P(\mathcal{R}) = ln\left(\frac{1+\frac{C_0}{a}}{A(\mathcal{R})}\right)^{\frac{K}{2}} \; ; \; \rho(\mathcal{R}) = K - P(\mathcal{R})$$

The general solution of Equation 17 is:

$$B(\mathcal{R}) = \frac{1}{1-\frac{\aleph}{\mathcal{R}}M(\mathcal{R})}$$

with

$$M(\mathcal{R}) = \int_{R_s}^{\mathcal{R}} u^2 \rho(u)\, du$$

Equation 14 gives

$$\left(\frac{A'}{A}+\frac{B'}{B}\right) = \aleph K B \mathcal{R}$$

Then

$$Ln(AB) = \aleph K \int_{R_s}^{\mathcal{R}} uB(u)\, du$$

That is,

$$AB = exp\left(\aleph K \int_{R_s}^{\mathcal{R}} uB(u)\, du\right)$$

For $\rho(\mathcal{R} \geq a) = 0$ *and* $P(\mathcal{R} \geq a) = 0$, the particular solution of the metric located outside the star is:
$A_0 B_0 = 1$

$$B_0(\mathcal{R}) = \frac{\mathcal{R}}{\mathcal{R}+C} = \left(1+\frac{C}{\mathcal{R}}\right)^{-1}$$

Hypothesis G.2

The density is constant: $\rho = \rho_0$

With $P(a) = 0$, *and* $\rho = \rho_0$, Equation 21 gives:

$$P + \rho_0 = \rho_0 \frac{\sqrt{1+\frac{C_0}{a}}}{\sqrt{A(\mathcal{R})}}$$

as $A(a) = A_0(a) = 1 + \frac{C_0}{a}$.

Condition $\sqrt{1+\frac{C_0}{a}} > \sqrt{A(R_s)} > 0$ will decide on the critical conditions for density ρ_0 (Haag, 1931).

Let calculate the general expression of the function $A(\mathcal{R})$.

Equation 14 can be written as:

$$\left(\frac{A'}{A}+\frac{B'}{B}\right) = \aleph B \mathcal{R} \rho_0 \frac{\sqrt{1+\frac{C_0}{a}}}{\sqrt{A(\mathcal{R})}} \qquad (22)$$

We deduce:

$$\frac{A'}{\sqrt{A}} = -\frac{B'\sqrt{A}}{B} + \aleph B \mathcal{R} \rho_0 \sqrt{1+\frac{C_0}{a}}$$

Let's $U = A^{1/2}$ then $2U' = A' A^{-1/2}$

then we have:

$$2U' = -U\frac{B'}{B} + \aleph B\mathcal{R}\rho_0\sqrt{1+\frac{C_0}{a}} \quad (23)$$

Case 1: letting $C = 0$ in Equation 18, and substituting the expression for B in Equation 23, we obtain a simple equation, we can easily solve for the Schwarzschild solution.

$$2U' = -\frac{\frac{2\aleph\rho_0}{3}\mathcal{R}}{1-\frac{\aleph\rho_0}{3}\mathcal{R}^2}U + \frac{\aleph\rho_0\mathcal{R}}{1-\frac{\aleph\rho_0}{3}\mathcal{R}^2}\sqrt{1+\frac{C_0}{a}}$$

whose solution is to the nearest constant:

$$U = \frac{-1}{2}\sqrt{1-\frac{\aleph\rho_0}{3}\mathcal{R}^2} + \frac{3}{2}\sqrt{1+\frac{C_0}{a}}$$

This solution was found by Schwarzschild in 1916 , with

$$2m = \frac{\aleph\rho_0}{3}a^3 \; and \; C_0 = -2m$$

$$A(\mathcal{R}) = \left(\frac{3}{2}\sqrt{1-\frac{2m}{a}} - \frac{1}{2}\sqrt{1-\frac{2m}{a^3}\mathcal{R}^2}\right)^2$$

Case 2: $C \neq 0$. The inhomogeneous equation to obtain function $A(\mathcal{R})$ is difficult to solve, and involves hyper elliptic integrals.

The general solution of Equation 23, is

$$U(\mathcal{R}) = \left(\frac{\aleph\rho_0\sqrt{1+\frac{C_0}{a}}}{2\sqrt{B}}\int_0^{\mathcal{R}} tB^{3/2}dt + \frac{1}{\sqrt{B}}\right)$$

and

$$A(\mathcal{R}) = \frac{1}{B(\mathcal{R})}\left(\frac{\aleph\rho_0\sqrt{1+\frac{C_0}{a}}}{2}\int_0^{\mathcal{R}} tB(t)^{3/2}dt + 1\right)^2$$

Remark

If $\rho_0 \to 0$, then:

$$A_0(\mathcal{R}) = \frac{1}{B_0(\mathcal{R})}$$

This is the particular solution of the Einstein equations out of the star.

Hypothesis G.3

We assume that the star is a special fluid, where $P + \rho = 0$ and $P - Q \neq 0$ are the energy conditions

If $\rho > 0$, then $P < 0$; that is, energy is negative, if $\rho < 0$, then $P > 0$; that is, mass is negative. Then, the energy conditions can be written as:

$$P' + \frac{(P-Q)}{\mathcal{R}} = 0$$

The solution of Equation 14 is

$$AB = 1$$

Assume that $Q = 2P_0\mathcal{R}$. Then

$$P(\mathcal{R}) = P_0\left(\mathcal{R} - \frac{a^2}{\mathcal{R}}\right) and \; P(a) = 0$$

Equation 17 gives

$$\frac{B'}{B} = \frac{1-B}{\mathcal{R}} - \aleph BP_0(\mathcal{R}^2 - a^2)$$

The interior solution is a particular De Sitter-Schwarzschild metric:

$$B(\mathcal{R}) = \frac{4\mathcal{R}}{4C + 4\mathcal{R} - 2\aleph P_0 a^2\mathcal{R}^2 + \aleph P_0\mathcal{R}^4}$$

$$A(\mathcal{R}) = \frac{1}{B(\mathcal{R})}$$

Meanwhile the exterior solution is:

$$A_0(\mathcal{R}) = \frac{1}{B_0(\mathcal{R})} = 1 + \frac{C}{\mathcal{R}}$$

Conclusion

After carefully reading the second article of Schwarzschild

(1916), and through reasonable mathematical assumptions, we showed that the physical interpretation of the variable r in the Schwarzschild metric is not a radial distance for a point located close to the star.

Also, we addressed the problem of the singularity in this metric by providing an explanation of gravitational collapse, as discussed by Hawking and Penrose (1970).

Starting from the principle of non-singularity, we constructed a metric from the Einstein tensor in a vacuum. This metric, asymptotic to a plane metric far away from the star and tangent to a throat sphere near the center, extends to a mirror metric. In the mirror metric, the masses are negative, and time is reversed.

Subsequently, we explained the general solution of the Einstein equation, including second term, considering various assumptions about the pressure and density inside the fluid ball representing the star.

If $\boldsymbol{P} + \boldsymbol{\rho} = \boldsymbol{K} \neq \mathbf{0}$, then

$$A(\mathcal{R})B(\mathcal{R}) = exp\left(\aleph K \int_{R_s}^{\mathcal{R}} uB(u)\, du\right)$$

If $\boldsymbol{P} + \boldsymbol{\rho} = \mathbf{0}$, then

$$A(\mathcal{R})B(\mathcal{R}) = 1$$

with

$$B(\mathcal{R}) = \frac{4\mathcal{R}}{4C + 4\mathcal{R} - 2\aleph P_0 a^2 \mathcal{R}^2 + \aleph P_0 \mathcal{R}^4}$$

This is a particular De Sitter-Schwarzschild's metric in a special fluid.

If $\boldsymbol{\rho} = \boldsymbol{\rho_0}$, then

$$\boldsymbol{B}(\boldsymbol{\mathcal{R}}) = \frac{\mathbf{3}\boldsymbol{\mathcal{R}}}{\mathbf{3}\boldsymbol{C} + \mathbf{3}\boldsymbol{\mathcal{R}} - \aleph\boldsymbol{\rho_0}\boldsymbol{\mathcal{R}}^{\mathbf{3}}}$$

$$\boldsymbol{A}(\boldsymbol{\mathcal{R}})\boldsymbol{B}(\boldsymbol{\mathcal{R}}) = \left(\frac{\aleph\boldsymbol{\rho_0}\sqrt{\mathbf{1} + \frac{\boldsymbol{C_0}}{\boldsymbol{a}}}}{\mathbf{2}} \int_{\mathbf{0}}^{\boldsymbol{\mathcal{R}}} \boldsymbol{tB}(\boldsymbol{t})^{\mathbf{3/2}}\boldsymbol{dt} + \mathbf{1}\right)^{\mathbf{2}}$$

This is the hyper-elliptic interior solution that generalizes the interior Schwarzschild's solution.

In the next paper, we will make new physical assumptions about the 3D throat sphere. We will examine the physics and nature of zero mass and relativistic particles, which can pass through that throat sphere.

The topology of the $\boldsymbol{S}^{\mathbf{3}}$ sphere embedded in a 4D space is not easily visualizable. Indeed, the stereographic projection of the $\boldsymbol{S}^{\mathbf{3}}$ sphere in our space is the entire $\mathbb{R}^3$ space plus a point at infinity according to our viewpoint. The gorge sphere is then isomorphic to $\mathbb{R}^3$, in which time has been stopped. The sphere $\boldsymbol{S}^{\mathbf{3}}$ forms the junction of our paraboloid space-time, with the mirror space-time where matter with property of negative mass exists.

CONFLICT OF INTERESTS

The authors have not declared any conflict of interests.

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