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Full Length Research Paper

Thick lenses systems

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The matrix optics formalism is applied to show that a simple Gaussian equation is enough to relate object and image distances in an optical system consisting of an arbitrary number of thick lenses in cascade immersed in air. First, the case of a single thick lens was studied. Applying sequentially the optical matrixes corresponding to refraction and displacement of the optical ray, and imposing certain conditions on the behavior of the bunch of rays being refracted by the lens, there were found characteristic parameters such as focal distance, back and front focal points, and principal planes. Then the equation relating object and image distances is found, which, after a coordinate transformation, becomes the well-known Gaussian equation, usually used to describe the more idealized case of thin lens. Further, the formalism is extended to compound systems of two, three and *N* thick lenses in cascade. It is also found that a simple Gaussian equation is sufficient to relate object and image distances no matter the number of lenses.

Key words: Matrix optics, thick lenses, back focal length, front focal length, principal planes, focus, multi-lenses system.

INTRODUCTION

The matrix optics formalism is a powerful tool that can be applied to the study of thick multi-lenses systems. It allows an intuitive approach in order to understand more deeply dispositive such as cameras, that may consist of several thick lenses in cascade. On the other hand, when the object-image equation for a single thick lens is deduced using matrix formalism, an expression that is complicated to deal with is obtained. However, through a coordinate transformation on the image and object distances, it is possible to obtain a very interesting result which simplifies the expression to a formula that is formally identical to the familiar expression used for thin lenses, that is, the Gaussian equation. In this work, such expression introducing optical parameters such as effective focal distance, front and back frontal length, and principal planes was fully derived. This is not a new result, as it has been pointed out in Hecht (2017), Jenkins and White (2001), and Born and Wolf (1999). Nevertheless, in this work this feature is demonstrated in a different way, by using the matrix formalism (not used in the references before mentioned for this specific result), which allows exploration of other situations. Further, this analysis was extended to a system formed by two thick lenses separated by a distance *d* which gave an interesting result that all optical parameters characterizing this system and the object-image equation

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are mathematically identical to the previously found expressions for a single thick lens. Furthermore, this analysis was extended to optical systems comprising three and *N* thick lenses, and formal identical results were once again found. Indeed, one of the more interesting results in this work is this generalization and, as a direct consequence, the possibility of expressing the relation between object and image distances for an arbitrary number of thick lenses in cascade just in a single equation. Actually, this general result is predicted by Feynman (2006) without demonstration.

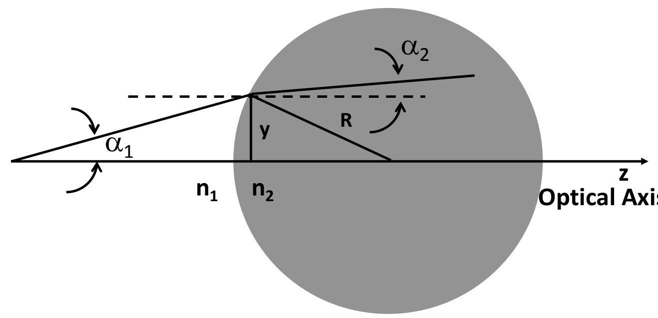

Figure 1. Refraction of an optical ray in a spherical surface.

MATRIX OPTICS

The optical matrices used for this study, that is, the refraction and displacement matrices, can be found in Hecht (2017). An important remark is that these matrixes are deduced for the paraxial approximation, making all our results to be valid in that case. In this paper, these results will be briefly derived and the interpretation and further application of these matrices to develop the expressions for spherical thick lenses will be shown. Also, the classical signal convention, which states that object distances to the left (right) of some reference interface will be positive (negative), whereas image distances placed to the right (left) of such interface will be positive (negative) was adopted. As for the curvature radios of surfaces, they will be positive (negative), if they were convex (concave). This convention is currently adopted in practically all specialized literature.

Let us suppose that an optical ray that makes an angle α_1 with respect to horizontal direction *z* (from here on, the optical axis of the system), is propagated in an optical medium with a refraction index n_1. It then hits a spherical medium with refraction index n_2 at a height *y* on the surface, measured from the optical axis, suffers refraction and, as a consequence, changes the angle with respect to the optical axes to α_2 (Figure 1).

In Figure 1, 'R' denotes the radium of the sphere. The equation that completely describes refraction is:

$$n_2\alpha_2 = n_1\alpha_1 + y\frac{n_1 - n_2}{R} \tag{1}$$

Note that in Equation 1, if 'R' tends to infinity, then the equation reduces to Snell's law (in paraxial form), as expected.

We can express Equation 1 in a matrix form, by defining the vector ($n\alpha$, y):

$$\begin{pmatrix} n_2\alpha_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{n_2 - n_1}{R} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} n_1\alpha_1 \\ y_1 \end{pmatrix} \tag{2}$$

Applying the usual operation of matrix product, the result

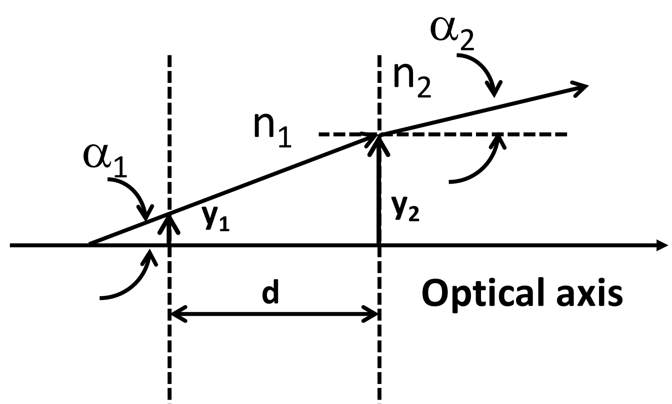

Figure 2. Propagation of an optical ray in an optical medium.

corresponding to the $n_2\alpha_2$ component reproduces Equation 1. The other term, $y_2 = y_1$, simply describes that no change in height was verified to the point of refraction.

Now we describe the change of height, that is, in '*y*' coordinate of the optical ray, as it propagates in an optical medium (Figure 2).

It is easy to see that, as the ray propagates a distance *d* measured over the optical axis in the medium with refraction index n_1, the changes in *y* coordinate is given by:

$$y_2 = y_1 + d\tan\alpha_1 . \tag{3}$$

As we work with the paraxial approximation, that is, $\alpha_1 \ll 1$, so, $\tan\alpha_1 \sim \alpha_1$, Equation 3 becomes:

$$y_2 = y_1 + d\alpha_1 . \tag{4}$$

Now, we can write Equation 4 in matrix form as:

$$\begin{pmatrix} n_2\alpha_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{d}{n_1} & 1 \end{pmatrix}\begin{pmatrix} n_1\alpha_1 \\ y_1 \end{pmatrix} . \tag{5}$$

Performing the product, the expression that corresponds to the element 'y_2', reproduces Equation 4, while the result corresponding to the component $n_2\alpha_2$ reproduces the Snell law, in paraxial form, at the interface between the two medium.

THICK LENSES IN AIR, FOCAL POINTS AND PRINCIPAL PLANES

The vantage of the matrix formalism is that it can be applied to several optical mediums in cascade by simply multiplying the matrixes corresponding to each element. Now, it will be applied to a spherical thick lens immersed in air, which consist in two refractive spherical surfaces of radios R_1 and R_2, separated by a distance *d* over the optical axis, which is the thickness of the lens, as shown in Figure 3.

In Figure 3, V_1 and V_2 are vertices points, localized at the intersections of the spherical surfaces of radio R_1 and R_2, with the optical axis, respectively. Also, it can be seen that an optical ray making an α_1 angle with respect to the optical axis is refracted by the first surface, at a height y_1, which propagates inside the lens (a distance *d* measured over the optical axis) and is then refracted by the second surface, leaving the lens with an angle of α_2 at a height y_2 with respect to the optical axis. In order to mathematically describe the journey of the optical ray through the lens, it can be written as:

$$\begin{pmatrix}\alpha_2\\ y_2\end{pmatrix}=\begin{pmatrix}1 & -\dfrac{1-n}{R_2}\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \dfrac{d}{n} & 1\end{pmatrix}\begin{pmatrix}1 & -\dfrac{n-1}{R_1}\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha_1\\ y_1\end{pmatrix} \quad (6)$$

As the lens is immersed in air, the indexes of refraction that multiply the angles in the vector components are both equal to one. Also, the elements corresponding to file 1 column 2 in both of the refraction matrixes in Equation 6 can be easily understood by looking to the general form of this element in Equation 2, where its numerator could be described in words as: "(minus) index of refraction to the right of the interface minus the index of refraction to the left of the interface (over R)". Then, the first interface (with radius 'R_1') is surrounded by air (index n_{ar}=1) to the left and glass (of index *n*) to the right. The same applies to the second surface (with radius 'R_2').

A convenient way to look at Equation 6 is from right to left: first, we have the input ray, hitting the lens with angle α_1 at a height y_1, it is refracted by the first surface, with radio R_1, then the ray is displaced a horizontal distance *d*, inside the lens, which has a refraction index *n*, and finally it is refracted by the second refractive surface, with radius R_2, and this gave the resultant optical ray leaving the lens, characterized by α_2 and y_2. Recalling that matrix product is not commutative, the correct order of the matrixes is fundamental in order to achieve the right result.

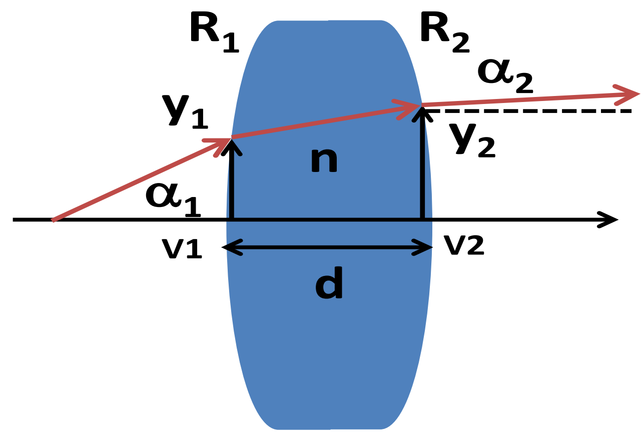

Figure 3. Thick lens.

Performing the usual operation of matrix product, Equation 6 results:

$$\begin{pmatrix}\alpha_2\\ y_2\end{pmatrix}=\begin{pmatrix}1+\dfrac{n-1}{n}\dfrac{d}{R_2}, & -\left[(n-1)\left(\dfrac{1}{R_1}-\dfrac{1}{R_2}\right)+\dfrac{(n-1)^2 d}{nR_1R_2}\right]\\ \dfrac{d}{n}, & 1-(n-1)\dfrac{d}{nR_1}\end{pmatrix}\begin{pmatrix}\alpha_1\\ y_1\end{pmatrix} \quad (7)$$

Note that, if *d* = 0, that is, when the thick lens becomes a thin lens, the element a_{12} in the matrix characterizing the lens, becomes equal to minus the inverse of the focal distance for a thin lens. From here on, this term will be written as -1/*f* (parameter *f* for thick lens will be analyzed later on).

Now, the back focal length, z_b, that is, the distance, measured from V_2, to which a thick lens focalizes a bunch of optical ray incident parallel to the optical axis, will be deduced (Figure 4).

Equivalently, it may be though as a plane wave front (being the wave fronts perpendicular to the optical rays) that is incident in the length, and, after refraction, the lens produced a convergent spherical wave, whose center is at back focal point. Some of these fronts are shown in Figure 4.

Now, the expression to find the height *y*, over the optical axis, of an optical ray that has travelled a horizontal distance *z* after leaving the lens will be written. In order to mathematically describe the whole process, the displacement matrix in air must be used in addition to the thick lens matrix in the following way:

$$\begin{pmatrix}\alpha_2\\ y\end{pmatrix}=\begin{pmatrix}1 & 0\\ z & 1\end{pmatrix}\begin{pmatrix}1+\dfrac{n-1}{n}\dfrac{d}{R_2} & -\dfrac{1}{f}\\ \dfrac{d}{n} & 1-(n-1)\dfrac{d}{nR_1}\end{pmatrix}\begin{pmatrix}\alpha_1\\ y_1\end{pmatrix} \quad (8)$$

Again, it is convenient to read Equation 8 from right to

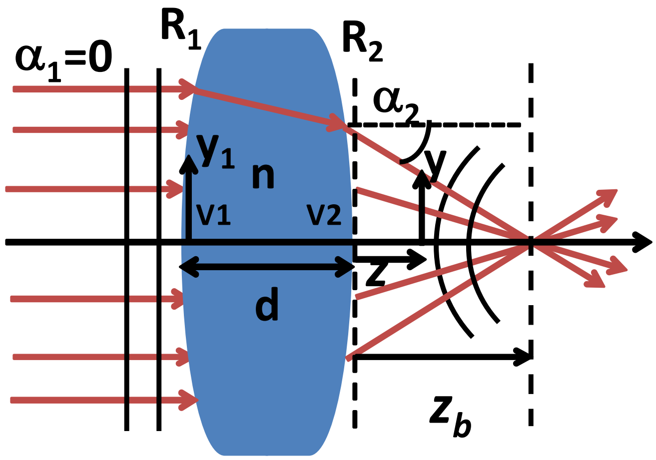

Figure 4. Finding back focal length for a thick lens.

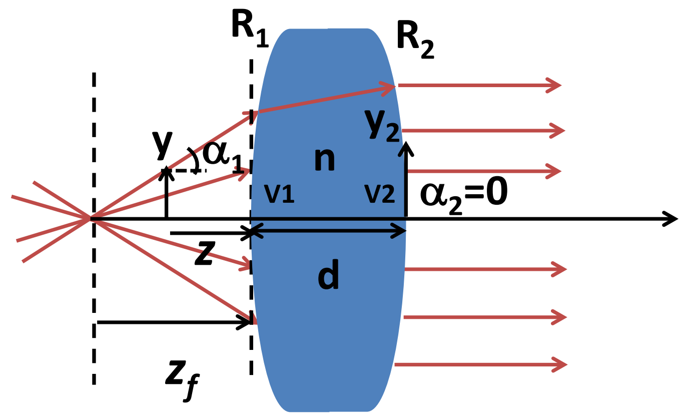

Figure 5. Finding front focal length for a thick lens.

left, that is, an input ray (α_1 y_1) is refracted by the lens, and then propagates an horizontal distance *z*. Performing the product of the matrixes, Equation 8 becomes:

$$\begin{pmatrix}\alpha_2\\ y\end{pmatrix}=\begin{pmatrix}1+\dfrac{n-1}{n}\dfrac{d}{R_2}, & -\dfrac{1}{f}\\ \left(1+\dfrac{n-1}{n}\dfrac{d}{R_2}\right)z+\dfrac{d}{n}, & 1-(n-1)\dfrac{d}{nR_1}-\dfrac{z}{f}\end{pmatrix}\begin{pmatrix}\alpha_1\\ y_1\end{pmatrix} \tag{9}$$

Now, the expression for *y*:

$$y=\left[\left(1+\frac{n-1}{n}\frac{d}{R_2}\right)z+\frac{d}{n}\right]\alpha_1+\left[1-(n-1)\frac{d}{nR_1}-\frac{z}{f}\right]y_1 \tag{10}$$

Noting that in our particular case α_1=0, we seek for that value of *z* that makes *y* null for all values of y_1 (Figure 4). This is by definition, the back focal point, z_b:

$$z_b=f\left[1-(n-1)\frac{d}{nR_1}\right] \tag{11}$$

Note that, if *d*=0, z_b=*f*, that is, the same result for thin lenses, as expected (and in this particular case, of course, the focal distance became the familiar expression for thin lenses).

A consideration of signal must be mentioned here. Back focal length is obviously an image point, and the corresponding signal convention should be applied, that is, if the result is positive, it should be localized to the right of V_2, as schematically shown in Figure 4, where a positive value of z_b is assumed. If negative, it should be localized to the left of V_2.

Now, we can deduce the front focal length, z_f, that is the distance measured from V_1 on the optical axis, in which an object point (or a source of spherical divergent wave fronts), must be located in order that a thick lens produce a bunch of parallel rays after refraction (corresponding to plane wave fronts). The situation is shown in Figure 5.

Now we write the expression to find (α_2,y_2):

$$\begin{pmatrix}\alpha_2\\ y_2\end{pmatrix}=\begin{pmatrix}1+\dfrac{n-1}{n}\dfrac{d}{R_2} & -\dfrac{1}{f}\\ \dfrac{d}{n} & 1-(n-1)\dfrac{d}{nR_1}\end{pmatrix}\begin{pmatrix}1 & 0\\ z & 1\end{pmatrix}\begin{pmatrix}\alpha_1\\ y\end{pmatrix} \tag{12}$$

Note the order in the matrixes product in Equation 12, indicating that a ray with height y and angle α_1 is propagated at a certain distance *z*, before it hits the length. Performing the product indicated in Equation 12:

$$\begin{pmatrix}\alpha_2\\ y_2\end{pmatrix}=\begin{pmatrix}1+\dfrac{n-1}{n}\dfrac{d}{R_2}-\dfrac{z}{f} & -\dfrac{1}{f}\\ \dfrac{d}{n}+\left(1-(n-1)\dfrac{d}{nR_1}\right)z & 1-(n-1)\dfrac{d}{nR_1}\end{pmatrix}\begin{pmatrix}\alpha_1\\ y\end{pmatrix} \tag{13}$$

Now, we analyze the term corresponding to α_2 from Equation 13:

$$\alpha_2=\left(1+\frac{n-1}{n}\frac{d}{R_2}-\frac{z}{f}\right)\alpha_1-\frac{y}{f}. \tag{14}$$

We must impose the conditions *y* = 0, and α_2 = 0, which are compatible with an object point over the optical axis producing, via the thick lens, an “image at infinity” (or a bunch of parallel rays), as indicated in Figure 5. As a consequence, the term between parentheses in Equation 14 should be equal to zero, for any value of α_1. The particular value of *z* that makes this condition be fulfilled

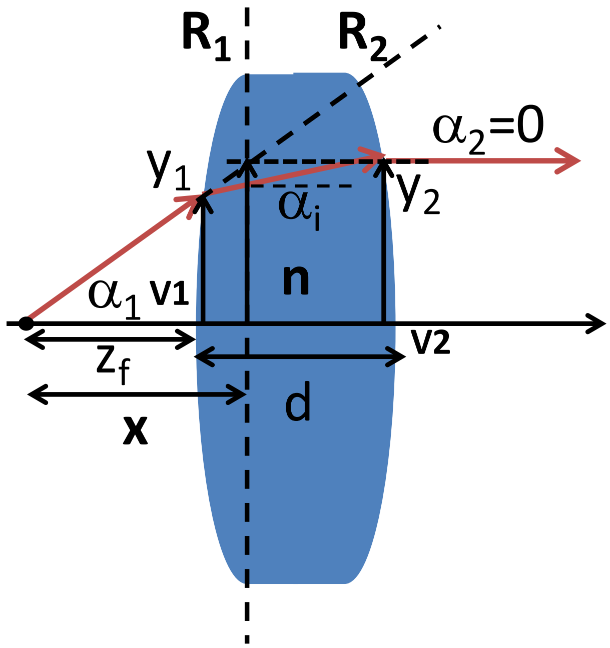

Figure 6. Finding *f* meaning for a thick lens.

is the z_f:

$$z_f = f\left[1+(n-1)\frac{d}{nR_2}\right]. \qquad (15)$$

If we set $d = 0$, that is, a thin lens, Equation 15 reduces, as expected, to the expression for thin lenses.

Front focal point is an object point and, if positive, should be located to the left of V_1 (as indicated in Figure 5, where that case was supposed). If it has negative signal, of course, it should be localized to the right of V_1.

We can now investigate the significance of *f* for thick lenses. In Figure 6, we see the front focal point for a given thick lens, and a generic optical ray, with α_1 angle, entering the length at y_1 and leaving the lens at y_2, with $\alpha_2 = 0$ (definition of z_f). We now project back the output ray until it intersects the projection of input ray. Doing this for every possible α_1 angle of those optical rays departing from z_f being refracted by the thick lens, we obtain a plane, perpendicular to the optical axis, at a distance x from z_f. This plane is called principal plane. In order to find the distance *x*, we observe that we can express the angle α_1, in the paraxial approximation as:

$$\alpha_1 \cong \frac{y_2}{x} \cong \frac{y_1}{z_f}. \qquad (16)$$

Then we express *x* as:

$$x = \frac{y_2}{y_1} z_f \qquad (17)$$

To find y_2 we write from Equation 7:

$$y_2 = \frac{d\alpha_1}{n} + y_1\left[1-\frac{(n-1)d}{nR_1}\right] \qquad (18)$$

Again, we must write the expression for α_1 in the paraxial approximation:

$$y_2 = \frac{dy_1}{nz_f} + y_1\left[1-\frac{(n-1)d}{nR_1}\right] \qquad (19)$$

And obtain:

$$\frac{y_2}{y_1} = \frac{d}{nz_f} + 1 - \frac{(n-1)d}{nR_1}. \qquad (20)$$

Now, using Equation 20 we can rewrite Equation 17 as:

$$x = \frac{y_2 z_f}{y_1} = \frac{d}{n} + z_f\left[1-\frac{(n-1)d}{nR_1}\right] \qquad (21)$$

Writing explicitly the expression for z_f, Equation 21 becomes:

$$x = \frac{d}{n} + f\left[1+(n-1)\frac{d}{nR_2}\right]\left[1-\frac{(n-1)d}{nR_1}\right] \qquad (22)$$

From Equation 22, and with a little algebra, it is easy to find that:

$$x = \frac{d}{n} + f\left[1-\frac{d}{nf}\right] = f \qquad (23)$$

In this way, we observed that the distance from z_f to the principal plane is given by *f*. We can also generate another plane, using the concept of back focal point in a totally symmetric way, and it can be found that the distance between this second plane and z_b is also *f* (Figure 7).

These two planes are called first principal plane (1PP), and second principal plane (2PP), and they can be seen in Figure 8.

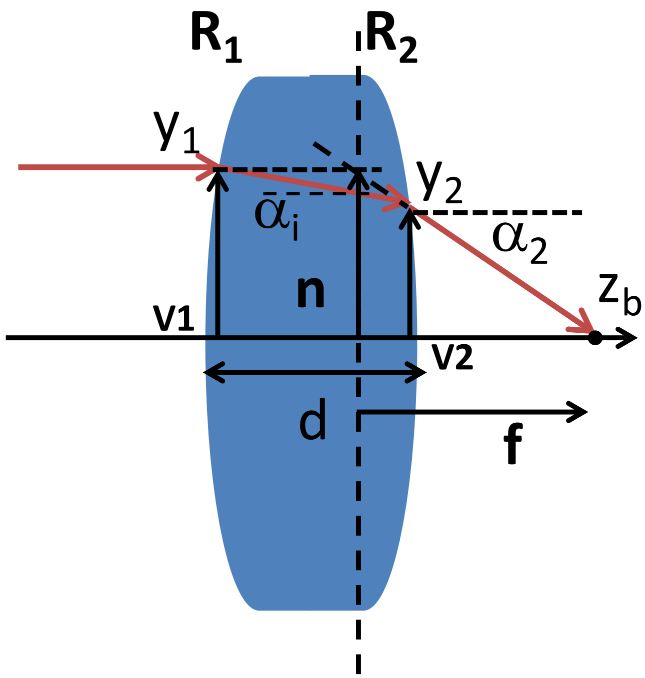

Figure 7. Focus and second principal plane for a thick lens.

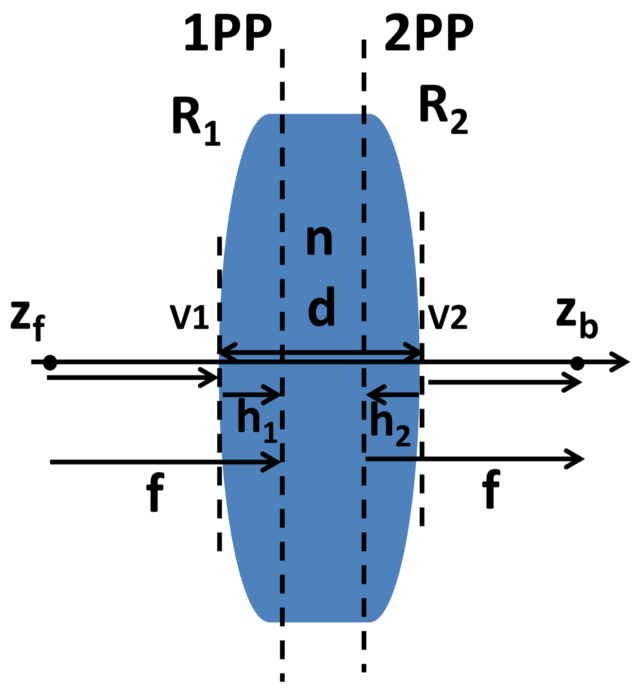

Figure 8. Principal planes for thick lenses.

It should be noted that the principal planes are seldom inside the lens, as shown in Figure 8. Depending on the specific lens data, they could be outside the lens.

For reasons that will be subsequently presented, it is convenient to localize the 1PP and 2PP with respect to the V_1 and V_2 vertices points, respectively; we call these distances h_1 and h_2 (Figure 8):

$$h_1 = f - z_f = -(n-1)\frac{df}{nR_2} \tag{24}$$

$$h_2 = z_b - f = -(n-1)\frac{df}{nR_1} \tag{25}$$

From the Equation 24, it can be seen that distance h_1 obeys the following signal convention: if positive, it is located to the right of V_1, but if negative, to the left of V_1. For h_2 (Equation 25), the signal convention for images with respect to V_2 holds, that is, if positive, h_2 must be located to the right of V_2 and if negative, to the left of V_2.

THICK LENS IN AIR AND IMAGE CONDITION

Now, the image condition for a thick lens in air will be deduced. We consider an object, with height h_o to a distance s_o from V_1. An image will be formed by the thick lens at distance s_i, measured with respect to V_2, and

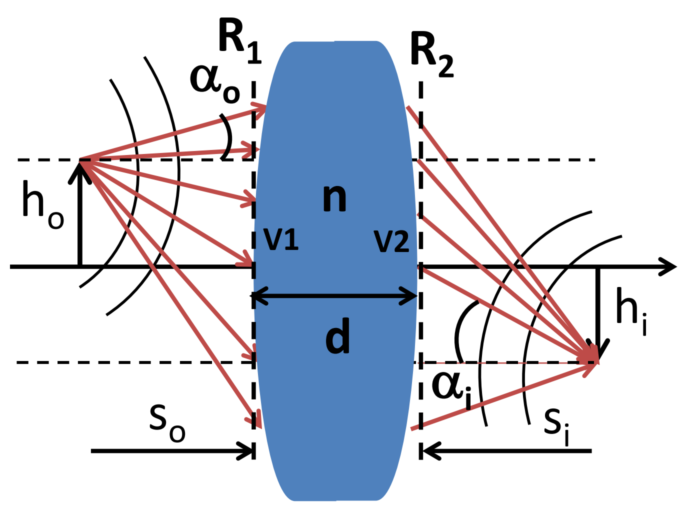

Figure 9. Image condition for a thick lens.

height h_i. The situation is visualized in Figure 9.

We see then that optical rays from object points will propagate a distance s_o over the optical axis; it will then be refracted by the thick lens and propagate a distance s_i, to the corresponding image point. In mathematical terms, we can write:

$$\begin{pmatrix}\alpha_i \\ h_i\end{pmatrix} = \begin{pmatrix}1 & 0 \\ s_i & 1\end{pmatrix}\begin{pmatrix}a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}1 & 0 \\ s_o & 1\end{pmatrix}\begin{pmatrix}\alpha_o \\ h_o\end{pmatrix} \tag{26}$$

Where the matrix *A* represent the thick lens, being its elements:

$$a_{11} = 1 + \frac{n-1}{n}\frac{d}{R_2} \quad (27)$$

$$a_{12} = -\frac{1}{f} = -\left[(n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) + \frac{(n-1)^2 d}{nR_1R_2}\right] \quad (28)$$

$$a_{21} = \frac{d}{n} \quad (29)$$

$$a_{22} = 1 - (n-1)\frac{d}{nR_1} \quad (30)$$

Performing the operation indicated in Equation 26, we arrive at:

$$\begin{pmatrix} \alpha_i \\ h_i \end{pmatrix} = \begin{pmatrix} a_{11} + a_{12}s_o & a_{12} \\ s_i(a_{11} + a_{12}s_o) + a_{21} + a_{22}s_o & s_i a_{12} + a_{22} \end{pmatrix}\begin{pmatrix} \alpha_o \\ h_o \end{pmatrix} \quad (31)$$

In order to mathematically find the image condition, we can consider an object point, for example, the one at the top of the object. This point is a front of spherical divergent waves (some of which are shown in Figure 9, perpendicular to the optical rays). The lens, in order to form an image, should convert this divergent wave front into a convergent one, whose convergence point is the image point corresponding to the object point at the top of the object. Then, we write, from Equation 31, the expression corresponding to h_i:

$$h_i = [s_i(a_{11} + a_{12}s_o) + a_{21} + a_{22}s_o]\alpha_o + (s_i a_{12} + a_{22})h_o \quad (32)$$

It is easy to see that the height of a specific image point (for example the one corresponding to the top), should not depend on the angle α_0. In this way, the image condition is:

$$\frac{\partial h_i}{\partial \alpha_o} = 0 \quad (33)$$

Applying Equation 33 in Equation 32

$$s_i(a_{11} + a_{12}s_o) + a_{21} + a_{22}s_o = 0 \quad (34)$$

We explore Equation 34 when *d*=0, that is, a thin lens. In that case, the matrix elements become: a_{11} = 1, a_{12} = -1/*f*, where *f* is now reduced to the focal distance for a thin lens, a_{21}=0 e a_{22}=1. With these values for the matrix elements, it is trivial to see that Equation 34 becomes the well-known Gaussian formula for thin lenses.

We rewrite Equation 34 as:

$$s_i a_{11} + a_{12}s_o s_i + a_{21} + a_{22}s_o = 0 . \quad (35)$$

Now, we seek for new coordinates, s'_i and s'_o, that allows to write Equation (35) in a simpler way. We can write:

$$s_o = s_o' + o . \quad (36)$$

$$s_i = s_i' + i . \quad (37)$$

Introducing Equations 36 and 37 in Equation 35, and rearranging in a convenient way:

$$s_i'(a_{11} + a_{12}o) + a_{12}s_o's_i' + s_o'(a_{12}i + a_{22}) + a_{12}oi + a_{21} + a_{22}o + ia_{11} = 0 \quad (38)$$

From Equation 38, it is easy to see that, in order to obtain an expression that looks like the Gaussian formula for thin lens, the following conditions should be imposed:

$$a_{11} + a_{12}o = 1 \quad (39)$$

$$a_{12}i + a_{22} = 1 \quad (40)$$

$$a_{12}oi + a_{21} + a_{22}o + ia_{11} = 0 \quad (41)$$

Solutions to Equations 39 and 40 give:

$$o = \frac{1 - a_{11}}{a_{12}} . \quad (42)$$

$$i = \frac{1 - a_{22}}{a_{12}} . \quad (43)$$

Equations 42 and 43 make Equation 41 to be fulfilled (this demonstration requires a certain amount of algebra, and it is shown in the Appendix). As a consequence, we were able to find new object and image distances that allow us to write Equation 35 in a familiar way. Replacing the corresponding elements of the lens matrix in Equations 42 and 43, we obtain:

$$o = f(n-1)\frac{d}{nR_2} \quad (44)$$

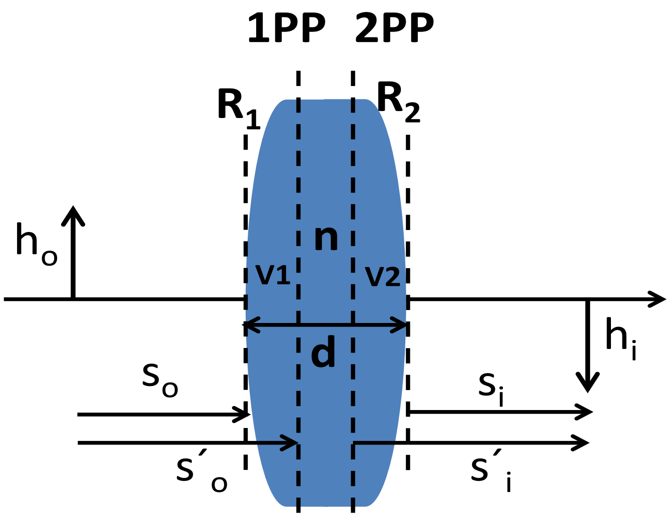

Figure 10. Principal planes and image condition for a thick lens.

$$i = -f(n-1)\frac{d}{nR_1}. \quad (45)$$

These expressions are closely related to Equations 24 and 25, that is, the distances of the principal planes to their respective vertices points. As the distances s_o and s_i are measured with respect to the vertices, this means that, in order to obtain a simpler lens equation, we must measure the object and image distances with respect to the 1PP and 2PP respectively, as shown in Figure 10.

As a consequence, we can replace the thick lens by the principal planes, and write the Gauss equation:

$$\frac{1}{s'_o} + \frac{1}{s'_i} = \frac{1}{f}. \quad (46)$$

s'_o follows the signal convention for objects; if positive (negative), it is located at the left (right) of 1PP, while s'_i follows the signal convention for images; if positive (negative), it is located at the right (left) of 2PP.

Of course, if we make d=0, that is, a thin lens, the two principal planes coalesce into one, and in that case, $s_o = s'_o$ and $s_i = s'_i$.

TWO THICK LENSES SYSTEMS

The formalism developed in the last sections can be generalized to a two lenses system.

Let us suppose a system of two thick lenses, characterized by matrixes A and B, and separated by a distance d. The lens A has refraction index n_a, thickness d_a, radius R_{1a}, and R_{2a} whereas for the lens B, index n_b, thickness d_b, radius R_{1b}, and R_{2b}. Matrix elements for each lens are the same as in Equations 27 to 30, with the specific parameters for each lens:

$$a_{11} = 1 + \frac{n_a - 1}{n_a}\frac{d_a}{R_{2a}}. \quad (47)$$

$$a_{12} = -\frac{1}{f_a} = -\left[(n_a - 1)\left(\frac{1}{R_{1a}} - \frac{1}{R_{2a}}\right) + \frac{(n_a - 1)^2 d_a}{n_a R_{1a} R_{2a}}\right] \quad (48)$$

$$a_{21} = \frac{d_a}{n_a}. \quad (49)$$

$$a_{22} = 1 - \frac{(n_a - 1)}{n_a}\frac{d_a}{R_{1a}}. \quad (50)$$

$$b_{11} = 1 + \frac{(n_b - 1)}{n_b}\frac{d_b}{R_{2b}}. \quad (51)$$

$$b_{12} = -\frac{1}{f_b} = -\left[(n_b - 1)\left(\frac{1}{R_{1b}} - \frac{1}{R_{2b}}\right) + \frac{(n_b - 1)^2 d_b}{n_b R_{1b} R_{2b}}\right]. \quad (52)$$

$$b_{21} = \frac{d_b}{n_b} \quad (53)$$

$$b_{22} = 1 - \frac{(n_b - 1)}{n_b}\frac{d_b}{R_{1b}} \quad (54)$$

The system is shown in Figure 11. Note that we have now four vertices points, V_{1a}, V_{2a}, V_{1b}, and V_{2b}.

We write for the matrix of the two lenses system:

$$M_{ab} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \quad (55)$$

Where the elements a_{ij} and b_{ij} are specified in the Equations 47 to 54.

The resultant matrix has elements:

$$M_{ab11} = b_{11}a_{11} + b_{12}(da_{11} + a_{21}). \quad (56)$$

$$M_{ab12} = b_{11}a_{12} + b_{12}(da_{12} + a_{22}). \quad (57)$$

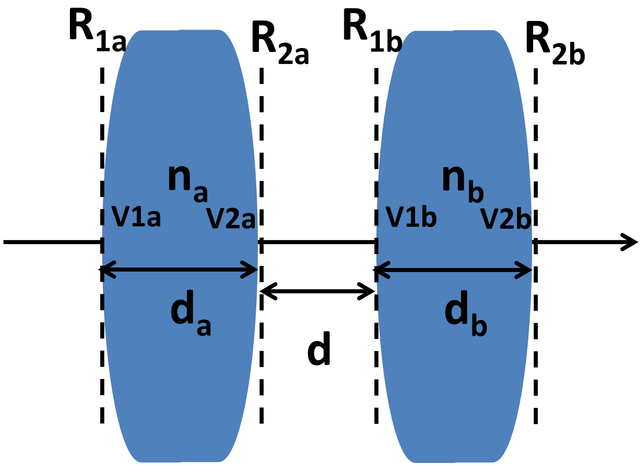

Figure 11. Two thick lenses separated a distance '*d*'.

$$M_{ab21} = b_{21}a_{11} + b_{22}\left(da_{11} + a_{21}\right). \quad (58)$$

$$M_{ab22} = b_{21}a_{12} + b_{22}\left(da_{12} + a_{22}\right). \quad (59)$$

It may be illustrative to develop the element M_{ab12} of the compound system using the matrix elements explicitly (Equations 47 to 54). As already seen for one single lens, the element "*1-2*" is related to the focal distance. After a little algebra, Equation 57 becomes:

$$M_{ab12} = -\frac{1}{f_a} - \frac{1}{f_b} + \frac{d}{f_a f_b} + \frac{(n_a - 1)d_a}{n_a R_{1a} f_b} - \frac{(n_b - 1)d_b}{n_b R_{2b} f_a} \quad (60)$$

It will be shown that this expression corresponds to the negative of the inverse of the focal distance for the two thick lenses, separated by a distance *d*. Here, it is referred to as focal distance, f_{ab}. Meanwhile, it is easy to see that, making $d_a = d_b = 0$, that is, two thin lenses, Equation 60 reduces to the well-known expression of the (negative of the inverse) focal distance for two thin lenses separated by a distance *d* (where f_a and f_b, of course, are reduced to the expressions of focal distances for thin lenses).

We can determine the back focal length for this system, proceeding in an analogous way as we did for the single thick lens (Equation 8), that is,

$$\begin{pmatrix} \alpha_2 \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}\begin{pmatrix} M_{ab11} & M_{ab12} \\ M_{ab21} & M_{ab22} \end{pmatrix}\begin{pmatrix} \alpha_1 \\ y_1 \end{pmatrix} \quad (61)$$

Performing the product of matrixes in Equation 61 and writing the expression for *y*:

$$y = \left(zM_{ab11} + M_{ab21}\right)\alpha_1 + \left(zM_{ab12} + M_{ab22}\right)y_1 \quad (62)$$

As we already know, for back focal point, α_1=0, and we must compute the value of *z* that makes *y* null for all values of y_1:

$$z_b = -\frac{M_{ab22}}{M_{ab12}} \quad (63)$$

Writing M_{ab12} as $-1/f_{ab}$, so:

$$z_b = f_{ab}M_{ab22} \quad (64)$$

Now, we take a look at the expression for z_b of the single thick lens (Equation 11). It may be written, in term of the matrix element of the lens, as $z_b = fa_{22}$.

Also, we can find the front focal length for the compound system of thick lenses, using a reasoning analogous to that used in Equations 12 to 15, the result is:

$$z_f = f_{ab}M_{ab11} \quad (65)$$

Once more, we observe that the expression is formally identical to the corresponding front focal length for a single thick lens (Equation 15), that may be written using the matrix element as $z_f = fa_{11}$.

The next logical step is to find the principal planes for the two thick lens system and find the relation with the focal distance, f_{ab} (Figure 12).

In Figure 12, we find the 1PP for the two thick lens system using, as before, the property of front focal point, zf, and projecting back the y_2 coordinate. Thereafter, we write for the angle α_1 (in the paraxial approximation):

$$\alpha_1 \cong \frac{y_2}{x} \cong \frac{y_1}{z_f}. \quad (66)$$

Then, we can write for x:

$$x = \frac{y_2}{y_1} z_f. \quad (67)$$

which is, of course, formally the same equation obtained when the single thick lens was studied. Now, in order to obtain an expression for y_2 in our two thick lenses systems, we write:

$$\begin{pmatrix} \alpha_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} M_{ab11} & M_{ab12} \\ M_{ab21} & M_{ab22} \end{pmatrix}\begin{pmatrix} \alpha_1 \\ y_1 \end{pmatrix} \quad (68)$$

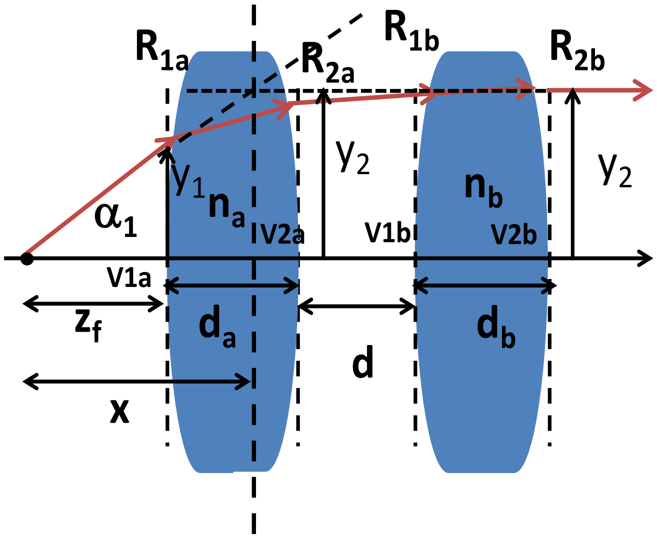

Figure 12. Finding principal planes for two thick lenses separated a distance '*d'*.

Note that this expression allows us to obtain the height and angle, α_2 and y_2, of a ray leaving the compound system at the last surface (R_{2b}), as a function of the corresponding parameters α_1 and y_1 of a ray hitting the systems at the first surface (R_{1a}).

From Equation 68, we obtain for y_2:

$$y_2 = M_{ab21}\alpha_1 + M_{ab22}y_1 \tag{69}$$

Substituting α_1 from Equation 66 in Equation 69:

$$y_2 = M_{ab21}\frac{y_1}{z_f} + M_{ab22}y_1 \tag{70}$$

from which it is trivial to obtain:

$$\frac{y_2}{y_1} = \frac{M_{ab21}}{z_f} + M_{ab22} \tag{71}$$

Now we can substitute Equation 71 in Equation 67 and obtain (remember that $z_f = f_{ab}M_{ab11}$):

$$x = M_{ab21} + f_{ab}M_{ab11}M_{ab22} \tag{72}$$

Our next task is to demonstrate that $x = f_{ab}$. It is a long algebra but also, one of the principal points of this work, so it will be developed here. We write Equation 72 explicitly as a function of the matrix elements:

$$x = b_{21}a_{11} + b_{22}(da_{11} + a_{21}) + f_{ab}[b_{11}a_{11} + b_{12}(da_{11} + a_{21})][b_{21}a_{12} + b_{22}(da_{12} + a_{22})] \tag{73}$$

Doing some basic algebra, Equation 73 can be written as:

$$x = b_{21}a_{11} + b_{22}da_{11} + b_{22}a_{21} + f_{ab}[b_{11}a_{11}b_{21}a_{12} + a_{11}db_{22}\left(b_{11}a_{12} + b_{12}da_{12} + b_{12}a_{22}\right) + b_{11}a_{11}b_{22}a_{22} + b_{12}da_{11}b_{21}a_{12} + \\ + b_{12}a_{21}b_{21}a_{12} + b_{12}a_{21}b_{22}da_{12} + b_{12}a_{21}b_{22}a_{22}] \tag{74}$$

We see, in Equation 74, that the term within the parentheses, multiplying the $a_{11}db_{22}$ factor, is the M_{ab12} matrix element (Equation 57), which is equal to -1/ f_{ab}. We write then:

$$x = b_{21}a_{11} + b_{22}da_{11} + b_{22}a_{21} + f_{ab}(b_{11}a_{11}b_{21}a_{12} - a_{11}\frac{d}{f_{ab}}b_{22} + b_{11}a_{11}b_{22}a_{22} + b_{12}da_{11}b_{21}a_{12} + b_{12}a_{21}b_{21}a_{12} + b_{12}a_{21}b_{22}da_{12} + b_{12}a_{21}b_{22}a_{22}) \tag{75}$$

In Equation 75, it is possible now to cancel the second and fifth terms. Doing this and after a rearrangement, we obtain:

$$x = b_{21}a_{11}[1 + f_{ab}\left(b_{11}a_{12} + b_{12}da_{12}\right)] + b_{22}a_{21} + f_{ab}[b_{11}a_{11}b_{22}a_{22} + b_{12}a_{21}b_{21}a_{12} + a_{21}b_{22}b_{12}\left(da_{12} + a_{22}\right)] \tag{76}$$

We can express the factors between parenthesis in the second and sixth terms using the fact that: $-1/f_{ab}=b_{11}a_{12} + b_{12}(da_{12} + a_{22})$, then:

$$x = b_{21}a_{11}[1 + f_{ab}(-\frac{1}{f_{ab}} - b_{12}a_{22})] + b_{22}a_{21} + f_{ab}[b_{11}a_{11}b_{22}a_{22} + b_{12}a_{21}b_{21}a_{12} + a_{21}b_{22}(-\frac{1}{f_{ab}} - b_{11}a_{12})] \tag{77}$$

After that, it can be obtained:

$$x = f_{ab}b_{12}b_{21}a_{21}a_{12} - f_{ab}b_{12}b_{21}a_{11}a_{22} + f_{ab}b_{11}a_{11}b_{22}a_{22} - f_{ab}b_{11}a_{12}a_{21}b_{22} \quad (78)$$

And, finally:

$$x = f_{ab}\left(a_{21}a_{12} - a_{11}a_{22}\right)\left(b_{12}b_{21} - b_{11}b_{22}\right) \quad (79)$$

Now, the elements a_{ij} and b_{ij} must be explicitly replaced in Equation 79, using Equations 47 to 54. Working with the first factor:

$$a_{21}a_{12} - a_{11}a_{22} = -\frac{d_a}{n_a}\frac{1}{f_a} - \left(1 + \frac{n_a - 1}{n_a}\frac{d_a}{R_{2a}}\right)\left(1 - \frac{n_a - 1}{n_a}\frac{d_a}{R_{1a}}\right). \quad (80)$$

After a simple algebra, it is obtained:

$$a_{21}a_{12} - a_{11}a_{22} = -1 - \frac{d_a}{n_a}\frac{1}{f_a} + \frac{d_a}{n_a}\left(\frac{n_a - 1}{R_{1a}} - \frac{n_a - 1}{R_{2a}} + \frac{(n_a - 1)^2}{n_a}\frac{d_a}{R_{2a}R_{1a}}\right) \quad (81)$$

The term between parentheses in Equation 81 is equal to $1/f_a$ (Equation 48), so:

$$a_{21}a_{12} - a_{11}a_{22} = -1 \quad (82)$$

Obviously, $b_{21}b_{12} - b_{11}b_{22} = -1$ also. With this, Equation 79 can be finally written as:

$$x = f_{ab} \quad (83)$$

With this demonstration, it has been shown that the interpretation of f_{ab} as the focal distance of the two thick lenses systems is correct, and totally analogous to the case of one single thick lens. We can now write the expression for focal distance of two thick lenses separated by a distance d as:

$$-\frac{1}{f_{ab}} = -\frac{1}{f_a} - \frac{1}{f_b} + \frac{d}{f_a f_b} + \frac{(n_a - 1)d_a}{n_a R_{1a} f_b} - \frac{(n_b - 1)d_b}{n_b R_{2b} f_a} \quad (84)$$

The principal plane found and shown in Figure 12 is the first principal plane. Of course, the second principal plane is easily found and is verified that its distance to the back focal point, z_b, is also f_{ab}.

The next steps now are, in close analogy with the single thick lens case, to find the distances of the 1PP and 2PP, h_1 and h_2, which will be measured with respect to the outermost vertices points of the system, that is, V_{1a} and V_{2b} respectively, and also determine the image condition.

In order to find the distances of the principal planes with respect to the outermost vertices points, we write:

$$h_1 = f_{ab} - z_f = f_{ab}\left(1 - M_{ab11}\right). \quad (85)$$

$$h_2 = z_b - f_{ab} = f_{ab}\left(M_{ab22} - 1\right). \quad (86)$$

Where Equations 64 and 65 were used to express front and back focal distances for the two thick lens system. As before, h_1 and h_2 follow the signal convention for object and images, respectively.

To find image condition, we measure the object distance, s_o, with respect to V_{1a} and image distance, s_i, with respect to V_{2b} (Figure 13).

Writing the matrix equation for the object-two thick lenses system-image:

$$\begin{pmatrix} \alpha_i \\ h_i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ s_i & 1 \end{pmatrix}\begin{pmatrix} M_{ab11} & M_{ab12} \\ M_{ab21} & M_{ab22} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ s_o & 1 \end{pmatrix}\begin{pmatrix} \alpha_o \\ h_o \end{pmatrix} \quad (87)$$

We see that Equation 87 is formally identical to Equation 26, thus, it is verified that the image condition is:

$$s_i M_{ab11} + M_{ab21}s_i s_o + M_{ab21} + s_o M_{ab22} = 0 \quad (88)$$

which is formally identical to Equation 35; therefore, we can now follow the same steps of Equations 36 to 45, to the extent of writing the image condition for the two thick lenses system in the simple form of a Gaussian equation. To achieve this, of course, object and image distances must be measured with respect to the principal planes. With this condition, a complicated system as shown in Figure 13 is reduced to the one shown in Figure 14.

Furthermore, that system is described analytically by:

$$\frac{1}{s'_o} + \frac{1}{s'_i} = \frac{1}{f_{ab}}. \quad (89)$$

THREE AND N THICK LENSES, IMAGE CONDITION

The expressions for three, four and N thick lenses systems follow the same general structure. Consider a three thick lenses system, A, B, and C, being the a, b lenses identical to the system seen in the last section, and C lens is located to a distance d_{bc} to the right of B lens. The c lens, of course has matrix elements mathematically identical to those represented in the Equations 47 to 50, and characterized with usual parameters, refraction index n_c, thickness d_c, radius R_{1c}, and R_{2c}. The matrix for this system is:

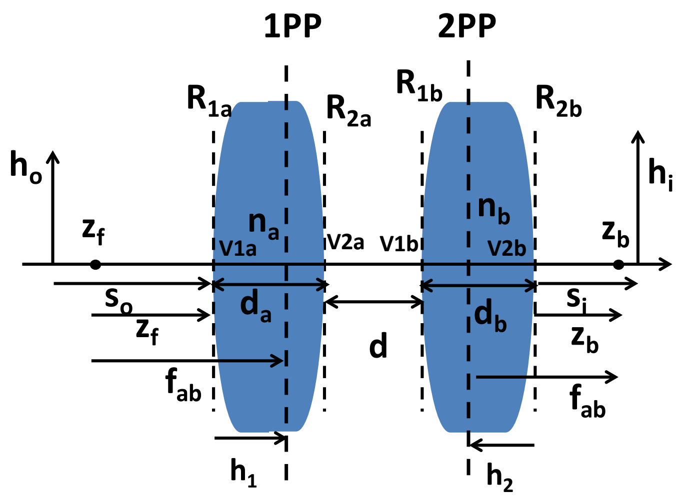

Figure 13. Image condition for two thick lenses. Principal planes are located respect to outermost vertices points, V_{1a} and V_{2b}.

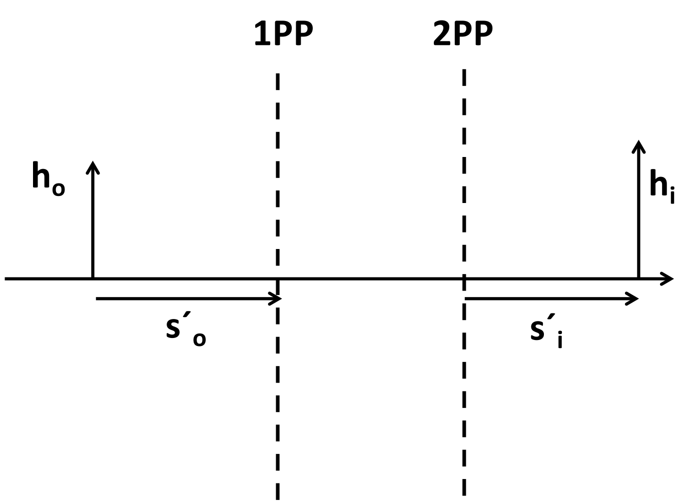

Figure 14. Object and Image coordinates for two thick lenses with respect to the principal planes.

$$M_{abc} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ d_{bc} & 1 \end{pmatrix} \begin{pmatrix} M_{ab11} & M_{ab12} \\ M_{ab21} & M_{ab22} \end{pmatrix} \tag{90}$$

And the matrix elements:

$$\begin{aligned} M_{abc11} &= c_{11}M_{ab11} + c_{12}\left(d_{bc}M_{ab11} + M_{ab21}\right) \\ M_{abc12} &= c_{11}M_{ab12} + c_{12}\left(d_{bc}M_{ab12} + M_{ab22}\right) \\ M_{abc21} &= c_{21}M_{ab11} + c_{22}\left(d_{bc}M_{ab11} + M_{ab21}\right) \\ M_{abc22} &= c_{21}M_{ab12} + c_{22}\left(d_{bc}M_{ab12} + M_{ab22}\right) \end{aligned} \tag{91}$$

All the characteristics parameters for this system, as the front focal point, back focal point, etcs, can be obtained from these elements, using the expressions already seen for the two thick lenses system, and substituting the corresponding matrix element to the three lenses system. Of course, the focal distance is associated to the M_{abc12} element as usual. A closer look at Equation 91 allows us to write them in a compact form:

$$M_{abc,ij} = c_{i1}M_{ab,1j} + c_{i2}\left(d_{bc}M_{ab,1j} + M_{ab,2j}\right) \tag{92}$$

Where ij represents the index of the matrix.

Finally, we can write the expression for the matrix elements of a system of N thick lenses:

$$M_{ab\ldots n,ij} = n_{i1}M_{ab\ldots n-1,1j} + n_{i2}\left(d_{n-1n}M_{ab\ldots n-1,1j} + M_{ab\ldots n-1,2j}\right) \tag{93}$$

Where n_{ij} represents the matrix elements corresponding to the last thick lens of the system (that is, the last on the right), and d_{n-1n}, the distance between this last lens and the one immediately at left, $N-1$. Of course, the other distances are contained in the $M_{ab\ldots n-1,ij}$ matrix.

As before, we can use the matrix elements of Equation 93 and the general expressions already derived, to find any parameter of interest of an optical system formed by N thick lenses. Also, we can determine the two principal planes for this system and find image distances with a simple Gaussian equation (Equation 89), keeping in mind that the focal distance is always related to the $M_{ab..n,12}$ element and the object and image distances should be measured with respect to the 1PP and 2PP, respectively.

CONCLUSIONS

In this work, thick lenses immersed in air using the matrix formalism in the paraxial approximation have been studied and their characteristic parameters, such as focal distance, back and front focal points, principal planes and object-image equation has been determined. A simple Gaussian equation, identical to the one used for the idealized case of thin lenses is found to relate the object and image distances, when these are measured with respect to the principal planes. Also, the analysis has been extended to two thick lenses separated by a distance d. Also, an expression for the focal distance of this system, and the back and front focal points and principal planes has been determined. It was settled that the mathematical expressions determining these parameters are formally identical to the corresponding to the single thick lens, as a function of the matrix elements characterizing both systems. Also in the case of the two thick lenses, as soon as the principal planes were determined, it was found that a simple Gaussian equation relates the object and image distances measured with respect to the principal planes. Finally, extending this analysis to three and *N* thick lenses systems, analogous results was found.

It is interesting to observe that all well-known expressions relating thin lenses systems can be deduced from the more realistic formulas derived in this work, simply by making the parameter d_a of the lens equal to zero, which is, passing from a thick to a thin lens.

This result is of particular interest given that in an optical system with an arbitrary number of lenses, allow us to find a single pair of principal planes associated to that system, and determine image distances (given the object distances) by resolving a simple Gaussian equation. This last result that was mentioned by Feynman (2006) in his lectures book, although well known to all specialists in the field, had not been formally derived in the basic or intermediate optic literature to the best of the researcher's knowledge.

CONFLICT OF INTERESTS

The author has not declared any conflict of interests.

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APPENDIX

Our task here is to demonstrate that the following identity is true:

$$a_{12}oi + a_{21} + a_{22}o + ia_{11} = 0 . \qquad \text{(A1)}$$

Where *o* and *i* are given by Equations 42 and 43. Substituting then those Equations in Equation (A1) we have:

$$a_{12}\left(\frac{1-a_{11}}{a_{12}}\right)\left(\frac{1-a_{22}}{a_{12}}\right) + a_{21} + a_{22}\left(\frac{1-a_{11}}{a_{12}}\right) + \left(\frac{1-a_{22}}{a_{12}}\right)a_{11} = 0 \qquad \text{(A2)}$$

After the basic algebra and further simplifications, Equation (A2) is reduced to:

$$\frac{1}{a_{12}} + a_{21} - \frac{a_{22}a_{11}}{a_{12}} = 0 \qquad \text{(A3)}$$

Substituting the expressions for the matrix elements a_{ij}, Equation (A3) becomes:

$$-f_a + \frac{d_a}{n_a} + f_a\left(1-(n_a-1)\frac{d_a}{n_a R_{1a}}\right)\left(1+\frac{n_a-1}{n_a}\frac{d_a}{R_{2a}}\right) = 0 \qquad \text{(A4)}$$

After the usual algebra:

$$f_a \frac{d_a}{n_a}\left[(n_a-1)\left(\frac{1}{R_{2a}} - \frac{1}{R_{1a}}\right) - \frac{(n_a-1)^2}{n_a}\frac{d_a}{R_{1a}R_{2a}}\right] + \frac{d_a}{n_a} = 0 \qquad \text{(A5)}$$

Clearly, the term between brackets in Equation (A5) is $-1/f_a$, and the identity is verified.