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Finite element analysis of nonlinear structures with Newmark method

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The Newmark method is an explicit method and the most important aspects of this subfamily are the possibility of unconditional stability for nonlinear systems and second-order accuracy. The possibility of unconditional stability and second-order accuracy allows the use of a large time step, and the explicitness of each time step involves no iterative procedure. To evaluate the numerical properties of the proposed family method in the solution of linear elastic and nonlinear systems, its computing sequence within a single time step must be realistically reflected in the analysis. In this paper, the concept of Newmark method for structure is explained and applied.

Key words: Nonlinear dynamic response optimization, structural, Newmark, nonlinear dynamic analysis.

INTRODUCTION

A Newmark diffusive scheme is presented for the time-domain solution of dynamic systems containing fractional derivatives. This scheme combines a classical Newmark time-integration method used to solve second-order mechanical systems, with a diffusive representation based on the transformation of the fractional operator into a diagonal system of linear differential equations, which can be seen as internal memory variables. The focus is given on the algorithm implementation into a finite element framework. This method is implicit and based on the assumption of the linear change of acceleration during each time step. Other types of variation of the acceleration during a time interval can also be assumed (Bernard, 2002). In general, these assumptions will indicate how much of the acceleration at the end of the interval enters into the relationships for velocity and displacement. In 1959, Newmark (Bradford, 1999) presented a method which permits different types of variation of the acceleration to be taken into account.

The main features of this method are given in the work. The Newmark-beta method is a method of numerical integration used to solve differential equations. It is used in finite element analysis to model dynamic systems, recalling the continuous-time equation of motion:

$$u = \overset{\bullet}{u}\,t + \frac{1}{2}\overset{\bullet\bullet}{u}\,t^2 \tag{1}$$

A differential equation is a mathematical equation for an unknown function of one or several variables that relates the values of the function itself and its derivatives of various orders. Differential equations play a prominent role in engineering, physics, economics and other disciplines (Nakahira, 1985). Visualization of the airflow into a duct was modelled using the Navier-Stokes equations, a set of partial differential equations. Differential equations arise in many areas of science and technology; whenever, a deterministic relationship involving some continuously changing quantities and their rates of change (expressed as derivatives) is known or postulated. Using the extended mean value theorem, the Newmark-β method states that the first time derivative (velocity in the equation of motion) can be solved as:

$$\overset{\bullet}{u}_{n+1} = \overset{\bullet}{u}_n + \Delta t\,\overset{\bullet\bullet}{u}_\gamma \tag{2}$$

Where

$$\overset{\bullet\bullet}{u}_\gamma = (1-\gamma)\overset{\bullet\bullet}{u}_n + \gamma \overset{\bullet\bullet}{u}_{n+1} \qquad 0 \le \gamma \le 1 \tag{3}$$

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Therefore,

$$\dot{u}_{n+1} = \dot{u}_n + (1-\gamma)\Delta t\, \ddot{u}_n + \gamma\Delta t\, \ddot{u}_{n+1} \tag{4}$$

Since the acceleration also varies with time, however, the extended mean value theorem must also be extended to the second time derivative to obtain the correct displacement. Thus,

$$u_{n+1} = u_n + \Delta t\, \dot{u}_n + \frac{1}{2}\Delta t^2\, \ddot{u}_\beta \tag{5}$$

Where again

$$\ddot{u}_\beta = (1-2\beta)\ddot{u}_n + 2\beta\ddot{u}_{n+1} \qquad 0 \le \beta \le 1 \tag{6}$$

Newmark showed that a reasonable value of γ is 0.5, therefore the update rules are,

$$\dot{u}_{n+1} = \dot{u}_n + \frac{\Delta t}{2}(\ddot{u}_n + \ddot{u}_{n+1}) \tag{7}$$

$$u_{n+1} = u_n + \Delta t\, \dot{u}_n + \frac{1-2\beta}{2}\Delta t^2\, \ddot{u}_n$$
$$+ \beta\Delta t^2\, \ddot{u}_{n+1} \tag{8}$$

Setting β to various values between 0 and 1 can give a wide range of results. Typically, β = 1 / 4, which yields the constant average acceleration method, is used (Nakahira, 1990).

NEWMARK'S RELATIONSHIPS FOR ACCELERATION, VELOCITY AND DISPLACEMENT

Newmark expressed the velocities and displacements at the end of a time increment in terms of the known parameters at the beginning and the unknown acceleration at the end of the time step as:

$$\dot{X}_{n+1} = \dot{X}_n + (1-\gamma)\ddot{X}_n\,\Delta t + \gamma\ddot{X}_{n+1}\,\Delta t \tag{9}$$

$$X_{n+1} = X_n + \dot{X}_n\,\Delta t + \left(\frac{1}{2}-\beta\right)\ddot{X}_n\,\Delta t^2$$
$$+ \beta\ddot{X}_{n+1}\,\Delta t^2 \tag{10}$$

Where γ and β are parameters which can be varied at will. The value of γ is taken to be equal to $\frac{1}{2}$ as other

values will produce numerical damping. Equation (9) can therefore be written as (Zampieri, 2006):

$$\dot{X}_{n+1} = \dot{X}_n + \frac{1}{2}\ddot{X}_n\,\Delta t + \frac{1}{2}\ddot{X}_{n+1}\,\Delta t \tag{11}$$

In addition to the expressions for the displacement and velocities, the condition of dynamic equilibrium at the end of the time interval was:

$$M\ddot{X}_{n+1} + C_{n+1}\dot{X}_{n+1} + K_{n+1}X_{n+1} = P_{n+1} \tag{12}$$

The following expression for the acceleration at the end of the time step was yielded:

$$\ddot{X}_{n+1} = M^{-1}\left[P_{n+1} - C_{n+1}\dot{X}_{n+1} - K_{n+1}X_{n+1}\right] \tag{13}$$

Equations (11), (12) and (13) from the basis is used for the non-linear analysis of the structural systems by Newmark method. In general, unless β is taken as zero, the calculation procedure is for one time increment.

ANALYSIS USING THE NEWMARK METHOD

Step 1: Assume value for the acceleration vector $\ddot{X}_{n+1}$ at the end of the time step.

Step 2: Compute the velocity and displacement vectors $\dot{X}_{n+1}$ and X_{n+1} at the end of the time step using Equations (12) and (14).

Step 3: Update the stiffness and damping matrices.

Step 4: Calculate the acceleration vector $\ddot{X}_{n+1}$ using Equation (13).

Step 5: Compare the computed acceleration vector $\ddot{X}_{n+1}$ with the assumed one. If these are equal or within a permissible difference, the calculations for the step have been completed, if not.

Step 6: Assume the last calculated value of $\ddot{X}_{n+1}$ to be the initial value in the next iteration to step 2.

The rate or the convergence of the process towards the equality of the derived and assumed acceleration is a function of the time increment Δt and is fully discussed in (Zampieri, 2006). The criterion of the convergence
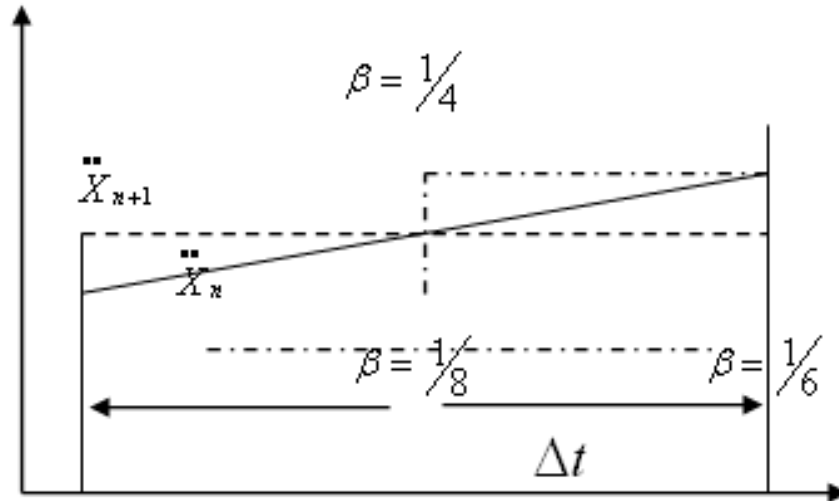
Figure 1. Acceleration during times.

given by Newmark is the equality of the assumed and calculated values of the acceleration at the end of the time step (Laier, 2010).

For MDOF systems, this involves the comparison of two vectors. Since it is highly unlikely that all the elements in the calculated vector will be equal to the corresponding elements in the assumed vector, a convergence criterion must be included in the process. The criterion may be based on a comparison of the values of the norm of the vectors and/or a comparison of the individual elements given either as a percentage or an absolute difference. The choice of type and magnitude of the permissible difference is a function of the required accuracy and is left for the experience and judgment of the analyst.

Interpretation of the parameter β

It is of interest to note how the acceleration during the time interval varies with variations in the values of β. Although, it is not possible to define a relationship for all values of β, for at least four values, the variation in the acceleration during the time step can be described. Three of these variations are shown in Figure 1.

It appears that a choice of $\beta = \frac{1}{2}$ corresponds to assuming a uniform value of the acceleration during the time interval which is equal to the mean of the initial and final value; a value of $\beta = \frac{1}{2}$ corresponds to assuming a step function with a uniform value to the initial value for the first half of the time increment and a uniform value equal to the final value for the second half; and a choice of $\beta = \frac{1}{6}$ corresponds to a linear change of acceleration during the time interval. The latter value of β results in the

basic equations as developed in the standard linear acceleration method. The main difference between the two algorithms is that in the Newmark method, equilibrium of the dynamic is forced for the total acceleration, velocity and displacement vectors at time t_{n+1}, whilst in the other equilibrium, it is only ensured for the incremental acceleration, velocity and displacement vectors. The latter may result in an accumulation of errors unless the acceleration is recalculated from the equations of motion at the end of the time step (Man-Chung, 2005).

The Newmark β = 0 method

The value β = 0 leads to an explicit algorithm for the Newmark method and is therefore discussed separately. When β = 0 the expression for the displacement at t_{n+1} is given as:

$$X_{n+1} = X_n + \dot{X}_n \Delta t + \frac{1}{2}\ddot{X}_n \Delta t^2 \qquad (14)$$

Substituting the expression for $\dot{X}_{n+1}$ given by Equation (12) into Equation (14) and solving for $\ddot{X}_{n+1}$ yields:

$$\ddot{X}_{n+1} = \left[M + \frac{1}{2}C_{n+1}\Delta t \right]^{-1}$$

$$\cdot \left[P_{n+1} - C_{n+1}\left(\dot{X}_n + \frac{1}{2}\ddot{X}_n \Delta t \right) - K_{n+1}X_{n+1} \right] \qquad (15)$$

X_{n+1} can be calculated from Equation (15). Once X_{n+1}

is known K_{n+1} can be calculated to form the product K_{n+1}. X_{n+1}. In some cases, this product can be built up as a column vector without formulating the global stiffness matrix (Nakahira, 1990; Peterson, 2008). The value of β = 0 corresponds to double pulses of acceleration at the beginning and end of the time interval with each double pulse consisting of a part equal to half of the acceleration times the time interval, one occurring just before the end of the preceding interval and the other just after the beginning of the next interval (Peterson, 2008).

Analysis using the Newmark (β = 0) method

With the value of K_n, C_n, X_n, $\dot{X}_n$ and $\ddot{X}_n$ given either at the end of the previous time step or from initial conditions, the calculation procedure of Newmark (β = 0) method for one time interval can be summed as follows:

Step 1: Calculate X_{n+1} from Equation (14).

Step 2: Set up K_{n+1} and C_{n+1}.

Step 3: Calculate $\ddot{X}_{n+1}$ using Equation (11).

Step4: Calculate $\dot{X}_{n+1}$ using Equation (12).

These steps complete the calculations for one time interval which may now be repeated for the next step. From the aforementioned, it can be seen that the calculation of $\ddot{X}_{n+1}$ as a function of the calculation of C_{n+1} is only possible for cases where the damping matrix is not a function of the velocity.

Stability and accuracy of the Newmark method

The Newmark method is of the second order accuracy and only conditionally stable. This means that the time interval Δt must be less than a certain value to ensure stability. The size of Δt is a function of the value of β and the smallest period of vibration of a system. Recommendations with respect to the choice of values for β and the size of time intervals are given in Benner and Martin (2010). A study concerning errors in the dynamic energy resulting from the use of the Newmark algorithms is investigated.

Potential dynamic work

This method is not commonly used for analysis because it is difficult but it is perfect in result and reality (Liqus,

2003).

In this method, total potential energy is used:

$$W = U + V \tag{16}$$

Where W = the total potential energy; U = the strain energy of the system and V = the potential energy of the loading.

The advantage of this method is decrease number of iterative per time step. There are a lot of ways for the analysis of the structure. Analyzing structure is divided into direct and indirect method. The method when applied to structural systems first predicts the displacement vector and then calculates the acceleration and velocity vectors at the middle of a time step. It then predicts the displacement vector and from that calculates the acceleration and velocity vectors at the end of the time step. The damping and stiffness matrices may or may not be update at this stage, depending upon the degree of non-linearity of the system. The equation of motion for a multi degree (MDOF) system can be written as:

$$MX"+ C (t) X' + k (T) X=P (T) \tag{17}$$

Where, M = mass matrix; C (t) = damping matrix; K (t) = stiffness matrix; X = displacement vector; X' = velocity vector; X" = acceleration vector; P(t) = load vector

Since m is a non-zero constant value, both sides of Equation (17) can be divided by m:

$$P = C (t) / m, Q = K (t) / m, F = P (t) / m$$

The equation can be written as:

$$X" + PX'+QX=F \tag{18}$$

Equation (18) is a linear differential equation if P and Q are independent of x.

$$X_{ji(k+1)} = X_{ji(k)} + S_{(k)}V_{ji(k)} \tag{19}$$

Where the suffices (k) and (k+1) denote the (k)th and (k+1)th iterate, respectively and where V_{ji} = the element of the direction vector, and $S_{(k)}$ = the step length which defines the position along $V_{ji(k)}$ where the total potential energy is a minimum. The expression for V_{ji} is, if the Fletcher-Reeves formulation of the conjugate gradients method is used, which is given by:

$$V_{ji(k)} = -g_{ji(k)} + \frac{\sum_{j=1}^{J}\sum_{i=1}^{3} g_{ji(k)} g_{ji(k)}}{\sum_{j=1}^{J}\sum_{i=1}^{3} g_{ji(k-1)} g_{ji(k-1)}} V_{ji(k-1)} \tag{20}$$

Calculation of the step length

The required polynomial for step length is found by substituting the expression for $X_{ji(k+1)}$ given in a suitable expression for the total potential energy:

$$e_{jn} = \frac{1}{2L_{jn} + e_{jn}}(a_1 + a_2 S + a_3 S^2) \qquad (21)$$

Where;

$$a_1 = \sum_{i=1}^{3}(2(x_{ni} - x_{ji})(x_{ni} - x_{ji}) + \qquad (21a)$$
$$(x_{ni} - x_{ji})(x_{ni} - x_{ji}))$$

$$a_2 = \sum_{i=1}^{3} 2((x_{ni} - x_{ji} + x_{ni} - x_{ji}) \qquad (21b)$$
$$(v_{ni} - v_{ji}))$$

$$a_3 = \sum_{i=1}^{3}(v_{ni} - v_{ji})^2 \qquad (21c)$$

And secondly to the expression for W in terms of the step length S and its derivative with respect to S as given:

$$W = C_1 S^4 + C_2 S^3 + C_3 S^2 + \qquad (22)$$
$$C_4 S + C_5$$

$$\partial W / \partial S = 4C_1 S^3 + 3C_2 S^2 + 2C_3 S + C_4 \qquad (23)$$

Where:

$$C_1 = \sum_{n=1}^{m}\left(\frac{EA}{2L(2L+e)^2}a_3^2\right)_n \qquad (23\text{-}a)$$

$$C_2 = \sum_{n=1}^{m}\left(\frac{EA}{L(2L+e)^2}a_2 a_3\right)_n \qquad (23\text{-}b)$$

$$C_3 = \sum_{n=1}^{m}\left(\frac{T_\circ}{2L+e}a_3 + \frac{EA}{2L(2L+e)^2}(a_2^2 + 2a_1 a_3)\right)_n \qquad (23\text{-}c)$$

$$C_4 = \sum_{n=1}^{m}\left(\frac{T_\circ}{2L+e}a_2 + \frac{EA}{L(2L+e)^2}a_1 a_2\right)_n - \qquad (23\text{-}d)$$
$$\sum_{j=1}^{J}\sum_{i=1}^{3}F_{ji}V_{ji}$$

$$C_5 = \sum_{n=1}^{m}\left(\frac{T_\circ}{2L+e}a_1 + \frac{EA}{2L(2L+e)^2}a_1^2\right)_n \qquad (23\text{-}f)$$
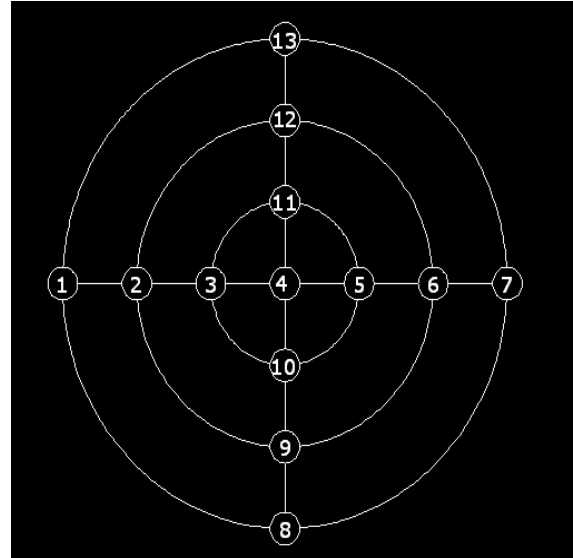$$-\sum_{j=1}^{J}\sum_{i=1}^{3}F_{ji}X_{ji}$$

Figure 2. Diagram of frame made.

NUMERICAL AND ANALYTICAL TESTING

The development of a mathematical control to ensure stability when using larger time steps is desirable. The mathematical model chosen is a circle flat net with 39 degrees of freedom. The circle flat net was also built as a finite element model and tested in order to verify the proposed theory given in this paper. The diagram is given in Figure 2. The model consisted of a circle net, with the beam element at 100 mm intervals. At the points of intersection, the circle net is clamped together with yield. Specifications of circle net are given as:

Overall dimensions: 600 x 600 mm; spacing of the cables: 100 mm; number of fixed boundary joints: 4; number of links: 12; Young's Modulus: 192.60 KN/mm^2 Size: 20 x 100 mm

The mass density influences the stability limit. Under some circumstance, scaling the mass density can potentially increase the efficiency of an analysis and the explicit dynamic uses a central difference rule to integrate the equation of motion explicitly through time. Deflections due to concentrated load at Node 4 is presented in Table 1.

The damping ratio of a practical structure depends on many factors, such as the structure system, the detail of joints, the foundation condition and so on. The damping ratio obtained from measurement of a practical structure varies not only with the detail of the structure itself, but also with the vibration amplitude, the measuring method and the data processing method. As a result, it is difficult to find out how the damping ratio is influenced by different factors 3D view of model is shown in figure 3.

In Figures 4 and 5, we presented the time history of the mean value of the displacement and mean value of the

Table 1. Deflections due to concentrated load at node 4.

Load (no) = 1000	FU method (T)	Finite element (E)	(T – E) / T*100
Z AXIS DEFLECTIONS (m) NODE 2 (LVDT)	263E-03	261E-03	0.76
Z AXIS DEFLECTIONS (m) NODE 3 (LVDT)	129E-03	127E-03	1.55
Z AXIS DEFLECTIONS (m) NODE 4 (LVDT)	95E-03	92E-03	3.16
Z AXIS DEFLECTIONS (m) NODE 5 (LVDT)	121E-03	120E-03	0.83
Z AXIS DEFLECTIONS (m) NODE 6 (LVDT)	97E-03	95E-03	2.06
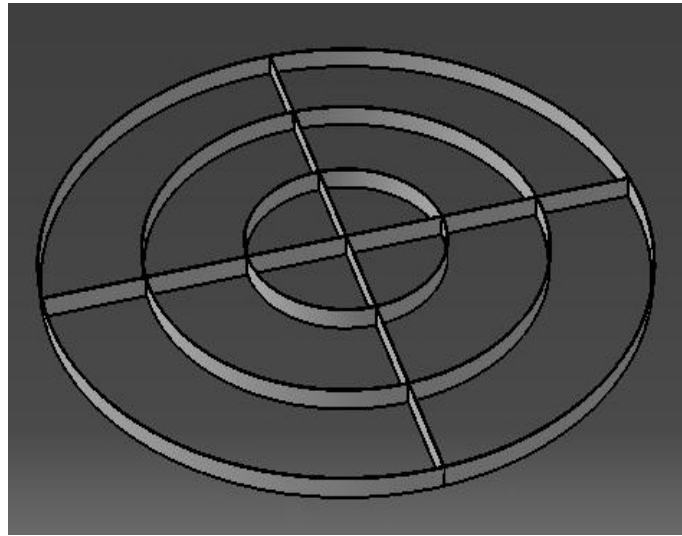
Figure 3. 3D view of model.
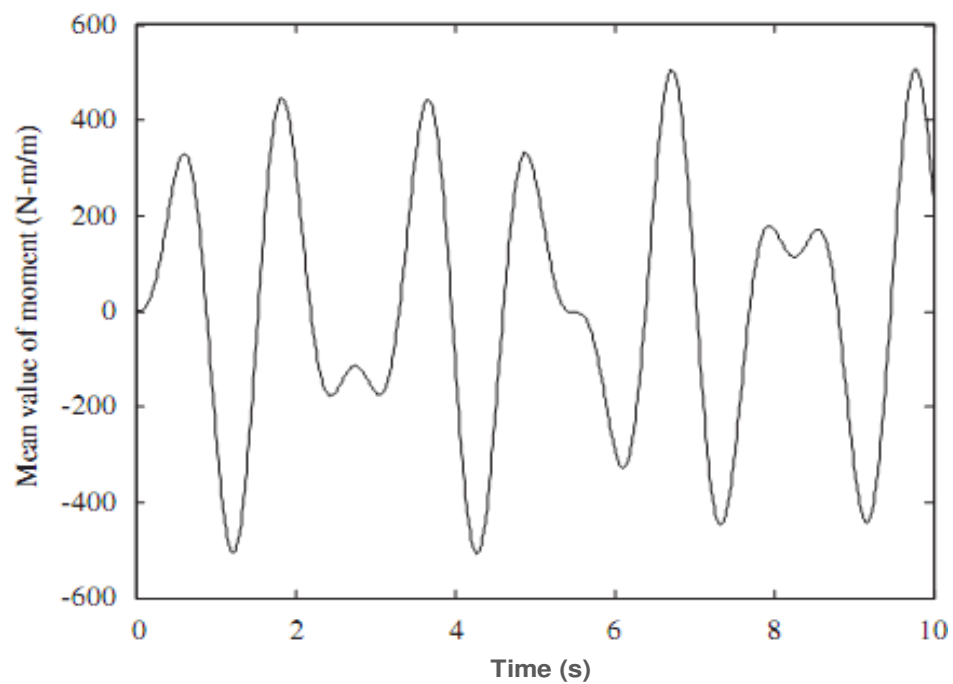
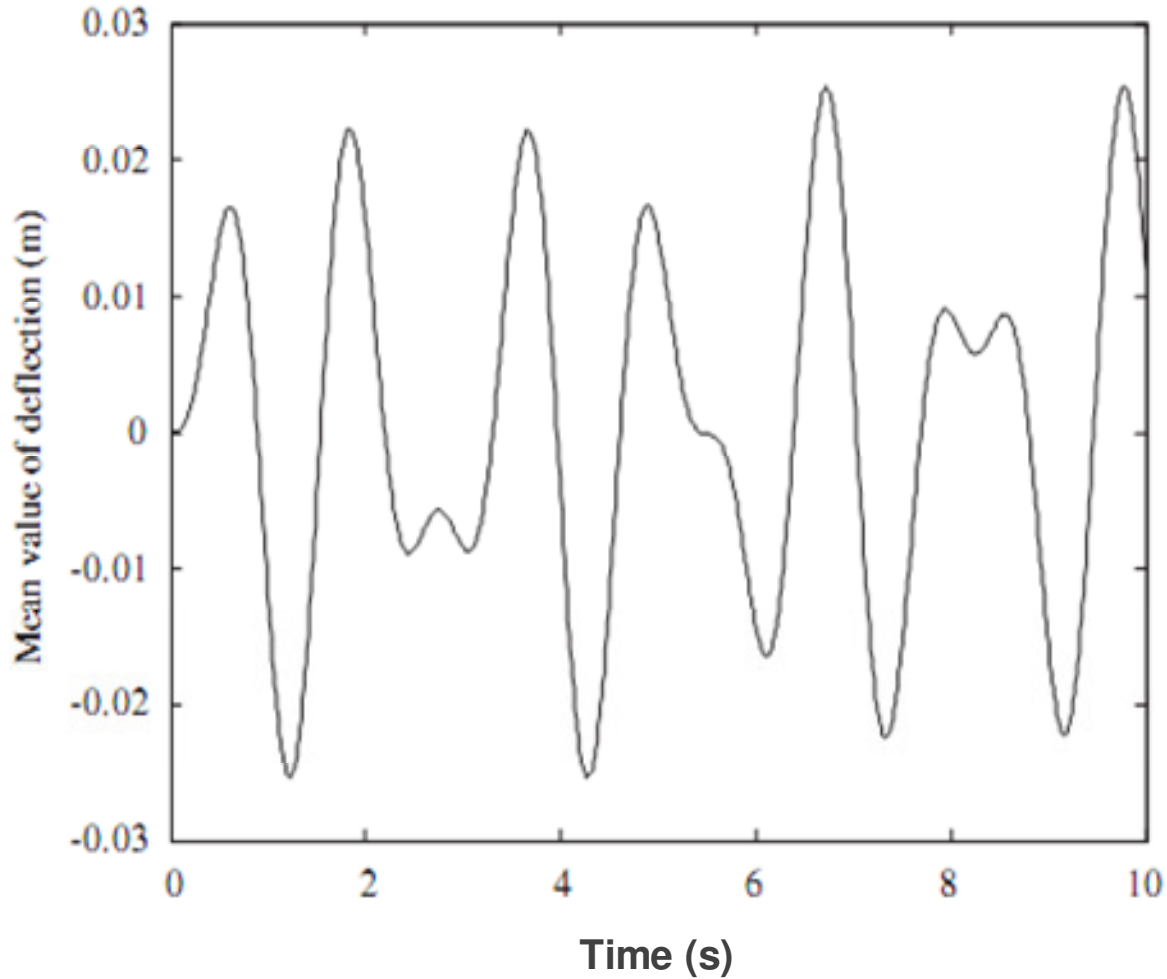**Figure 4.** Standard deviation of defection against the height of centre.

Figure 5. Time history of mean value of moment.

Table 2. Frequency on 5 mode shape.

Mode	Frequency of Fu method	Frequency (Hz) of finite element	Differentials percentage
1	1.4083	1.4321	1.69
2	1.42645	1.4151	0.8
3	1.5645	1.4945	4.47
4	1.7224	1.7041	1.06
5	2.1568	2.0823	3.45

moment at the center of the structures, which is in the direction of the longer side of the structures, when the shorter sides are simply supported. The differential percentage results are given in Table 2.

In Figures 6 and 7, the variations of the mean values of defection as well as the standard deviations of defection at any two arbitrary time instants are nonlinear when the height of center of the structures increases. This phenomenon is understandable since the stiffness of the

structures reduces as the height of center of the structures increases.

CONCLUSIONS

This proposed family method is very competitive with other integration methods for solving general structural dynamic problems, where the responses are dominated
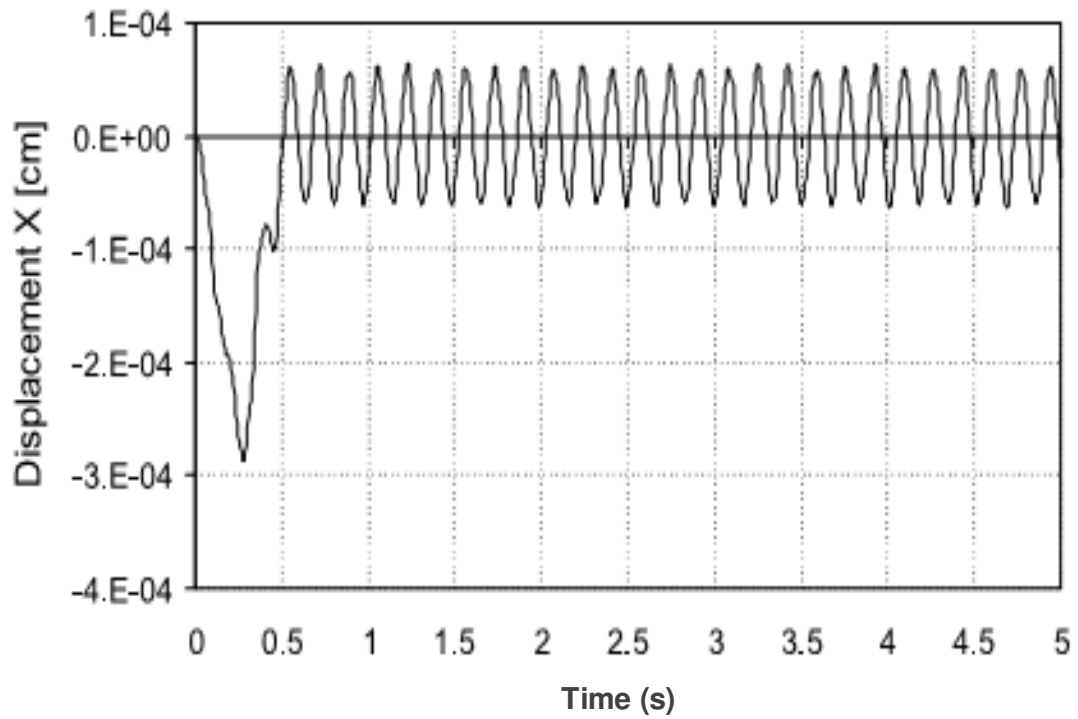
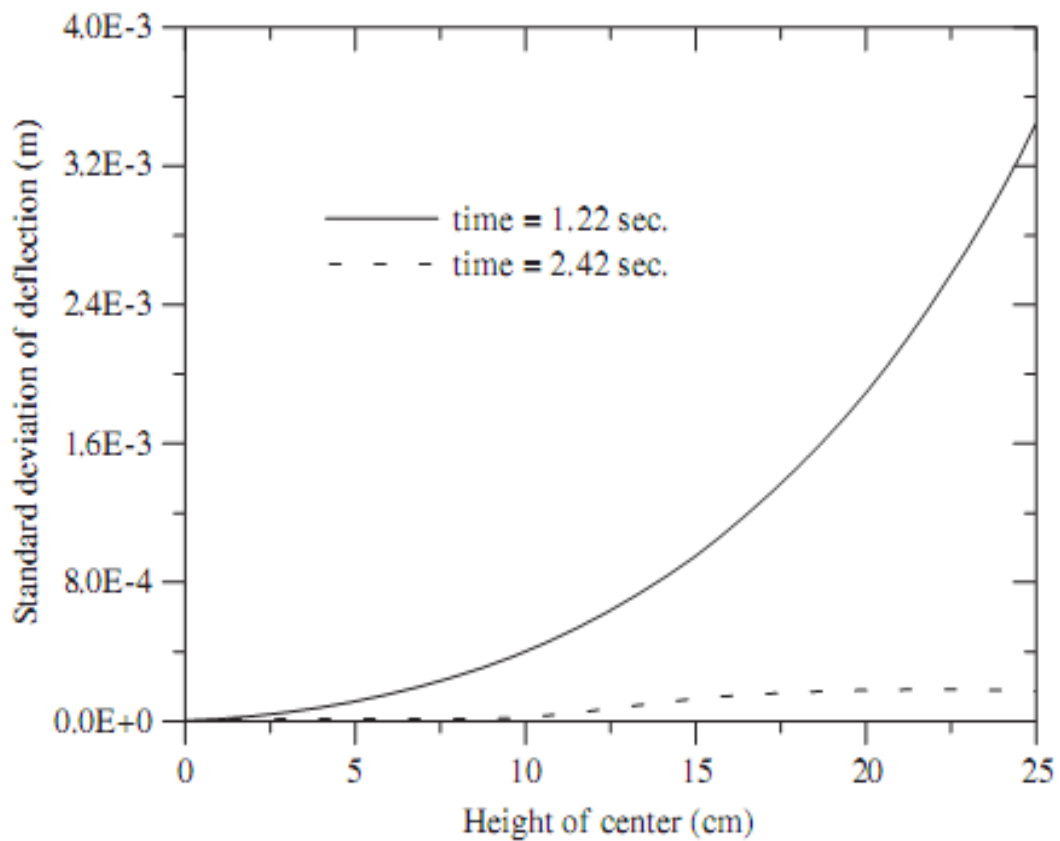**Figure 6.** Time history of mean value of defection.

Figure 7. Dynamic response in X direction.

by low frequency modes. This is because it can integrate the unconditional stability and the explicitness of each time step. Unconditional stability will allow it to use a larger time step to perform step-by-step integration and the explicitness of each time step involves no need of numerical iterations. The stability analysis indicates the presence of numerical damping and shows an effective stability limit of t which is the smallest natural period of a system. These stability conditions are closely related to the instantaneous degree of nonlinearity. The objective of this work was principally to develop a Newmark algorithm analysis theory and verify it by numerical and finite element testing. The proposed method was found to be stable for time steps equal to or less than half of the smallest periodic time of the system.

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