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Full Length Research Paper

Completed Beltrami-Michell formulation in polar coordinates

Surya N. Patnaik [1] and Dale A. Hopkins[2]

[1]Ohio Aerospace Institute, Brook Park, Ohio 44142
[2]NASA Glenn Research Center, Cleveland, Ohio 44135

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A set of conditions had not been formulated on the boundary of an elastic continuum since the time of Saint-Venant. This limitation prevented the formulation of a direct stress calculation method in elasticity for a continuum with a displacement boundary condition. The missed condition, referred to as the boundary compatibility condition, is now formulated in polar coordinates. The augmentation of the new condition completes the Beltrami-Michell formulation in polar coordinates. The completed formulation that includes equilibrium equations and a compatibility condition in the field as well as the traction and boundary compatibility condition is derived from the stationary condition of the variation functional of the integrated force method. The new method is illustrated by solving an example of a mixed boundary value problem for mechanical as well as thermal loads.

Key words: Elasticity, Boundary, Compatibility, Variational, Derivation.

INTRODUCTION

The stress-strain law, the equilibrium equation (EE), and the compatibility condition (CC) are the three fundamental relations in elasticity. The material law was formulated in the mid-seventeenth century by Hooke (1635–1703). The equilibrium equation or the stress formulation is credited to Cauchy (1789–1857). Saint-Venant (1797–1886) developed the CC, or the strain formulation. It is a general belief that the fundamental elasticity relations were known for over a century. The thrust, therefore, was to develop approximate solution techniques because a closed-form solution cannot be generated for the vast majority of the solid mechanics problems. Such techniques included Airy's method (Love, 1927), Ritz's method (Ritz, 1909), the moment distribution technique (Cross, 1932), Kani's method (Thadani, 1964), the finite element technique (Gallagher, 1974) and others.

It is surprising that the strain formulation was not known on the boundary of an elastic continuum, even though Cauchy's stress formulation explicitly contained the boundary conditions also known as the traction conditions. Because of this deficiency, problems with displacement boundary conditions could not be solved using the direct stress calculation method, popularly referred to as the Beltrami-Michell formulation (BMF) (Sokolnikoff, 1956). The strain formulation that was missed on an elastic boundary is referred to as the boundary compatibility condition (BCC). The BCC has been derived. Now the stress and strain formulations are parallel in form; each contains field equations and boundary condi-tions. Earlier, we derived the BCC for two-dimensional (Patnaik, 1986) and three-dimensional (Patnaik et al., 2004) problems in elasticity in Cartesian coordinates. The BMF was completed by adding the new BCC to the classical method. The completed Beltrami-Michell formu-lation (CBMF) can be used to solve displacement as well as mixed boundary value problems in elasticity. The CBMF stress formulation is as versatile as the Navier displacement method, yet its equation structure is simpler. Solutions to plate and shell problems via the CBMF are discussed (Patnaik and Nagaraj, 1987; Patnaik and Satish, 1990; Patnaik et al., 1996).

A conservative elastician, believing the set of existing equations to be sufficient, may be reluctant to accept the new BCCs. However, it should be realized that some for-

*Corresponding author. E-mail: surya.n.patnaik@nasa.gov.

[1]Principal Scientist, 22800 Cedar Point Road.

mulae and equations of the solid mechanics discipline were not completed in the first attempt, but were perfected eventually. For example, perfecting the flexure formulae required a century between Galileo, Bernoulli, and Coulomb. Saint-Venant completed the shear stress formula that was initiated by Navier. Cauchy formulated the stress equilibrium equation that was also developed by Navier in terms of displacement, but it contained only a single material constant instead of two.

The formulation of the BCC in polar coordinates is the primary contribution of this paper. The use of the new condition is also illustrated through the solution of a mixed boundary value problem for thermomechanical loads. The CBMF containing the BCC is obtained from the stationary condition of the variational functional π_s of the integrated force method (IFM) (Patnaik and Hopkins, 2004). The variational calculation is performed in two distinct steps:

(1) The terms of the functional π_s are transformed to obtain integrands, whose coefficients are either displacement variables, stress function, or reactions.
(2) The stationary condition (Washizu, 1968) of the functional $\delta\pi_s$ with respect to displacement, stress function, and reaction yield all the expressions of the CBMF.

The BCC is the coefficient of the variational stress function in the line integral term. Variational calculus in polar coordinates is more difficult than that in the Cartesian system (Patnaik, 1986) because the coefficients of the terms in the functional are functions of the r-coordinate. Also the Jacobian ($J = r$) has to be used. A nonvariational approach or carelessness can easily miss an expression because of the tensorial nature of stress and strain. The accuracy of CBMF derivation is essential because solution of elasticity problems in polar coordinates is very popular. Solutions have been obtained by Novozhilov (1961) and Timpe (1924) for a number of problems in polar coordinates, especially for symmetrically loaded circular domain. Many existing elasticity solutions can be verified by back-substituting into the CBMF. To demonstrate the use of the new condition, two mixed boundary value problems are solved. The first example is for mechanical load, while the second is for thermal load.

This paper is organized as follows: First, a variational derivation is given for the CBMF. Green's theorem is used for a quick validation of the new boundary condition. Then the CBMF is used to solve a problem with stress and displacement boundary conditions. This is followed by discussion and conclusions. Appendix A is a listing of symbols and acronyms found in this paper. Appendix B presents the major steps of the variational derivation, which can be used by the reader to verify the BCC. Appendix C presents the solution of a structure using the IFM, which is the discrete analogue of the CBMF.

Completed Beltrami-Michell Formulation in polar coordinates

The CBMF in polar coordinates is obtained from the stationary condition of the variational functional (Patnaik, 1986) of the IFM. The functional π_s has three terms (Equation 1a). The first term $A(\sigma, u)$ represents the strain energy, expressed in terms of stress σ and displacement u. The second term $B(\varepsilon, \varphi)$ is the complementary strain energy written in terms of the strain ε and the stress function φ. The third term W is the potential of the work done. Basic steps of the derivation are given in Appendix B. The functional is transformed into integrals with integrands whose coefficients are displacement, stress function, or reaction variables. Symbolically it can be represented as follows:

$$\pi_s = A + B - W \qquad (1a)$$

$$\pi_s = \iint_D \left[(\text{field EE})\{u\} + (\text{field CC})\varphi\right] ds + \int_{1_1} \left[(\text{boundary EE})\{u\}\right] d1 + \int_{1} \left[(\text{boundary CC})\varphi\right] d1 + \int_{1_2} (\text{continuity condition})\{\text{reaction}\}\, d1 = 0 \qquad (1b)$$

Where the two displacement components $\begin{Bmatrix} u \\ v \end{Bmatrix}$ are represented as $\{u\}$. Likewise, $\{\text{reaction}\}$ represents the two reactions $\begin{Bmatrix} \text{reaction along } r \\ \text{reaction along } \theta \end{Bmatrix}$. Also, D is the plate domain; 1_1 and 1_2 are boundary segments where traction is pre-scribed and reaction is induced, respectively; and 1 is the line segment where stress is indeterminate. The stationary condition of the functional in Equation (1b) with respect to displacement, stress function, and reaction can be represented by the following symbolic expression:

$$\delta\pi_s = \iint_D \left[(\text{field EE})\,\delta\{u\} + (\text{field CC})\,\delta\varphi\right] ds + \int_{1_1} \left[(\text{boundary EE})\,\delta\{u\}\right] d1 + \int_{1} \left[(\text{boundary CC})\,\delta\varphi\right] d1 + \int_{1_2} (\text{continuity condition})\,\delta\{\text{reaction}\}\, d1 = 0 \qquad (1c)$$

The field EE and field CC are the coefficients of the variational displacement and stress function, respectively, in the surface integral terms of the functional (see also app. B). Likewise the boundary EE and boundary CC are the coefficients of the variational displacement and stress function, respectively, in the line integral terms. The continuity conditions are the coefficients of the variational reactions. The field equations and boundary conditions of the CBMF recovered from the stationary condition of the variational functional in Equation (1c) are as follows:

Equilibrium equations

Field:

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau}{\partial \theta} + \frac{(\sigma_r - \sigma_\theta)}{r} + b_r = 0 \qquad (2a)$$

$$\frac{\partial \tau}{\partial r}+\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta}+\frac{2\tau}{r}+b_\theta=0 \qquad (2b)$$

Boundary:

$$n_r\sigma_r+n_\theta\tau=\bar{P}_r \qquad (3a)$$

$$n_r\tau+n_\theta\sigma_\theta=\bar{P}_\theta \qquad (3b)$$

Compatibility conditions

Field:

$$\left(\frac{1}{E}\right)\left[\frac{1}{r^2}\frac{\partial^2\sigma_r}{\partial\theta^2}-\upsilon\frac{\partial^2\sigma_r}{\partial r^2}-\frac{(1+2\upsilon)}{r}\frac{\partial\sigma_r}{\partial r}+\frac{\partial^2\sigma_\theta}{\partial r^2}-\frac{\upsilon}{r^2}\frac{\partial^2\sigma_\theta}{\partial\theta^2}+\frac{(1+2\upsilon)}{r}\frac{\partial\sigma_\theta}{\partial r}-\frac{(1+\upsilon)}{r}\frac{\partial^2\tau}{\partial r\partial\theta}-\frac{(1+\upsilon)}{r^2}\frac{\partial\tau}{\partial r}\right]=0 \qquad (4)$$

Boundary:

$$\left[\frac{\partial}{\partial r}(\sigma_\theta-\upsilon\sigma_r)-\frac{(1+\upsilon)}{r}\left(\frac{\partial\tau}{\partial\theta}+\sigma_\theta-\sigma_r\right)\right]n_r+\left[(1+\upsilon)\frac{\partial\tau}{\partial r}-(1+\upsilon)\frac{\partial\sigma_\theta}{\partial\theta}+\frac{2(1+\upsilon)}{r}\tau\right]n_\theta=0 \qquad (5)$$

The kinematics boundary conditions are

$$u-\bar{u}=0 \qquad (6a)$$

$$\upsilon-\bar{\upsilon}=0 \qquad (6b)$$

and

r and θ are the polar coordinates
σ_r, σ_θ, and τ are the stress components
ε_r, ε_θ, and γ are the strain components
u and υ are the displacements
b_r and b_θ are body forces
$\bar{P}_r$ and $\bar{P}_\theta$ are tractions applied along boundary segment 1_1
$\bar{u}$ and $\bar{\upsilon}$ are initial displacements along boundary segment 1_2
φ is the stress function

The CBMF in Equations (2) to (6) contains the following:

(1) The stress formulation of Cauchy as coefficients of variational displacements δu and $\delta\upsilon$ in the surface integral. It consists of two EEs in the field (Equations (2a) and (2b)) and two on the boundary [Equations (3a) and (3b)] that are popularly known as the traction conditions. The stress formulation has two distinct components: the field equations and the boundary conditions.

(2) Strain formulation as the coefficient of variational stress function$\delta\varphi$. It is written in terms of stress for an isotropic material with Young's modulus E and Poisson's ratio υ. It consists of the single field CC [Equation (4)], along with one new BCC [Equation (5)]. Saint-Venant, unlike Cauchy, formulated only the field condition. He missed the boundary conditions that we have completed. Now, both the stress and strain formulations are consistent, containing the field equations [Equations. (2a), (2b), and (4)) as well as the boundary conditions (Equations (3a), (3b), and (5)]. In Appendix B, the strain formulation is derived in terms of the strains.

(3) Displacement boundary conditions. Two kinematics displacement boundary conditions (Equations (6a) and (6b)) are obtained as coefficients of the variational reactions. A rigorous derivation of the continuity condition is more difficult than the stress and the strain formulations, which are straightforward.

The three-component stress tensor (σ_r, σ_θ, and τ) is indeterminate in the field and on the boundary because the state of equilibrium provides only two equations. To achieve determinacy of the stress state, we must add one CC in the field as well as one on the boundary. Saint-Venant has given us the field CC. We have formulated the BCC. For the derivation of elasticity equation, the variational technique is an elegant method because of the tensorial nature of stress and strain. A nonvariational approach may miss an equation.

The CC should be imposed only when the domain is indeterminate, whether it is the field or the boundary. The CC has no relevance for a determinate domain or a determinate boundary. A plane stress problem is one degree indeterminate in the field because there are three stresses and two displacements. It has one field CC. A BCC should not be imposed on a free or a determinate boundary, where at least one stress component is zero. A clamped boundary is typically indeterminate; thus, one BCC is imposed.

The solution of an elasticity problem using the CBMF has two distinct steps:

(1) The stress state is calculated first using the Equations (1) to (5) for an elastic continuum with stress and displacement boundary conditions. The displacement boundary conditions [Equation (6)] are not used, but the BCC is used.
(2) Displacements are back-calculated by integrating the strain field. The kinematics boundary conditions given by Equation (6) are used to evaluate the constants of integration in the displacement function.

The CBMF recognizes that displacement does not induce stress. The derivative of displacement, which becomes the strain that induces stress, is accounted for through the BCC. Displacement conditions are used to eliminate rigid body movement as explained in step (2).

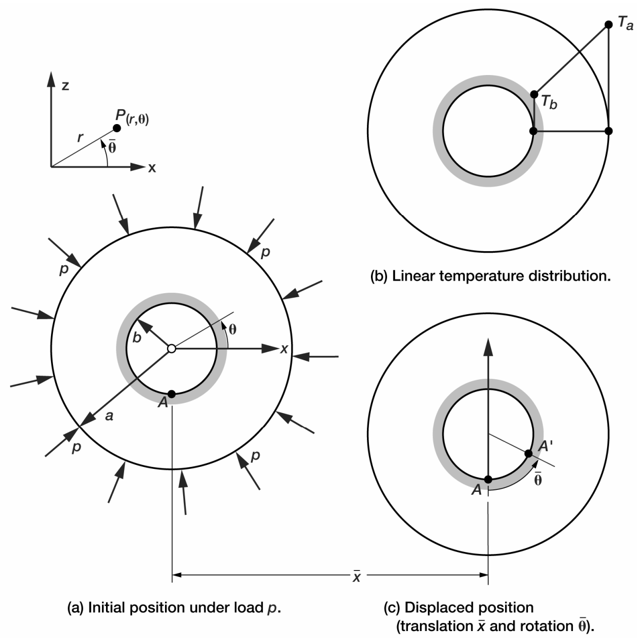

Figure 1. Annular plate.

The BCC expressed in strain, stress, and displacement is as follows:

Expressed in strain:

$$\left[\frac{\varepsilon_r}{r}-\frac{\partial}{r\partial r}(r\varepsilon_\theta)+\frac{\partial\gamma}{2r\partial\theta}\right]n_r+\left[\frac{\partial\varepsilon_r}{r\partial\theta}-\frac{\partial\gamma}{2\partial r}-\frac{\gamma}{r}\right]n_\theta=0 \quad (7a)$$

Written in stress for an isotropic material with Poisson's ratio υ:

$$\left[\frac{\partial}{\partial r}(\sigma_\theta-\upsilon\sigma_r)-\frac{(1+\upsilon)}{r}\left(\frac{\partial\tau}{\partial\theta}+\sigma_\theta-\sigma_r\right)\right]n_r+\left[(1+\upsilon)\frac{\partial\tau}{\partial r}-(1+\upsilon)\frac{\partial\sigma_\theta}{\partial\theta}+\frac{2(1+\upsilon)}{r}\tau\right]n_\theta=0 \quad (7b)$$

In terms of displacements u and v:

$$\left[\frac{\partial^2 u}{\partial\theta^2}-r\frac{\partial^2\upsilon}{\partial r\partial\theta}-\frac{\partial\upsilon}{\partial\theta}\right]n_r+\left[r\frac{\partial^2 u}{\partial r\partial\theta}+\upsilon-r^2\frac{\partial^2\upsilon}{\partial r^2}-r\frac{\partial\upsilon}{\partial r}-r\frac{\partial u}{\partial\theta}\right]n_\theta=0 \quad (7c)$$

The BCC expression contains either all three strains, or three stresses, or it contains derivatives of two displacement components. The BCC is not a continuity condition in displacement, stress, or strain; however, it is a function of the variables. As such, the BCC is expressed in the derivatives of stress, strain, and displacement, but it is not a component of rotation. The BCC is an independent condition. It forms a new elasticity expression that was missed since the time of Saint-Venant. The field CC is a second-order differential equation, while the boundary counterpart is a first-order equation. This characteristic is applicable to the stress formulation. The field EEs is first-order equations, while the boundary (or traction) conditions are algebraic equations.

Annular plate subjected to thermomechanical load

We will now illustrate the CBMF calculation strategy through the solution of a radially symmetrical annular plate with mixed boundary conditions for mechanical and thermal loads. Consider a plate made of an isotropic material with Young's modulus E and Poisson's ratio υ. It has thickness h (considered unity) with outer and inner radii of a and b, respectively, as shown in Figure 1(a). The mechanical load case consists of a uniform radial load of intensity p applied at the outer boundary $r = a$. The inner boundary is restrained: $u = 0$ at $r = b$. The CBMF for the mixed boundary value problem is generated from a special case of the variational functional. It is obtained using the condition of symmetry or by setting the shear stress τ and transverse displacement υ to zero (τ = 0, υ = 0) as well as by neglecting variation with respect to the angle θ: $\left(\frac{\partial f}{\partial\theta}=0\right)$. Also, a simpler stress function ψ is used.

$$\sigma_r=\psi-U \quad (8a)$$

$$\sigma_\theta=r\left(\frac{d\psi}{dr}+\frac{\psi}{r}\right)-U \quad (8b)$$

Where $U = 0$ for this example.

Solution for mechanical load only

The equations of CBMF for a symmetrical annular plate subjected to a uniform mechanical load of intensity p are listed in Equations. (9) to (11):

Equilibrium equations

Field:

$$\frac{\partial\sigma_r}{\partial r}+\frac{(\sigma_r-\sigma_\theta)}{r}=0 \quad (9a)$$

Boundary:

$$\sigma_r=p \text{ at } r=a \quad (9b)$$

Compatibility conditions

Field:

$$\frac{\partial}{\partial r}(\sigma_\theta-\upsilon\sigma_r)+\frac{(1+\upsilon)}{r}(\sigma_\theta-\sigma_r)=0 \quad (10a)$$

Boundary:

$$\sigma_\theta-\upsilon\sigma_r=0 \text{ at } r=b \quad (10b)$$

Where the displacement boundary condition is

$$u=0 \text{ at } r=b \quad (11)$$

There are two stresses σ_r and σ_θ and one displacement u. In the field, there is one EE [Equation (9a)] and one CC [Equation (10a)]. On the boundary there is one traction condition [Equation. (9b)] and a single BCC [Equation (10b)]. Also, at the inner boundary there is one kinematics condition [Equation (11)]. Solution to Equations (9a), (9b), (10a), and (10b) yield the stress response. The single displacement u is back-calculated by integrating the strain and evaluating the integration constant using the kinematics condition [Equation (11)].

The EE and CC in the field are arranged to obtain the following two simpler uncoupled equations:

$$\frac{d}{dr}(\sigma_r + \sigma_\theta) = 0 \quad (12a)$$

$$\frac{d\sigma_r}{dr} + \frac{(\sigma_r - \sigma_\theta)}{r} = 0 \quad (12b)$$

Integration of the first Equation (12a) yields the sum of the stresses $\sigma_r + \sigma_\theta$ to be a constant. The second Equation (12b) is uncoupled and solved. The two constants in the stress variables are determined from the traction condition [Equation (9b)] and the new BCC [Equation (10b)]. The stress solution follows:

$$\sigma_r = \frac{\left[(1+\upsilon)r^2 + (1-\upsilon)b^2\right]}{(1+\upsilon)a^2 + (1-\upsilon)b^2}\left(\frac{a^2}{r^2}\right)p \quad (13a)$$

$$\sigma_\theta = \frac{\left[(1+\upsilon)r^2 - (1-\upsilon)b^2\right]}{(1+\upsilon)a^2 + (1-\upsilon)b^2}\left(\frac{a^2}{r^2}\right)p \quad (13b)$$

$$\sigma_r + \sigma_\theta = \frac{2p}{1 + \frac{(1-\upsilon)}{(1+\upsilon)}\left(\frac{b}{a}\right)^2} \quad (13c)$$

$$\sigma_r + \sigma_\theta = \frac{2p}{1 + 0.54\left(\frac{b}{a}\right)^2} \quad \text{for } \upsilon = 0.3 \quad (13d)$$

For a plate with $E = 30\times10^6$ psi, $\upsilon = 0.3$, $p = 1$ psi, $a = 20$ in., and $b = 10$ in., the stresses at the outer boundary are:

$\sigma_r = 1.0$, while $\sigma_\theta = 0.763$ psi and $\sigma_r + \sigma_\theta = 1.763$ psi. At the inner boundary these are $\sigma_r = 1.356$ psi, $\sigma_\theta = 0.407$ psi, and $\sigma_r + \sigma_\theta = 1.763$ psi. The sum of the stresses $\sigma_r + \sigma_\theta = 1.763$ psi is independent of the r coordinate of the plate. The BCC $\sigma_\theta - \upsilon\sigma_r = 0.62 - \frac{62}{r^2}$ has an inverse quadratic variation with respect to the radius, with a minimum value of zero at the restrained boundary ($r = 10$ in.) and a maximum value of 0.46 psi at the outer free boundary ($r = 20$ in.). The BCC should not be imposed on the free boundary at $r = a = 20$ in. The stress state in the mixed boundary value problem is obtained without any use of the prescribed displacement boundary condition. The displacement function u is obtained following the standard elasticity solution strategy. Stress is changed into strain using Hooke's law. It is integrated to obtain the displacement function that contained a constant c.

$$u = \frac{\left(1-\upsilon^2\right)\left(r^2 - b^2\right)}{E\left(1+\upsilon\right)a^2 + \left(1-\upsilon\right)b^2}\left(\frac{a^2}{r}\right)p + c \quad (14a)$$

The constant c does not affect the stress state. For the problem, the constant is calculated to be zero ($c = 0$) from the homogeneous kinematics boundary condition $u = 0$ at $r = b$ [Equation (11)]:

$$u = \frac{\left(1-\upsilon^2\right)\left(r^2 - b^2\right)}{E\left(1+\upsilon\right)a^2 + \left(1-\upsilon\right)b^2}\left(\frac{a^2}{r}\right)p \quad (14b)$$

The values of displacements are $u = 0$ at the inner boundary, $r = b$, and $u = 3.5\times10^{-7}$ in. at the outer boundary, $r = a$.

The CBMF produced the solution to the mixed boundary value problem in two steps: First the stress state was calculated using the field EE and CC, along with the traction condition as well as the BCC. Then the displacement function was back-calculated. Solution to the mixed boundary value problem could not have been obtained by the classical BMF stress formulation. Solution to the mixed boundary value problem is not available in standard textbooks in elasticity (Sokolnikoff, 1956; Timoshenko and Goodier, 1969; Saada, 1983)

Solution for thermal load only

The CBMF solution for the annular plate is obtained for a temperature distribution given as

$$T = T_b + \frac{(T_a - T_b)}{(a-b)}(r-b) \quad (15)$$

The temperature distribution is shown in Figure 1(b). It has a linear variation with values T_a and T_b at $r = a$ and $r = b$, respectively. The coefficient of thermal expansion is α. The CBMF equations for the annular plate subjected to a thermal load are given below:

Equilibrium equations

Field:

$$\frac{\partial\sigma_r}{\partial r} + \frac{(\sigma_r - \sigma_\theta)}{r} = 0 \quad (16a)$$

Traction on boundary:

$$\sigma_r = 0 \ \text{ at } \ r = a \quad (16b)$$

Compatibility conditions

Field:

$$\frac{\partial}{\partial r}(\sigma_\theta - \upsilon\sigma_r) + \frac{(1+\upsilon)}{r}(\sigma_\theta - \sigma_r) = -\alpha E\frac{dT}{dr} \tag{17a}$$

Boundary:

$$\sigma_\theta - \upsilon\sigma_r = -\alpha ET \quad \text{at} \quad r = b \tag{17b}$$

where the displacement boundary condition is

$$u = 0 \quad \text{at} \quad r = b \tag{18}$$

Both the field and the boundary CCs [Equations (10a) and (10b)] for the mechanical load are modified for the temperature load to obtain Equations (17a) and (17b). The field EE is not changed. The mechanical load is set to zero (*p* = 0) in the traction Equation (16b). The EE and CC in the field are rearranged to obtain the following two simpler working equations:

$$\frac{d}{dr}(\sigma_r + \sigma_\theta) = -\alpha E\frac{dT}{dr} \tag{19a}$$

$$\frac{d\sigma_r}{dr} + \frac{(\sigma_r - \sigma_\theta)}{r} = 0 \tag{19b}$$

The field equations are solved for the boundary conditions to obtain the response, consisting of σ_r, σ_θ, and u:

$$\begin{aligned}\sigma_r = &\left\{\frac{E\alpha(a-r)}{3r^2(a-b)\left[a^2+b^2+\upsilon\left(a^2-b^2\right)\right]}\right\} \\ &\times\Big(T_a\Big\{r^2\left[a^2+b^2+\upsilon\left(a^2-b^2\right)\right]+rb^2\left[a(1-\upsilon)-b(2-\upsilon)\right]+ab^2\left[a(1-\upsilon)-b(2-\upsilon)\right]\Big\} \\ &+T_b\Big\{-r^2\left[a^2+b^2+\upsilon\left(a^2-b^2\right)\right]+rb^2\left[a(2+\upsilon)-b(1+\upsilon\right]+ab^2\left[-b(1+\upsilon)+a(2+\upsilon)\right]\Big\}\Big)\end{aligned} \tag{20a}$$

$$\begin{aligned}\sigma_\theta = &\left\{\frac{E\alpha}{3r^2(b-a)\left[a^2+b^2+\upsilon\left(a^2-b^2\right)\right]}\right\} \\ &\times\Big(T_a\Big\{2r^3\left[a^2+b^2+\upsilon\left(a^2-b^2\right)\right]-r^2\left[a^3+2b^3+\upsilon\left(a^3-b^3\right)\right]+a^2b^2\left[a(1-\upsilon)-b(2-\upsilon)\right]\Big\} \\ &+T_b\Big\{-2r^3\left[a^2+b^2+\upsilon\left(a^2-b^2\right)\right]+r^2\left[a^3-b^3+3ab^2+\upsilon\left(a^3-b^3\right)\right]+a^2b^2\left[a(2+\upsilon)-b(1+\upsilon)\right]\Big\}\Big)\end{aligned} \tag{20b}$$

$$\begin{aligned}u = &\left\{\frac{\alpha(1+\upsilon)(b-r)}{3r^2(b-a)\left[a^2+b^2+\upsilon\left(a^2-b^2\right)\right]}\right\} \\ &\times\Big(T_a\Big\{r^2\left[a^2+b^2+\upsilon\left(a^2-b^2\right)\right]+ra^2\left[a-2b-\upsilon(a-b)\right]+a^2b\left[a(1-\upsilon)-b(2-\upsilon)\right]\Big\} \\ &+T_b\Big\{-r^2\left[a^2+b^2+\upsilon\left(a^2-b^2\right)\right]+ra^2\left[2a-b+\upsilon(a-b)\right]+a^2b\left[a(2+\upsilon)-b(1+\upsilon)\right]\Big\}\Big)\end{aligned} \tag{20c}$$

The numerical values of the response parameters for T_a = 100 °C, T_b = 50 °C, and $\alpha = 12\times10^{-6}$/°C are

(1) at $r = a$: σ_r = 0 ksi, σ_θ = –17.5 ksi, and u = 0.012 in.
(2) at $r = b$: σ_r = 14.2 ksi, σ_θ = –13.7 ksi, and u = 0 in.

The sum of the stresses $\sigma_r + \sigma_\theta = 18508 - 1800r$ has a linear variation with respect to the r coordinate because of a similar distribution of temperature [see Equation (15)]. The BCC $\sigma_\theta - \upsilon\sigma_r + \alpha ET = 6.478 + 0.780\,r - \frac{1427.797}{r^2}\,\text{ksi}$ is zero at the inner boundary.

The CBMF solved the thermal load problem for a mixed boundary value problem. Superposition of solutions for mechanical and thermal loads yields the result for thermomechanical combined load. The CBMF solution satisfying the field equations and boundary conditions given by equation sets (2-6) can be considered accurate because of simultaneous compliance of the equilibrium equations and the compatibility conditions.

DISCUSSIONS

This section examines the CBMF concept. Attributes of the CC are also given. The annular plate example is supplemented with an eight-bar discrete truss structure. The solution to the truss problem using the integrated force method (IFM), which is the discrete analogue of CBMF, is given in Appendix C.

Completed Beltrami-Michell Formulation

Hooke's law, which is common to all analysis methods, relates stress to strain through the material matrix [*G*]:

$$\{\sigma\} = [G]\{\varepsilon\} \tag{21}$$

Stress σ must satisfy the state of equilibrium in the field as well as on the boundary of an elastic continuum. Likewise, strain has to comply with the condition of compatibility in the domain as well as on the boundary. The stress and strain formulations are sufficient for the determination of the stress state in an elastic continuum with stress and displacement boundary conditions. The equations that are required to calculate the stress state can be conceptualized in the following symbolic expression:

$$\begin{bmatrix}\text{Equilibrium equations} \\ \text{-------------------} \\ \text{Compatibility conditions}\end{bmatrix}\{\text{stress}\} = \begin{Bmatrix}\text{Mechanical load} \\ \text{---------------} \\ \text{Initial deformation}\end{Bmatrix} \tag{22}$$

The state of equilibrium and compatibility is sufficient for the determination of the stress state. Displacement is not required to calculate stress. An elastic body can undergo rigid body displacement and rotation that does not induce stress. Total displacement can be decomposed into an elastic component and a rigid body component: $disp = disp_{\text{elastic}} + disp_{\text{rigid}}$. The stress calculation in the CBMF accounts for the elastic component via the strain in the field and on the boundary. Recovery of the displacement from the stress state uses the kinematics or the rigid body displacement component.

Calculating stress by combining the equilibrium and compatibility was envisioned by Michell, and it is described by Love (1927) in the following quotation.

"It is possible by taking account of these relations [compatibility conditions] to obtain a complete system of equations (Equation 19) which must be satisfied by stress components, and thus the way is open for a direct determination of stress without the intermediate steps of forming and solving differential equations to determine the components of displacements."

The proposition of Beltrami and Michell can be realized now with the availability of the new BCC. A direct method is now available to calculate the stress state in a general elastic continuum with displacement as well as stress boundary conditions. The stress state is obtained without any recourse to displacement, which is back-calculated by integrating the strains. In the quotation, "intermediate steps" refers to Navier's displacement method that contains higher order differential equations. For the annular plate example, the CBMF required the solution of two uncoupled differential equations. Navier's method in contrast would have required the solution of a third-order differential equation.

An IFM of structural analysis has previously been formulated (Patnaik et al., 2004). IFM is the discrete analogue of the CBMF in elasticity. In IFM, forces are calculated from a set of equations $[S]\{F\} = \{P^*\}$ that include the EE and the CC. Displacements are back-calculated. The IFM solution to a truss problem is given in Appendix C.

Nature of compatibility condition

The CC is a controller type of relation. Strains are controlled, $f(\varepsilon_r, \varepsilon_\theta, \gamma) = 0$, in elasticity (or the strain formulation); likewise the deformations β are balanced, $f(\beta_1, \beta_2, \ldots, \beta_n) = 0$, in a discrete structural system. The controller type of relation cannot be derived from an application of the standard concepts of mechanics, like "action equal to reaction" (leading to the EE), or the "cause effect relation" (that has given us Hooke's law), or the "displacement continuity concept" (the "strain continuity" is conceptually incorrect). This is probably one important reason for the late development of these CCs. In elasticity, the field CC (or Saint-Venant's "strain formulation") can be derived by simply eliminating the displacements from the strain displacement relations. However, the derivation of the BCC requires the use of variational calculus. For structures, a direct application of Saint-Venant's "strain formulation" would have been sufficient for the derivation of the CC (Patnaik and Hopkins, 2004) No calculus would have been required because, like EE, the CC is also an algebraic equation. But such a procedure was not adopted, and the CC was not developed as a deformation balance concept. Variational calculus is the right tool to derive the BCC because of the tensorial nature of stress and strain.

Nontriviality property of boundary compatibility condition

The field CCs f_{CC} are satisfied automatically when expressed in continuous displacement functions u and υ: $f_{CC}(u, \upsilon) = \xi(u, \upsilon) - \xi(u, \upsilon) = 0$. However, the BCC, when expressed in terms of displacements, produces a nontrivial condition:

$$\left(\frac{\partial^2 u}{\partial\theta^2} - r\frac{\partial^2 \upsilon}{\partial r\partial\theta} - \frac{\partial\upsilon}{\partial\theta}\right)n_r + \left(r\frac{\partial^2 u}{\partial r\partial\theta} + \upsilon - r^2\frac{\partial^2\upsilon}{\partial r^2} - r\frac{\partial\upsilon}{\partial r} - r\frac{\partial u}{\partial\theta}\right)n_\theta = 0 \quad (23)$$

The Navier displacement method should account for the BCC because this is not a trivial condition in displacement. The BCC should be enforced along the interelement boundaries in a finite element model. The role of BCC should be investigated further in the Navier displacement method. A two-span plate made of two different materials supported on an elastic foundation may be an ideal example for the investigation.

Rotation and compatibility condition

The BCC should not be confused with rotation. An elastic body under load moves from its initial position to occupy the final form by undergoing strain, a $\bar{x}$ translation, and a $\bar{\theta}$ rotation (see Figure 1c). Only strain (not translation or rotation) induces stress. Strain is zero when the body is rigid. Rotation and strain are independent of each other, even though both quantities are defined in terms of the derivatives of displacement. For example, the BCC, which is a function of the strains is defined in polar coordinates as

$$\left[\frac{\varepsilon_r}{r} - \frac{\partial}{r\partial r}(r\varepsilon_\theta) + \frac{\partial\gamma}{2r\partial\theta}\right]n_r + \left(\frac{\partial\varepsilon_r}{r\partial\theta} - \frac{\partial\gamma}{2\partial r} - \frac{\gamma}{r}\right)n_\theta = 0 \quad (24)$$

The BCC enforces an equality constraint on the strain components; it imposes no restriction on either translation or rotation. The annular disk requires the BCC for analysis even while it is undergoing translation and rotation on a flat surface (with $z = 0$), as shown in Figure 1(c).

Stability of structure

Consider the discrete truss shown in Figure 2. Its analysis is given in Appendix C. The truss has one field CC and one boundary CC as follows:

Field CC:

$$\beta_1 + \beta_2 - \sqrt{2}\beta_3 - \sqrt{2}\beta_4 + \beta_5 + \beta_6 = 0 \quad \text{(expressed in bar deformation, } \beta\text{)} \quad (25a)$$

$$\sigma_1 + \sigma_2 - 2\sigma_3 - 2\sigma_4 + \sigma_5 + \sigma_6 = 0 \quad \text{(expressed in bar stress, } \sigma\text{)} \quad (25b)$$

Boundary CC:

$$\beta_2 + \beta_7 = 0 \quad \text{(in deformation)} \quad (25c)$$

$$\sigma_2 + \sigma_7 = 0 \quad \text{(in stress)} \quad (25d)$$

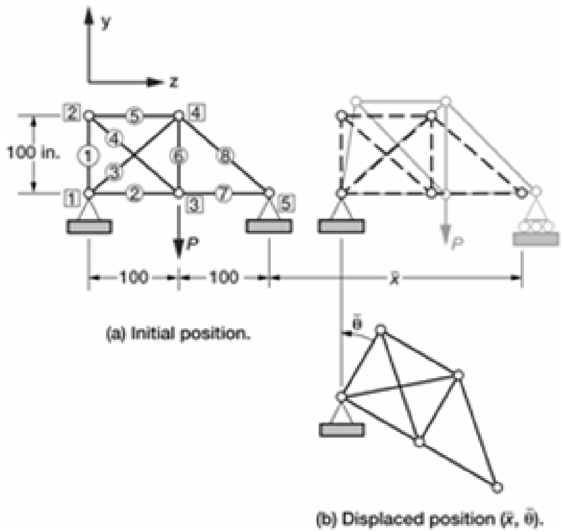

Figure 2. Eight-bar truss.

The field CC in Equation (25a) restrains the six bars stresses. This is the discrete analogue of Saint-Venant's strain formulation (Equation (4)). On the boundary, two member stresses are related [Equation (25c)]. The two stresses σ_2 and σ_7 cannot assume independent values in the lower boundary chord in Figure 2(a). The situation is similar to the BCC of the annular plate. The stresses σ_r and σ_θ cannot assume independent values along the inner boundary because of the BCC $\sigma_\theta - \upsilon\sigma_r = 0 \Rightarrow 0.4068 - 0.3 \times 1.356 = 0, \text{or } \sigma_\theta = \upsilon\sigma_r$. Compatibility conditions are required for the analysis of indeterminate structures, which are more stable than their determinate counterpart. Stability of a structure may concern the state of equilibrium. The CC does not degrade the stability of a structure. The original truss shown in Figure 2(a) is displaced in two steps. The axial constraint at node 5 is released first. This process eliminates the BCC but the structure is still stable. It is one degree indeterminate and the support at node 5 can move along the *x*-coordinate direction. If the transverse restraint is released at that node, then the structure rotates. The truss undergoes a $\bar{x}$ translation and a $\bar{\theta}$ rotation as depicted in Figure 2(b). It can be analyzed as a mechanism by accounting for the field CC [given by Equation (25a)].

Number of boundary compatibility conditions

There are two displacement boundary conditions in a plane elasticity problem. The question is "should there be two BCCs?" The answer is "no." The number of BCCs is equal to the number of field CCs, which is equal to the indeterminacy *r*, defined as the difference in the number of stress *n* and displacement *m* variables ($r = n - m$). The elasticity problem in polar coordinates has three stresses and two displacements. It is one degree indeterminate, and it has one field CC and one boundary CC. The number of displacement boundary conditions in the Navier displacement method is not equal to the number of BCCs. Consider the displaced position for the annular plate and the truss shown in Figures. 1(c) and 2(b), respectively. The plate undergoes translation and rotation, referred to as the kinematics conditions. The plate is restrained at the inner boundary, which is the elastic condition. The calculation of the displacement function in the Navier displacement method requires the simultaneous compliance of both types of conditions: kinematics as well as the elastic conditions. In the CBMF, the kinematics and the elastic conditions are enforced in two steps. First, the elastic condition, which essentially is the BCC, is used to calculate the stress response. In the annular plate example, the BCC is used to calculate stress in the CBMF. The kinematics condition $u = 0$ is then used to evaluate the integration constant in the displacement function.

Verification of existing elasticity solutions

Existing elasticity solutions (Mushkelishvili, 1953) should be examined and adjusted for the compliance of boundary compatibility conditions. The compliance can be verified by calculating the residue in the new condition given in equation 7. The solution should be adjusted when the residue is not zero.

Concluding remarks

The boundary compatibility condition (BCC) for an elastic continuum has been derived in polar coordinates using a variational approach. The BCC in essence is a constraint that is imposed on the strain or the stress state. The new boundary condition completes the stress formulation in elasticity. The completed Beltrami-Michell stress formulation can be used to calculate the stress state in a general elastic continuum without any reference to the displacement in the field or on the boundary. The displacement is back-calculated from the stress state. The BCC when expressed in displacements yields a nontrivial condition.

Appendix A—Symbols and Acronyms

Symbols:

A strain energy; *a*, *b* plate outer and inner radii, respectively; *B* complementary strain energy; b_r, b_θ body forces; *c* displacement function constant; *D* plate domain; *E* Young's modulus; f function; {*F*} member force vector; G_r, G_θ Green's functions; [*G*] material matrix; *h* plate thickness; *J* Jacobian; Γ domain boundary; Γ_1, Γ_2 boundary segments; n_r, n_θ direction cosines; $\{P^*\}$ load vector; $\overline{P}_r, \overline{P}_\theta$ prescribed loads or tractions; p mechanical load; R_r, R_θ reactions; r, z, θ polar coordinates; $[S]$ IFM governing matrix; T temperature distribution; U potential function; u, v displacements; $\overline{u}, \overline{v}$ prescribed displacements; V body force potential; W potential of work done; $\overline{x}$ translation; α coefficient of thermal expansion; β deformation; ε_r, ε_θ, γ plain strain components; $\overline{\theta}$ rotation; ξ function of displacement; π_s variational functional of the integrated force method; σ_i bar stress; σ_r, σ_θ, τ plain stress components; υ Poisson's ratio; ψ simple stress function; φ stress function

Acronyms:

BCC boundary compatibility condition; BMF Beltrami-Michell formulation; CBMF completed Beltrami-Michell formulation; CC compatibility condition; EE equilibrium equation; IFM integrated force method;

Appendix B—Variational Formulation for the Completed Beltrami-Michell Formulation

This appendix provides the variational derivation of the completed Beltrami-Michell formulation in polar coordinates that includes the new boundary compatibility condition (BCC). The equations are obtained from the stationary condition of the integrated force method (IFM) functional π_s, defined previously in Equation (1a) as

$$\pi_s = A + B - W \qquad \text{(B1)}$$

Where

$$A = \iint_D \left(\boxed{1}\, \frac{\sigma_r \partial u}{\partial r} + \boxed{2}\, \frac{\tau\, \partial u}{r\partial\theta} + \boxed{3}\, \sigma_\theta u + \boxed{4}\, \frac{\tau\, \partial\upsilon}{\partial r} + \boxed{5}\, \frac{\sigma_\theta \partial\upsilon}{r\partial\theta} - \boxed{6}\, \frac{\tau\, \upsilon}{r} \right) hr\,dr\,d\theta \qquad \text{(B2a)}$$

$$B = \iint_D \left(\boxed{7}\, \frac{\varepsilon_r \partial^2\varphi}{r^2\partial\theta^2} + \boxed{8}\, \frac{\varepsilon_r \partial\varphi}{r\partial r} + \boxed{9}\, \frac{\varepsilon_\theta \partial^2\varphi}{\partial r^2} + \boxed{10}\, \frac{\gamma\partial\varphi}{r^2\partial\theta} - \boxed{11}\, \frac{\gamma\partial^2\varphi}{r\partial r\partial\theta} \right) hr\,dr\,d\theta \qquad \text{(B2b)}$$

$$W = h\left[\oint_{1_1} \left(\boxed{12}\, u\overline{P}_r + \boxed{13}\, \upsilon\overline{P}_\theta \right) d1_1 + \oint_{1_2} \left(\boxed{14}\, \overline{u}R_r + \boxed{15}\, \overline{\upsilon}R_\theta \right) d1_2 + \boxed{16} \iint_D (b_r u + b_\theta\upsilon)(r\,dr\,d\theta) \right] \qquad \text{(B2c)}$$

The plate domain D has boundary λ, that is separated into segments 1_1 and 1_2, $1 = 1_1 + 1_2$. Body forces are b_r and b_θ. Along the boundary segment λ_1, loads $\overline{P}_r$ and $\overline{P}_\theta$ are prescribed, and displacements u and υ are free. The segment 1_2 has prescribed displacements $\overline{u}$ and $\overline{\upsilon}$ that can induce reactions R_x and R_r. The derivation sets the uniform plate thickness to unity (h = 1) without any consequence.

The term A represents the strain energy, and it is expressed in stress and displacement, which are considered independent of each other. The strain energy term B is expressed in strain and stress function φ, which are also considered independent of each other. The potential of the work done is W. Body force potential V is defined as $b_r = \frac{\partial V}{\partial r}$ and $b_\theta = \frac{\partial V}{r\partial\theta}$. The stress function φ is defined as

$$\sigma_r = \frac{\partial\varphi}{r\partial r} + \frac{\partial^2\varphi}{r^2\partial\theta^2} - V \qquad \text{(B3a)}$$

$$\sigma_\theta = \frac{\partial^2\varphi}{\partial r^2} - V \qquad \text{(B3b)}$$

$$\tau = -\frac{\partial}{\partial r}\left(\frac{\partial\varphi}{r\partial\theta}\right) \qquad \text{(B3c)}$$

Each term of the functional is reduced to obtain new terms that contain two factors. The second factor can be displacement, a stress function, or reaction. The first factor is an expression in terms of stress, strain, and load. The stationary condition of the functional with respect to displacement, stress function, and reactions will yield the following expressions:

(1) Field equilibrium equations (EEs) in stress. They are the coefficients of the variational displacements δu and $\delta\upsilon$ in the surface integral terms.
(2) Boundary EEs, or traction conditions. They are the coefficients of δu and $\delta\upsilon$ in the line integral terms.
(3) Field CC in strains. It is the coefficients of the variational stress function ($\delta\varphi$) in the surface integral term.
(4) Boundary CC. It is the coefficient of ($\delta\varphi$) in the line integral term.
(5) The displacement continuity condition. It is the coefficient of the variational reactions in the line integral term.

Derivation of equations stated in items (1) to (4) listed above (see Equation (B4) below) is straightforward. The derivation of the continuity condition (item (5)) required back-calculation (see Equations (B10) to (B12) below).
The first 11 terms of the functional reduced using techniques of calculus are given in Equation (B4). The other five terms (12 through 16) are retained without any operation.

$$\boxed{1}\ \iint \left(\sigma_r \frac{\partial u}{\partial r} \right) ds = \oint \sigma_r n_r [u]\, d1 - \iint \left(\frac{\partial\sigma_r}{\partial r} + \frac{\sigma_r}{r} \right) [u]\, ds$$

$$\boxed{2}\ \iint \left(\frac{\tau\partial u}{r\partial\theta} \right) ds = \oint \tau n_\theta [u]\, d1 - \iint \left(\frac{\partial\tau}{r\partial\theta} \right) [u]\, ds$$

$$\boxed{3}\ \iint \left(\frac{\sigma_\theta u}{r} \right) ds = \iint \left(\frac{\sigma_\theta}{r} \right) [u]\, ds$$

$$\boxed{4}\ \iint \left(\frac{\tau\partial\upsilon}{\partial r} \right) ds = \iint \tau n_r [\upsilon]\, d1 - \iint \left(\frac{\partial\tau}{\partial r} + \frac{\tau}{r} \right) [\upsilon]\, ds$$

$$\boxed{5}\ \iint\left(\frac{\sigma_\theta \partial\upsilon}{r\partial\theta}\right)ds=\oint \sigma_\theta n_\theta\,[\upsilon]\,d1-\iint\left(\frac{\partial\sigma_\theta}{r\partial\theta}\right)[\upsilon]\,ds$$

$$\boxed{6}\ -\iint\left(\frac{\tau\upsilon}{r}\right)ds=-\iint\left(\frac{\tau}{r}\right)[\upsilon]\,ds$$

$$\boxed{7}\ \iint\left(\frac{\varepsilon_r}{r^2}\right)\frac{\partial^2\varphi}{\partial\theta^2}(ds)=-\oint\left(\frac{\varepsilon_r}{r}\right)n_\theta\left[\frac{\partial\varphi}{\partial\theta}\right]d1+\oint\frac{1}{r}\frac{\partial}{\partial\theta}(\varepsilon_r)\,n_\theta\,[\varphi]d1+\iint\left(\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}(\varepsilon_r)\right)[\varphi]\,ds$$

$$\boxed{8}\ \iint\frac{\varepsilon_r\partial\varphi}{r\partial r}\,ds=\oint\frac{\varepsilon_r n_r}{r}[\varphi]\,d1-\iint\frac{\partial\varepsilon_r}{r\partial r}[\varphi]\,ds$$

$$\boxed{9}\ \iint\left(\frac{\varepsilon_\theta\partial^2\varphi}{\partial r^2}\right)ds=\oint(\varepsilon_\theta)n_r\left[\frac{\partial\varphi}{\partial r}\right]d1-\oint\frac{n_r\partial(r\varepsilon_\theta)}{r\partial r}[\varphi]\,d1+\iint\frac{\partial^2(r\varepsilon_\theta)}{r\partial r^2}[\varphi]\,ds$$

$$=\oint(\varepsilon_\theta)n_r\left[\frac{\partial\varphi}{\partial r}\right]d1-\oint\frac{n_r\partial(r\varepsilon_\theta)}{r\partial r}[\varphi]\,d1+\iint\left(\frac{2}{r}\frac{\partial\varepsilon_\theta}{\partial r}+\frac{\partial^2(\varepsilon_\theta)}{\partial r^2}\right)[\varphi]\,ds$$

$$\boxed{10}\ \iint\left(\frac{\gamma\partial\varphi}{r^2\partial\theta}\,ds\right)=-\oint\left(\frac{\gamma}{r}\right)n_\theta\,[\varphi]\,d1-\iint\frac{\partial\gamma}{r^2\partial\theta}[\varphi]\,ds$$

$$\boxed{11}\ \iint-\left(\frac{\gamma}{r}\frac{\partial^2\varphi}{\partial r\partial\theta}\right)ds=\oint\frac{1}{2}\left(-\frac{\partial\gamma}{\partial r}n_\theta+\frac{\partial\gamma}{r\partial\theta}n_r\right)[\delta\varphi]d1$$

$$-\oint\left(\frac{\gamma}{2r}\right)n_r\left[\frac{\partial\varphi}{\partial\theta}\right]d1+\oint\left(\frac{\gamma}{2}\right)n_\theta\left[\frac{\partial\varphi}{\partial r}\right]d1-\iint\frac{\partial^2\gamma}{r\partial r\partial\theta}[\varphi]\,ds \tag{B4}$$

All 11 terms are combined to obtain the following form of the functional:

$$\begin{aligned}\pi_s=&-\iint_D\left\{\left(\frac{\partial\sigma_r}{\partial r}+\frac{\partial\tau}{r\partial\theta}-\frac{\sigma_r-\sigma_\theta}{r}+b_r\right)[u]+\left(\frac{\partial\tau}{\partial r}+\frac{\partial\sigma_\theta}{r\partial\theta}+\frac{2\tau}{r}+b_\theta\right)[\upsilon]\right\}ds\\&+\iint_D\left(\frac{\partial^2\varepsilon_r}{r^2\partial\theta^2}-\frac{\partial\varepsilon_r}{r\partial r}+\frac{\partial^2\varepsilon_\theta}{\partial r^2}+\frac{2\partial\varepsilon_\theta}{r\partial r}-\frac{\partial\gamma}{r^2\partial\theta}-\frac{\partial^2\gamma}{r\partial r\partial\theta}\right)[\varphi]\,ds\\&+\oint_{1_1}\left((\sigma_r n_r+\tau n_\theta-\overline{P}_r)[u]+(\tau n_r+\sigma_\theta n_\theta-\overline{P}_\theta)[\upsilon]\right)d1\\&+\oint_{1}\left(\left\{\frac{\varepsilon_r}{r}-\frac{\partial(r\varepsilon_\theta)}{r\partial r}+\frac{\partial\gamma}{2r\partial\theta}\right\}n_r+\left\{\frac{\partial\varepsilon_r}{r\partial\theta}-\frac{\partial\gamma}{2\partial r}-\left(\frac{\gamma}{r}\right)\right\}n_\theta\right)[\varphi]d1\\&+\oint_{1_2}\left\{\left(\varepsilon_\theta n_r+\frac{\gamma}{2}n_\theta\right)\left[\frac{\partial\varphi}{\partial r}\right]-\left(\frac{\gamma}{2r}n_r+\frac{\varepsilon_r}{r}n_\theta\right)\left[\frac{\partial\varphi}{\partial\theta}\right]\right\}d1+\oint_{1_2}(\overline{u}R_r+\overline{\upsilon}R_\theta)\,d1_2\end{aligned} \tag{B5}$$

The variation of the functional with respect to displacements δu and $\delta\upsilon$ yields the field EEs

$$\frac{\partial\sigma_r}{\partial r}+\frac{\partial\tau}{r\partial\theta}-\frac{\sigma_r-\sigma_\theta}{r}+b_r=0 \tag{B6a}$$

$$\frac{\partial\tau}{\partial r}+\frac{\partial\sigma_\theta}{r\partial\theta}+\frac{2\tau}{r}+b_\theta=0 \tag{B6b}$$

Likewise, the field CC is obtained as the coefficient of the variation of the stress function$\delta\varphi$:

$$\frac{\partial^2\varepsilon_r}{r^2\partial\theta^2}-\frac{\partial\varepsilon_r}{r\partial r}+\frac{\partial^2\varepsilon_\theta}{\partial r^2}+\frac{2\partial\varepsilon_\theta}{r\partial r}-\frac{\partial\gamma}{r^2\partial\theta}-\frac{\partial^2\gamma}{r\partial r\partial\theta}=0 \tag{B7}$$

Along the boundary segment 1_1, the variation of the displacements δu and $\delta\upsilon$ yields the EEs or the traction conditions

$$\sigma_r n_r+\tau n_\theta=\overline{P}_r \tag{B8a}$$

$$\tau n_r+\sigma_\theta n_\theta=\overline{P}_\theta \tag{B8b}$$

Along an indeterminate boundary, the BCC is obtained as the coefficient of the variation of the stress function $\delta\varphi$:

$$\left(\frac{\varepsilon_r}{r}-\frac{\partial(r\varepsilon_\theta)}{r\partial r}+\frac{\partial\gamma}{2r\partial\theta}\right)n_r+\left(\frac{\partial\varepsilon_r}{r\partial\theta}-\frac{\partial\gamma}{2\partial r}-\frac{\gamma}{r}\right)n_\theta=0 \tag{B9}$$

In summary, stress equilibrium is enforced in the field [Equation (B6)] and on the boundary (Equation (B8)). Likewise strain compliance is achieved in the field (Equation (B7)) and on the boundary [Equation (B9)].

Displacement continuity

The displacement boundary conditions $u-\overline{u}=0$ and $\upsilon-\overline{\upsilon}=0$ are routinely used in analysis. Their derivations are shown through back-calculation. This strategy is followed to avoid artificiality in a direct derivation process. The expression $(u-\overline{u})\delta(P_r=\sigma_r n_r+\tau n_\theta)+(\upsilon-\overline{\upsilon})\delta(P_\theta=\tau n_r+\sigma_\theta n_\theta)=0$ yields the continuity conditions. Because $\overline{u}$ and $\overline{\upsilon}$ are contained in terms 14 and 15 in Equation (B2c), we have to prove the following formula along boundary segment λ_2:

$$\oint_{1_2}\left[\left(\varepsilon_\theta n_r+\frac{\gamma}{2}n_\theta\right)\left(\frac{\partial\varphi}{\partial r}\right)-\left(\frac{\gamma}{2r}n_r+\frac{\varepsilon_r}{r}n_\theta\right)\left(\frac{\partial\varphi}{\partial\theta}\right)\right]d1=\oint_{1_2}u(\sigma_r n_r+\tau n_\theta)+\upsilon(\tau n_r+\sigma_\theta n_\theta)\,d1 \tag{B10a}$$

The variational form of Eq. (B10a) can be written as

$$\oint_{1_2}\left[\left(\varepsilon_\theta n_r+\frac{\gamma}{2}n_\theta\right)\delta\left(\frac{\partial\varphi}{\partial r}\right)-\left(\frac{\gamma}{2r}n_r+\frac{\varepsilon_r}{r}n_\theta\right)\delta\left(\frac{\partial\varphi}{\partial\theta}\right)\right]d1=\oint_{1_2}u\delta(\sigma_r n_r+\tau n_\theta)+\upsilon\delta(\tau n_r+\sigma_\theta n_\theta)\,d1 \tag{B10b}$$

Consider the reduction of the first of the two right-hand terms in Equation (B10a):

$$\oint u(\sigma_r n_r+\tau n_\theta)d1=\oint u\left\{{}^{a}\left(\frac{\partial\varphi}{r\partial r}n_r\right)+{}^{b}\left(\frac{\partial^2\varphi}{r^2\partial\theta^2}\right)n_r-{}^{c}\left[\frac{\partial}{\partial r}\left(\frac{\partial\varphi}{r\partial\theta}\right)\right]n_\theta\right\}d1 \tag{B11a}$$

The terms *b* and *c* that contain higher derivatives of the stress function are reduced to obtain terms in the first derivative of the stress function:

$$\oint^{b}\left(\frac{\partial^2\varphi}{r^2\partial\theta^2}n_r\right)u\,d1=-\oint^{b}\left[\frac{\partial\varphi}{r^2\partial\theta}\left(\frac{\partial u}{\partial\theta}\right)n_r\right]d1-\oint\frac{\partial}{\partial r}\left(\frac{\partial\varphi}{r\partial\theta}\right)un_\theta\,d1=\oint\left(\frac{\partial\varphi}{r\partial\theta}\right)\frac{\partial u}{\partial r}(n_\theta d1) \tag{B11b}$$

The variation of the first right-hand term of Equation (B10a) becomes

$$\oint u\,\delta(\sigma_r n_r+\tau n_\theta)d1=\oint\left\{\left[\left(\frac{u\delta\partial\varphi}{r\partial r}\right)-\left(\frac{\partial u}{r^2\partial\theta}\right)\left(\delta\frac{\partial\varphi}{\partial\theta}\right)\right]n_r+\left(\frac{\partial u}{r\partial r}\right)\left(\delta\frac{\partial\varphi}{\partial\theta}\right)n_\theta\right\}d1 \tag{B11c}$$

Likewise, the second right-hand term is reduced:

$$\oint\delta(\tau n_r+\sigma_\theta n_\theta)\upsilon\,d1=\oint\left\{\left[\left(\frac{\upsilon\delta\partial\varphi}{r^2\partial\theta}\right)+\frac{\partial\upsilon}{r\partial\theta}\left(\frac{\delta\partial\varphi}{\partial r}\right)\right]n_r-\left(\frac{\partial\upsilon}{\partial r}\right)\left(\frac{\delta\partial\varphi}{\partial r}\right)n_\theta\right\}d1 \tag{B11d}$$

Verification of the formula given by Equation (B10a) is obtained by combining the two Equations (B11c) and (B11d):

$$\oint_{l_2} \left(u\delta(\sigma_r n_r + \tau n_\theta) + \upsilon\delta(\tau n_r + \sigma_\theta n_\theta)\right) dl \tag{B12}$$

$$= \oint_{l_2} \left\{ \left[\left(\frac{u}{r} + \frac{\partial \upsilon}{r\partial\theta} \right) n_r - \left(\frac{\partial \upsilon}{\partial r} \right) n_\theta \right] \delta\left(\frac{\partial \varphi}{\partial r} \right) - \left[\left(\frac{\partial u}{r\partial\theta} - \frac{\upsilon}{r} \right) n_r - \frac{\partial u}{\partial r} n_\theta \right] \delta\left(\frac{\partial \varphi}{r\partial\theta} \right) \right\} dl$$

$$= \oint_{l_2} \left\{ \left[(\varepsilon_\theta) n_r + \left(\frac{\gamma}{2} \right) n_\theta \right] \delta\left(\frac{\partial \varphi}{\partial r} \right) - \left[\left(\frac{\gamma}{2} \right) n_r - \varepsilon_r n_\theta \right] \delta\left(\frac{\partial \varphi}{r\partial\theta} \right) \right\} dl$$

Verification of boundary compatibility condition

Green's theorem is used for a quick verification of the BCC. The BCC is inserted in the line integral coefficient to recover the well known field CC in the surface integral term. The integral theorem in polar coordinates can be written as

$$\iint \left(\frac{1}{r} \frac{\partial}{\partial r}(rG_r) - \frac{\partial G_\theta}{r\partial\theta} \right) ds = \oint (G_r n_r + G_\theta n_\theta) dl \tag{B13}$$

Where G_r and G_θ are the coefficients of direction cosines n_r and n_θ in Equation (B9), respectively:

$$G_r = \frac{\varepsilon_r}{r} - \frac{\partial(r\varepsilon_\theta)}{r\partial r} + \frac{\partial\gamma}{2r\partial\theta} \tag{B14a}$$

$$G_\theta = \frac{\partial\varepsilon_r}{r\partial\theta} - \frac{\partial\gamma}{2\partial r} - \frac{\gamma}{r} \tag{B14b}$$

The surface integral terms are generated as

$$\frac{\partial(rG_r)}{r\partial r} = \left(\frac{\partial\varepsilon_r}{r\partial r} \right) - \left(\frac{2\partial\varepsilon_\theta}{r\partial r} + \frac{\partial^2\varepsilon_\theta}{\partial r^2} \right) + \frac{\partial^2\gamma}{2r\partial r\partial\theta} \tag{B15a}$$

$$\frac{\partial G_\theta}{r\partial\theta} = \frac{\partial^2\varepsilon_r}{r^2\partial\theta^2} - \frac{\partial^2\gamma}{2r\partial r\partial\theta} - \frac{\partial\gamma}{r^2\partial\theta} \tag{B15b}$$

$$\iint \left\{ \frac{\partial(rG_r)}{r\partial r} - \frac{\partial G_\theta}{r\partial\theta} \right\} ds = -\iint \left(\frac{\partial^2\varepsilon_r}{r^2\partial\theta^2} - \frac{\partial\varepsilon_r}{r\partial r} + \frac{\partial^2\varepsilon_\theta}{\partial r^2} + \frac{2\partial\varepsilon_\theta}{r\partial r} - \frac{\partial\gamma}{r^2\partial\theta} - \frac{\partial^2\gamma}{r\partial r\partial\theta} \right) ds \tag{B16}$$

The coefficient within the bracket is the field CC. The compatibility concept applies to the field as well as to the boundary. The nature of the compatibility expression changes in compliance with the domain and the boundary. The same interpretation is true for Cauchy's field EEs.

Appendix C—Solution to an eight-bar truss

The solution to the eight-bar truss shown in Figure 2(a) is obtained using the integrated force method (IFM), (Patnaik, 1986) which is the discrete analogue of the completed Beltrami-Michell formulation (CBMF). The IFM, like the CBMF, generates the force solution by coupling the equilibrium equations (EEs) to the compatibility conditions. Displacements are back-calculated from the force solution. The truss is made of steel with Young's modulus E = 30 000 ksi. Each of the eight bars has an area of 1 in^2. Nodes 1 and 5 are fully restrained. It is subjected to a gravity load of magnitude P = –10 kip at the midspan location. The problem is to calculate the force and displacement response.

The six EEs of the structure can be written in terms of bar forces F as

$$\begin{bmatrix} 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 1 & 0 & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 1 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -10 \\ 0 \\ 0 \end{Bmatrix} \tag{C1}$$

The two compatibility conditions (CCs) in bar deformations β can be written as

$$\begin{bmatrix} 1 & 1 & -\sqrt{2} & -\sqrt{2} & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \\ \beta_7 \\ \beta_8 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{C2}$$

The CC is rewritten in member forces using the flexibility relation $\left(\beta = \frac{FL}{AE}\right)$ for bar length L, area A, modulus E, and member force F:

$$\begin{bmatrix} 1 & 1 & -2 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{C3}$$

Simultaneous solution of the six EEs and the two CCs yields the eight member forces {F}. The six displacements {X} are back-calculated:

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{Bmatrix} = \begin{Bmatrix} -2.6 \\ -1.3 \\ -3.3 \\ 3.7 \\ -2.6 \\ 7.4 \\ 1.3 \\ -7.1 \end{Bmatrix}_{\text{kip}} \quad \text{and} \quad \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \end{Bmatrix} = \begin{Bmatrix} 0.021 \\ -0.009 \\ -0.004 \\ -0.059 \\ 0.012 \\ -0.034 \end{Bmatrix}_{\text{in.}} \tag{C4}$$

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