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Full Length Research Paper

Direct solution of Riccati equation arising in inventory production control in a Stochastic manufacturing system

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We studied the production inventory problem in a manufacturing system with degenerate Stochastic demand. We derived the dynamics of inventory-demand ratio that evolves according to Stochastic neoclassical differential equation through Ito's Lemma. We established the Riccati based solution of the Hamilton- Jacobi-Bellman (HJB) equation associated with this problem.

Key words: Inventory production models, Stochastic differential equations, manufacturing systems.

INTRODUCTION

Many manufacturing enterprises use a production inventory system to manage fluctuations in consumers' demand for the product. Such a system consists of a manufacturing plant and a finished goods warehouse to store those products which are manufactured but not immediately sold. The advantages of having products in inventory are: first they are immediately available to meet demand; second, by using the warehouse to store excess production during low demand periods to be available for sale during high demand periods. This usually permits the use of a smaller manufacturing plant than would otherwise be necessary and also reduces the difficulties of managing the system.

We are concerned with the optimization problem to minimize the expected discounted cost control of production planning in a manufacturing system with degenerate stochastic demand.

$$J(p) = E\left[\int_0^\infty e^{-\rho t}\left\{h\left(x_t - \overline{x}\right)^2 + c\left(p_t - \overline{p}\right)^2\right\}dt\right]$$

On simplification if c = h = 1, $p^- = x^- = 0$, then the above production inventory problem becomes:

$$J(p) = E\left[\int_0^\infty e^{-\rho t}\left\{x_t^{\,2} + p_t^{\,2}\right\}dt\right] \tag{1.1}$$

subject to the dynamics of the state equation which says that the inventory at time t is increased by the production rate and decreased by the demand rate can be written as according to

$$dx_t = (p_t - y_t)\,dt, \quad x_0 = x, \quad x > 0, \qquad (1.2) \tag{1.2}$$

and the demand equation with the production rate is described by the Brownian motion

$$dy_t = Ay_t + \sigma y_t dw_t, \quad y_0 = y, \quad y > 0 \tag{1.3}$$

in the class P of admissible controls of production processes pt with non-negative constant $p_t \geq 0$ defined on a complete probability space (Ω, F , P) endowed with the natural filtration Ft generated by σ(ws , s ≤ t) carrying a one-dimensional standard Brownian motion wt , xt is the inventory level for production rate at time t (state variable), yt is the demand rate at time t, pt is the production rate at time t (control variable), ρ > 0 is the constant non-negative discount rate, A is the non-zero constant, σ is non-zero constant diffusion coefficient, x0 is the initial value of inventory level, y0 is the initial value of demand rate, h is the inventory holding cost

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coefficient h > 0, c is production cost coefficient c > 0, p^- is the value of factory-optimal production level, and x^- is the value of factory-optimal inventory level.

The purpose of the paper is to give an optimal production cost control by an existence Riccati solution associated with the reduced (one-dimensional) HJB equation. We first applied the technique of dynamic programming principle (Bellman, 1957) to obtain the general (two-dimensional) HJB equation corresponding to production inventory control problem. We also developed revised optimal inventory-production control problem.

The control problem of production planning in a manufacturing systems with discount rate has been studied by many authors like Fleming et al. (1987), Sethi and Zhang (1994). The Bellman equation associated with production inventory control problem is quite different from them and it is treated by Bensoussan et al. (1984) for the one-dimensional manufacturing systems with the unbounded control region. This type of optimization problem studied also by Morimoto and Kawaguchi (2002) for renewable resources, Baten and Morimoto (2005) for linear degenerate systems, Baten and Sobhan (2007) for one-sector neoclassical growth model with the CES function where the dynamics of the capital-labor ratio can be described by a diffusion-type stochastic process. Generally speaking, the similar types of linear control problems investigated for the stochastic differential systems with invariant measures like Bensossan (1988), Borker (1989).

This paper was organized as follows. In section 2 we obtained the general (two- dimensional) HJB equation by the Bellman principle of optimality and then we reduced the two-dimensional HJB equation to one-dimensional second-order differential equation. Then we derived the dynamics of inventory-demand ratio that evolves according to stochastic neoclassical differential equation through Ito's Lemma. In section we established the Riccati based solution of production inventory control problem that was satisfied by the value function of this optimization problem. Finally conclusions were made in the last section.

MATERIALS AND METHODS

Development of optimal inventory production control problem

The Hamilton-Jacobi-Bellman equation

Suppose u(x, y, t) : Rn × R × Rn → R is a function whose value is the minimum value of the objective function of the production inventory control problem for the manufacturing system given that we start it at time t in state x, and y. That is,

$$u(x,y,t) = \inf_p J(p)$$

where the value function u is finite valued and twice continuously differentiable on (0, ∞). We initially assume that the u(x, y, t) exists for all x, y and t in the relevant ranges.

Since (1.2) and (1.3) is a scalar equation, the subscript t here means only time t. Thus, x and y will not cause any confusion and, at the same time, will eliminate the need of writing many parentheses. Thus, dwt is a scalar.

To solve the problem defined by (1.1), (1.2) and (1.3), let u(x, y, t) known as the value function, be the expected value of the objective function (1.1) from t to infinity, when an optimal policy is followed from t to infinity, given xt = x, yt = y. Then by the principle of optimality [Richard Bellman (1957)],

$$u(x,y,t) = \min_p E\left[\left\{x_t^2 + p_t^2\right\}dt + u\left[x(t+dt), y(t+dt), t+dt\right]\right] + o(dt) \tag{2.1}$$

We assume that u(x, y, t) is a continuously differentiable of its arguments. By Taylor's expansion, we have

$$u\left[x(t+dt), y(t+dt), t+dt\right] = u(x,y,t) - \rho u(x,y,t)\,dt + u_y dy_t + u_x dx_t + \frac{1}{2}u_{yy}(dy_t)^2 + o(dt) \tag{2.2}$$

From (1.2), we can formally write

$$(dx_t)^2 = (p_t)^2(dt)^2 + (y_t)^2(dt)^2 - 2p_t y_t (dt)^2 \tag{2.3}$$

$$(dy_t)^2 = (Ay_t)^2(dt)^2 + (\sigma y_t)^2(dw_t)^2 + 2(Ay_t)(\sigma y_t)(dt)(dw_t) \tag{2.4}$$

$$dx_t dt = p_t(dt)^2 - y_t(dt)^2 \tag{2.5}$$

and

$$dy_t dt = (Ay_t)(dt)^2 + (\sigma y_t)\,dt\,dw_t. \tag{2.6}$$

The exact meaning of these expressions comes from the theory of stochastic calculus; Arnold (1974), Karatzas and Shreve (1991). For our purposes, it is sufficient to know the multiplication rules of stochastic calculus

$$(dw_t)^2 = dt,\ dw_t dt = 0,\ dt^2 = 0 \tag{2.7}$$

Substitute (2.2) into (2.1) and use (2.3), (2.4) (2.5), (2.6) and (2.7) to obtain

$$u(x,y,t) = \min_p\left\{p^2 + pu_x\right\}dt + u(x,y,t) - \rho u(x,y,t)\,dt + \frac{1}{2}\sigma^2 y^2 u_{yy}dt + Ayu_y dt - yu_x dt + x^2 + o(dt) \tag{2.8}$$

Note that we suppressed the arguments of the functions involved in (2.2).

Canceling the term u on both sides of (2.8), dividing the remainder by dt, and letting t → 0, we obtain the dynamic programming partial differential equation or Hamilton- Jacobi-Bellman equation

$$-\rho u(x,y,t) + \frac{1}{2}\sigma^2 y^2 u_{yy} + Ayu_y - yu_x + F^{\bullet}(u_x) + x^2 = 0$$
$$u(0,y) = 0, \qquad x, y > 0, \tag{2.9}$$

where $F^*(x)$ is the Legendre transform of $F(x)$, that is,

$$F^{\bullet}(x) = \min_{p}\left\{p^2 + px\right\} = -\frac{x^2}{4}$$

and u_x, u_y, u_{xx}, u_{yy} are partial derivatives of $u(x, y, t)$ with respect to x and y. The main feature of the HJB equation (2.9) is the vanishing of the coefficient of uyy for y = 0 in partial differential equation terminology, then the equation is degenerate elliptic. Generally speaking, the difficulty stems from the degeneracy in the second order term of the HJB equation (2.9).

RESULTS AND DISCUSSION

A reduction to 1-dimensional case

In this subsection, the general (two dimensional) HJB equation reduced to a one-dimensional second-order differential equation. From the two dimensional state space form (one state x for inventory level and the other state y for demand rate) it reduced to one-dimensional form for $(z = x/y)$ the ratio of inventory to demand.

There exists a $v \in (0,\infty)$ such that $u(x,y) = y^2 v(x/y),\ y > 0.$
Since

$$u_x = yv^{'}(x/y)1/y, \quad u_y = 2yv(x/y) - xv^{'}(x/y),$$
$$u_{yy} = 2v(x/y) - 2(x/y)v^{'}(x/y) + (x/y)^2 v^{''}(x/y).$$

Setting z = x/y and substituting these in (2.9), we have

$$-\rho v(z) + \frac{1}{2}\sigma^2\left[2v(z) - 2v'(z)z + z^2 v''(z)\right] + 2Av(z) - Azv'(z) - v'(z)$$
$$+ \min_{p}\left\{p^2 + pyv'(z)\right\} + z^2 = 0 \tag{2.10}$$

Since $\min_{p}\left\{p^2 + pyv'(z)\right\} = y^2 \min_{p}\left\{(p/y)^2 + (p/y)v'(z)\right\} = y^2 \min_{q}\left\{q^2 + qv'(z)\right\}$
and

$$\min_{q\geq 0}\left\{q^2 + qv'(z) - v'(z)\right\} = \min_{k+1\geq 0}\left\{(k+1)^2 + kv'(z)\right\}.$$

Then the HJB equation (2.10) became

$$\tilde{\rho}v(z) + \frac{1}{2}\sigma^2 z^2 v''(z) + \tilde{A}zv'(z) + \min_{k+1\geq 0}\left\{(k+1)^2 + kv'(z)\right\} + z^2 = 0$$
$$v(0) = 0, \quad z > 0, \tag{2.11}$$

where $\tilde{\rho} = \rho - 2A, \quad \text{and } \tilde{A} = -\left(A + \sigma^2\right).$

Stochastic neoclassical differential equation and value function.

In this subsection, the dynamics of the state equation of inventory level (1.2) reduced to a one-dimensional process by working in intensive (per capita) variables.

Define $z_t = x_t / y_t$: inventory-demand ratio and $q_t = p_t / y_t$: per-capita production.

To determine the stochastic differential for the inventory-demand ratio,

$z_t \equiv x_t / y_t$, we apply Ito's formula as follows

$$z = \frac{x}{y} \equiv G(y,t), \quad \text{then } \frac{\partial G}{\partial y} = -\frac{x}{y^2} = -\frac{z}{y}, \quad \frac{\partial^2 G}{\partial y^2} = 2\frac{x}{y^3} = 2\frac{z}{y^2}, \quad \frac{\partial G}{\partial t} = \frac{\dot{x}}{y} = \frac{p}{y} - 1.$$

From Ito's formula,

$$dz = \frac{\partial G}{\partial y}dy + \frac{\partial G}{\partial t}dt + \frac{1}{2}\frac{\partial^2 G}{\partial y^2}(dy)^2. \tag{2.12}$$

From (1.2), we have that (dy) 2 = σ2 y2 dt. Substituting the above expressions into (2.12), we have that the dynamics of zt to be the inventory-demand ratio at time t which evolves according to the stochastic neoclassical differential equation for demand y

$$\begin{aligned} dz_t &= \left(-\frac{z_t}{y_t}\right)(Ay_t dt + \sigma y_t dw_t)(q_t - 1)dt + z_t\sigma^2 dt \\ &= \left[-Az_t + (q_t - 1) + \sigma^2 z_t\right]dt - \sigma z_t dw_t \\ &= \left[(-A + \sigma^2)z_t + (q_t - 1)\right]dt - \sigma z_t dw_t \\ &= \left[Bz_t + k_t\right]dt - \sigma z_t dw_t, \quad z_0 = z,\ z > 0, \end{aligned} \tag{2.13}$$

where

$$B = -A + \sigma^2, \quad k = q - 1.$$

The inventory production control problem became

$$\tilde{J}(k_t) = E\left[\int_0^\infty e^{-\tilde{\rho}t}\left\{z_t^2 + (k_t + 1)^2\right\}dt\right] \tag{2.14}$$

subject to degenerate stochastic differential equation

$$dz = \left[Bz_t + k_t\right]dt - \sigma z_t dw_t, \quad z_0 = z,\ z > 0. \tag{2.15}$$

Let us consider the minimum value of the payoff function is a function of this initial point. The value function defined as a function whose value is the minimum value of the objective function of the production inventory control problem (2.14) for the manufacturing system, that is,

$$V(z) = \inf_{k} E\left[\int_0^\infty e^{-\tilde{\rho}t}\left\{z_t^2 + (k_t + 1)^2\right\}dt\right] = \inf_{k}\tilde{J}(k_t) \tag{2.16}$$

The value function V (z) is a solution to the reduced (one dimensional) HJB equation (2.11) and the solution of this HJB equation can be used to test controller for optimality or perhaps to construct a feedback controller. Riccati Based Solution to optimal control problem.

This section finally dealt with the Riccati-based solution of the reduced one-dimensional HJB equation (2.11) corresponding to the production inventory control problem (2.16) subject to (2.13) using the dynamic programmming principle [Richard Bellman, 1957].

To find the Riccati based solution of HJB equation (2.11), we referred to Prato (1984), Prato and Ichikawa (1990) for the degenerate linear control problems related to Riccati equation.

By taking the derivative of (2.11) with respect to k and setting it to zero, we minimized the expression inside the bracket of (2.11).

This procedure built up

$$k_t^{\bullet} = -\left(v'\left(z\right)/2\right) - 1. \quad (3.1)$$

Substituting (3.1) into (2.11) yields the equation

$$\tilde{\rho}v\left(z\right) + \frac{1}{2}\sigma^2 z^2 v''\left(z\right) + Bzv'\left(z\right) - \left\{v'\left(z\right)\right\}^2 / 4 - v'\left(z\right) + z^2 = 0 \quad (3.2)$$

known as the HJB equation. This is a partial differential equation which has a solution form

$$v\left(z\right) = az^2\left(t\right). \quad (3.3)$$

Then

$$v'\left(z\right) = 2az\left(t\right), \quad v''\left(z\right) = 2a. \quad (3.4)$$

Substituting (3.3) and (3.4) into (3.2) yields

$$\left(1 - \tilde{\rho}a + \sigma^2 a - a^2 + 2aB\right) z^2 - 2az = 0 \quad (3.5)$$

Since (3.5) must hold for any value of z, we must have

$$a^2 - a\left(2B + \sigma^2 + \tilde{\rho}\right) - 1 = 0,$$

is called a Riccati equation from which we obtain

$$a = -\left(2B + \sigma^2 + \tilde{\rho}\right) \pm \sqrt{\left(2B + \sigma^2 + \tilde{\rho}\right)^2 + 4} \Big/ 2 = K_1 \text{ (constant)}$$

So, (3.3) is a solution form of (3.2).

Conclusion

In general we can further study a stochastic optimal inventory production control problem for linear degenerate systems to minimize the discounted expected cost:

$$J\left(p\right) = E\left[\int_0^\infty e^{-\rho t}\left\{h\left(x_t\right) + p_t^{\,m}\right\}dt\right]$$

over p ∈P subject to (1.2), (1.3) and (1.4) and in addition a continuous, non-negative, convex function h ∈ R satisfying the polynomial growth condition such that

$$0 \le h\left(x\right) \le K\left(1 + |\,x\,|^m\right),\ x \in \Re,\ m \in \mathrm{N}_+$$

for some constant $K > 0.$

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REFERENCES

Arnold A (1974). Stochastic differential equations: Theory and applications, Wiely, New York.

Baten MA, Morimoto H (2005). An extension of the linear regulator for degenerate diffusions. IEEE Trans. on Auto. Con., 50(11): 1822-1826.

Baten MA, Sobhan A (2007). Optimal consumption in a growth model with the CES production function. Stochas. Ana. and Appli., 25: 1025-1042.

Bellman R (1957). Dynamic programming, Princeton N.J.: Princeton University Press.

Bensoussan A (1988). Perturbation methods in optimal control, Gauthier-Villars, Paris.

Bensoussan A, Sethi SP, Vickson R, Derzko N (1984). Stochastic production planning with production constraints. SIAM J. Control and Optim., 22: 920-935.

Borkar V (1989). Optimal control of diffusion processes, Pitman Research Notes in Math., pp. 203.

Fleming WH, Sethi SP, Soner HM (1987). An optimal stochastic production planning with randomly fluctuating demand. SIAM J. Control Optim., 25: 1494-1502.

Karatzas I, Shreve SE (1991). Brownian motion and stochastic calculus, Springer- Verlag. New York.

Morimoto H, Kawaguchi K (2002). Optimal exploitation of renewable resources by the viscosity solution method. Stoc. Ana. App., 20: 927-946.

Prato G (1984). Direct solution of a Riccati equation arising in stochastic control theory. Appl. Math. Optim., 11: 191-208.

Prato G, Ichikawa A (1990). Quadratic control for linear time-varying systems. SIAM J. Control Optim., 28: 359-381.

Sethi SP, Zhang Q (1994). Hierarchical decision making in a stochastic manufacturing systems. Birkh¨auser, Boston.