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The improved Riccati equation mapping method for constructing many families of exact solutions for a nonlinear partial differential equation of nanobiosciences

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In this paper we apply the improved Riccati equation mapping method to construct many families of general exact solutions of a nonlinear partial differential equation involving parameters of special significant for nanobiosciences and biophysics, namely, the equation of nano-ionic currents along microtublues. Comparison between our new results and the well-known results are given. The nonlinear equation elaborated here is quite original and proposed in the context of important nanosciences problems related with cell signaling. It could be even of basic importance for explanation of congnitive processes in neurons. We can successfully recover the previously known exact solutions that have been found by other methods. The proposed method in this article can be applied to many other nonlinear partial differential equations.

Key words: Improved Riccati equation mapping method, exact traveling wave solutions, nonlinear partial differential equation of nano-ionic currents along microtublues.

INTRODUCTION

In recent years, the exact traveling wave solutions of nonlinear partial differential equations (PDEs) have been investigated by many authors who are interested in nonlinear physical phenomena. Many powerful methods have been presented by those authors such as the inverse scattering transform method (Ablowitz and Clarkson, 1991), the Hirota's bilinear method (Hirota, 1971), the Painleve expansion method (Weiss et al., 1983; Kudryashov, 1988, 1990, 19991), the Backlund truncated method (Miura, 1978), the exp-function method (He and Wu, 2006; Yusufoglu, 2008; Bekir, 2009, 2010; Aslan, 2011), the tanh-function method (Abdou, 2007; Fan, 2000; Zhang and Xia, 2008; Yusufoglu and Bekir, 2008), the Jacobi elliptic function method (Chen and Wang, 2005; Liu et al., 2001; Lu, 2005), the (G'/G)-expansion method (Wang et al., 2008; Zhang et al., 2008; Zayed, 2009, 2010; Bekir, 2008; Ayhan and Bekir, 2012; Aslan, 2010, 2011, 2012a, b), the generalized Riccati equation mapping method (Zhu 2008; Zayed and Arnous, 2013), and so on.

In the present paper, we shall use the improved Riccati equation mapping method to find the exact solutions of a nonlinear PDE of special significant for nanobiosciences. The main idea of this method is that the traveling wave solutions of nonlinear equations can be expressed by

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polynomials in Q where $Q = Q(\xi)$ satisfies the generalized Riccati equation $Q' = r + pQ + qQ^2$ where $\xi = kx + \omega t$ where r, p, k, ω and q are constants. The degree of this polynomial can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in the given nonlinear equation, the coefficients of this polynomial can be obtained by solving a set of algebraic equations resulted from the process of using the proposed method.

The objective of this paper is to apply the improved Riccati equation mapping method for finding many families of exact traveling wave solutions of the following nonlinear PDE which describes the problem of transfer of ions along microtubules (Sekulic et al., 2011, Sataric et al., 2010):

$$\frac{L^2}{3}u_{xxx} + \frac{Z^{3/2}}{L}(\chi G_0 - 2\delta C_0)uu_t + 2u_x + \frac{ZC_0}{L}u_t + \frac{1}{L}(RZ^{-1} - G_0 Z)u = 0 \quad (1)$$

Here $R = 0.34 \times 10^9 \Omega$ is the resistance of the electric elementary unit (EEU), $L = 8 \times 10^{-9} m$ is the length of one tubulin heterodimer protein (EEU), $C_0 = 1.8 \times 10^{-15} F$ is the total maximal capacitance of the EEU. $G_0 = 1.1 \times 10^{-13} si$ is the conductance of pertaining the nano-pores (NPs) and $Z = 5.56 \times 10^{10} \Omega$ represents the characteristic impedance of the system. The parameters δ and χ describe the nonlinearity of the EEU's capacitance and conductance of existing between protofilaments of microtubule, respectively. The detailed consideration of microtubules in the context of nonlinear transmission lines is presented in (Sekulic et al., 2011, Sataric et al., 2009). The physical details of the derivation of Equation (1) describing the time-space voltage of ionic pulse is elaborated in (Sataric et al., 2010). Recently, Equation (1) has been discussed in (Sekulic et al., 2011) by using the modified extended tanh-function method, where its exact solutions have been found. Comparison between our new results in this article and the well - known results obtained in (Sataric et al., 2010) will be investigated later.

Description of the improved Riccati equation mapping method

We suppose that a nonlinear PDE in the following from:

$$P(u, u_x, u_t, u_{xx}, u_{tt}, \ldots) = o, \quad (2)$$

where $u = u(x,t)$ is an unknown function, P is a polynomial in $u = u(x,t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved, Let us now give the main steps for solving Equation (1) using the improved Riccati equation mapping method (Zhu, 2008; Zayed and Arnous, 2013):

Step 1: We look for its traveling wave solution in the form

$$u(x,t) = u(\xi), \quad \xi = kx + \omega t, \quad (3)$$

where k, ω are constants. Substituting Equation (3) into Equation (2) gives the nonlinear ordinary differential equation (ODE) for $u(\xi)$ as follows:

$$H(u, u', u'', \ldots) = 0, \quad (4)$$

Here H is a polynomial in $u(\xi)$ and its total derivatives $u', u'', u''', \ldots$ such that $u' = \frac{du}{d\xi}, u'' = \frac{d^2u}{d\xi^2}, \ldots$.

Step 2: We suppose that the solution of the ODE (2.3) can be expressed as follows:

$$u(\xi) = \sum_{i=-m}^{m} a_i Q^i(\xi), \quad (5)$$

where $a_i (i = 0, \pm 1, \pm 2, \ldots, \pm m)$ are constants to be determined later such as $a_m \neq 0$ or $a_{-m} \neq 0$, and $Q = Q(\xi)$ is the solution of generalized Riccati equation:

$$Q' = r + pQ + qQ^2, \quad (6)$$

where r, p and q are constants , such that $q \neq 0$.

Step 3: We determine the positive integer m in Equation (5) by balancing the nonlinear terms and the highest order derivatives of $u(\xi)$ in Equation (4).

Step 4: Substituting Equation (5) and along with Equation (5) into Equation (4) and then equating all the coefficients of $Q^i (i = 0, \pm 1, \pm 2, \ldots, \pm m)$ to zero yield a system of algebraic equations which can be solved by using the Maple or Mathematica to find the values of the constants $a_i(-m, \ldots, m)$ and k, ω.

Step 5: It is well-known (Zhu, 2008; Zayed and Arnous, 2013) that Equation (6) has many families of solutions:

Type 1: When $\Delta = p^2 - 4qr > 0$ and $pq \neq 0$ or

$qr \neq 0$ we have

$$\Phi_1(\xi) = -\frac{1}{2q}[p + \sqrt{\Delta}\tanh(\frac{\sqrt{\Delta}}{2}\xi)],$$

$$\Phi_2(\xi) = -\frac{1}{2q}[p + \sqrt{\Delta}\coth(\frac{\sqrt{\Delta}}{2}\xi)],$$

$$\Phi_3(\xi) = -\frac{1}{2q}[p + \sqrt{\Delta}(\tanh(\sqrt{\Delta}\xi) \pm i\,\mathrm{sech}(\sqrt{\Delta}\xi))], \quad i\sqrt{-1}$$

$$\Phi_4(\xi) = -\frac{1}{2q}[p + \sqrt{\Delta}(\coth(\sqrt{\Delta}\xi) \pm \mathrm{csch}(\sqrt{\Delta}\xi))],$$

$$\Phi_5(\xi) = -\frac{1}{4q}[2p + \sqrt{\Delta}(\tanh(\frac{\sqrt{\Delta}}{4}\xi) \pm \coth(\frac{\sqrt{\Delta}}{4}\xi))],$$

$$\Phi_6(\xi) = \frac{1}{2q}[-p + \frac{\sqrt{\Delta(A^2 + B^2)} - A\sqrt{\Delta}\cosh(\sqrt{\Delta}\xi)}{A\sinh(\sqrt{\Delta}\xi) + B}],$$

$$\Phi_7(\xi) = \frac{1}{2q}[-p - \frac{\sqrt{\Delta(B^2 - A^2)} + A\sqrt{\Delta}\cosh(\sqrt{\Delta}\xi)}{A\sinh(\sqrt{\Delta}\xi) + B}],$$

where *A* and *B* are two non-zero real constants satisfying $B^2 - A^2 > 0$,

$$\Phi_8(\xi) = \frac{2r\cosh(\frac{\sqrt{\Delta}}{2}\xi)}{\sqrt{\Delta}\sinh(\frac{\sqrt{\Delta}}{2}\xi) - p\cosh(\frac{\sqrt{\Delta}}{2}\xi)},$$

$$\Phi_9(\xi) = \frac{-2r\sinh(\frac{\sqrt{\Delta}}{2}\xi)}{p\sinh(\frac{\sqrt{\Delta}}{2}\xi) - \sqrt{\Delta}\cosh(\frac{\sqrt{\Delta}}{2}\xi)},$$

$$\Phi_{10}(\xi) = \frac{2r\cosh(\frac{\sqrt{\Delta}}{2}\xi)}{\sqrt{\Delta}\sinh(\sqrt{\Delta}\xi) - p\cosh(\sqrt{\Delta}\xi) \pm i\sqrt{\Delta}}, \quad i = \sqrt{-1}$$

$$\Phi_{11}(\xi) = \frac{2r\sinh(\frac{\sqrt{\Delta}}{2}\xi)}{-p\sinh(\sqrt{\Delta}\xi) + \sqrt{\Delta}\cosh(\sqrt{\Delta}\xi) \pm \sqrt{\Delta}},$$

$$\Phi_{12}(\xi) = \frac{4r\sinh(\frac{\sqrt{\Delta}}{4}\xi)\cosh(\frac{\sqrt{\Delta}}{4}\xi)}{-2p\sinh(\frac{\sqrt{\Delta}}{4}\xi)\cosh(\frac{\sqrt{\Delta}}{4}\xi) + 2\sqrt{\Delta}\cosh^2(\frac{\sqrt{\Delta}}{2}\xi) - \sqrt{\Delta}},$$

Type 2: When $\Delta = p^2 - 4qr < 0$ and $pq \neq 0$ or $qr \neq 0$ we have:

$$\Phi_{13}(\xi) = \frac{1}{2q}[-p + \sqrt{-\Delta}\tan(\frac{\sqrt{-\Delta}}{2}\xi)],$$

$$\Phi_{14}(\xi) = -\frac{1}{2q}[p + \sqrt{-\Delta}\cot(\frac{\sqrt{-\Delta}}{2}\xi)],$$

$$\Phi_{15}(\xi) = \frac{1}{2q}[-p + \sqrt{-\Delta}(\tan(\sqrt{-\Delta}\xi) \pm \sec(\sqrt{-\Delta}\xi))],$$

$$\Phi_{16}(\xi) = -\frac{1}{2q}[p + \sqrt{-\Delta}(\cot(\sqrt{-\Delta}\xi) \pm \csc(\sqrt{-\Delta}\xi))],$$

$$\Phi_{17}(\xi) = \frac{1}{4q}[-2p + \sqrt{-\Delta}(\tan(\frac{\sqrt{-\Delta}}{4}\xi) - \cot(\frac{\sqrt{-\Delta}}{4}\xi))],$$

$$\Phi_{18}(\xi) = \frac{1}{2q}[-p + \frac{\pm\sqrt{-\Delta(A^2 - B^2)} - A\sqrt{-\Delta}\cos(\sqrt{-\Delta}\xi)}{A\sin(\sqrt{-\Delta}\xi) + B}],$$

$$\Phi_{19}(\xi) = \frac{1}{2q}[-p - \frac{\pm\sqrt{-\Delta(A^2 - B^2)} - A\sqrt{-\Delta}\sin(\sqrt{-\Delta}\xi)}{A\sin(\sqrt{-\Delta}\xi) + B}],$$

where *A* and *B* are two non-zero real constants satisfying $A^2 - B^2 > 0$,

$$\Phi_{20}(\xi) = -\frac{2r\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{\sqrt{-\Delta}\sin(\frac{\sqrt{-\Delta}}{2}\xi) + p\cos(\frac{\sqrt{-\Delta}}{2}\xi)},$$

$$\Phi_{21}(\xi) = \frac{2r\sin(\frac{\sqrt{-\Delta}}{2}\xi)}{-p\sin(\frac{\sqrt{-\Delta}}{2}\xi) + \sqrt{-\Delta}\cos(\frac{\sqrt{-\Delta}}{2}\xi)},$$

$$\Phi_{22}(\xi) = -\frac{2r\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{\sqrt{-\Delta}\sin(\sqrt{-\Delta}\xi) + p\cos(\sqrt{-\Delta}\xi) \pm \sqrt{-\Delta}},$$

$$\Phi_{23}(\xi) = \frac{2r\sin(\frac{\sqrt{-\Delta}}{2}\xi)}{-p\sin(\sqrt{-\Delta}\xi) + \sqrt{-\Delta}\cos(\sqrt{-\Delta}\xi) \pm \sqrt{-\Delta}},$$

$$\Phi_{24}(\xi) = \frac{4r\sin(\frac{\sqrt{-\Delta}}{4}\xi)\cos(\frac{\sqrt{-\Delta}}{4}\xi)}{-2p\sin(\frac{\sqrt{-\Delta}}{4}\xi)\cos(\frac{\sqrt{-\Delta}}{4}\xi) + 2\sqrt{-\Delta}\cos^2(\frac{\sqrt{-\Delta}}{2}\xi) - \sqrt{-\Delta}},$$

Type 3: When $r = 0\, and\ \ pq \neq 0$ we have:

$$\Phi_{25}(\xi) = \frac{-pd}{q[d + \cosh(p\xi) - \sinh(p\xi)]},$$

$$\Phi_{26}(\xi) = -\frac{p[\cosh(p\xi) + \sinh(p\xi)]}{q[d + \cosh(p\xi) + \sinh(p\xi)]},$$

where *d* is an arbitrary constant.

Type 4: When $r = p = 0\, and\ \ q \neq 0$ we have
$\Phi_{27}(\xi) = \frac{-1}{q\xi + c_1}$, where c_1 is an arbitrary constant.

Step 6: Substituting the well known solutions of Equation (6) listed in Step 5 into Equation (5), we have many families of exact solutions of Equation (2).

Many families of exact traveling wave solutions for Equation (1)

In this section, we apply the proposed method of the description of the improved Riccati equation mapping method, to find many families of new exact traveling wave solutions of Equation (1). To this end, we use the wave transformation

$$u(x,t)=u(\xi),\qquad \xi=\frac{1}{L}x-\frac{c}{\tau}t\ , \tag{7}$$

where $\tau = RC_0 = 0.6\times10^{-6}s,$ represents the characteristic charging (discharging) time of an EEU's capacitor C_0 through the resistance R and c is the dimensionless velocity of the wave, to reduce Equation (1) into the following ODE:

$$u''' + A_1 cuu' + (6-B_1c)u' + C_1u = 0\ . \tag{8}$$

Here the set of abbreviations are arranged as follows:

$$A_1=\frac{3Z^{3/2}}{\tau}(2\delta C_0-\chi G_0),\ B_1=\frac{3ZC_0}{\tau},\ C_1=3(RZ^{-1}-G_0Z)$$

By balancing $u'''\ with\ uu'$, we have m = 2. Hence the formal solution of Equation (7) takes the form

$$u(\xi)=a_2Q^2+a_1Q+a_0+a_{-1}Q^{-1}+a_{-2}Q^{-2}\ , \tag{9}$$

where $a_2,\ a_1, a_0,\ a_{-1}, a_{-2}$ are parameters to be determined later, such that $a_{-2}\neq 0$ or $a_2\neq 0$.

Inserting Equation (9) with the aid of Equation (6) into Equation (8), we get the following system of algebraic equations:

$$\begin{aligned}
Q^{-5}:&\quad -24a_{-2}r^3-2a_{-2}^2rA_1c=0,\\
Q^{5}:&\quad 24a_2q^3+2a_2^2qA_1c=0,\\
Q^{-4}:&\quad -54a_{-2}pr^2-6a_{-1}r^3-A_1c(2a_{-2}^2p+3a_{-1}a_{-2}r)=0,\\
Q^{4}:&\quad 54a_2pq^2+6a_1q^3+A_1c(2a_2^2p+3a_1a_2q)=0,
\end{aligned}$$

$$\begin{aligned}
Q^{-3}:&\quad -40a_{-2}qr^2-38a_{-2}p^2r-12a_{-1}pr^2-A_1c(2a_{-2}^2q+3a_{-1}a_{-2}p+a_{-1}^2r+2a_0a_{-2}r)-2a_{-2}r(6-B_1c)=0,\\
Q^{3}:&\quad 40a_2q^2r+38a_2p^2q+12a_1pq^2+A_1c(2a_2^2r+3a_1a_2p+a_1^2q+2a_0a_2q)+2a_2q(6-B_1c)=0,
\end{aligned}$$

$$\begin{aligned}
Q^{-2}:&\quad -52a_{-2}pqr-8a_{-1}qr^2-7a_{-1}p^2r-8a_{-2}p^3-A_1c(3a_{-1}a_{-2}q+a_1a_{-2}r+a_{-1}^2p+2a_0a_{-2}p+a_0a_{-1}r)\\
&\quad -(2a_{-2}p+a_{-1}r)(6-B_1c)+C_1a_{-2}=0,\\
Q^{2}:&\quad 52a_2pqr+8a_1q^2r+7a_1p^2q+8a_2p^3+A_1c(3a_1a_2r+a_{-1}a_2q+a_1^2p+2a_0a_2p+a_0a_1q)\\
&\quad +(2a_2p+a_1q)(6-B_1c)+C_1a_2=0,
\end{aligned}$$

$$\begin{aligned}
Q^{-1}:&\quad -16a_{-2}q^2r-8a_{-1}pqr-14a_{-2}p^2q-a_{-1}p^3-A_1c(a_1a_{-2}p+a_{-1}^2q+2a_0a_{-2}q+a_0a_{-1}p)\\
&\quad -(2a_{-2}q+a_{-1}p)(6-B_1c)+C_1a_{-1}=0,\\
Q\ :&\quad 16a_2qr^2+8a_1pqr+14a_2p^2r+a_1p^3+A_1c(a_{-1}a_2p+a_1^2r+2a_0a_2r+a_0a_1p)\\
&\quad +(2a_2r+a_1p)(6-B_1c)+C_1a_1=0,
\end{aligned}$$

$$\begin{aligned}
Q^{0}:&\quad -6a_{-2}pq^2-2a_{-1}q^2r-a_{-1}p^2q+a_1p^2r+6a_2pr^2+2a_1qr^2\\
&\quad +A_1c(-a_1a_{-2}q+a_{-1}a_2r-a_0a_{-1}q+a_0a_1r)+(a_1r-a_{-1}q)(6-B_1c)+C_1a_0=0
\end{aligned}$$

By solving these algebraic equations with the aid of Maple or Mathematica we have the following cases:

Case 1

$$p=p,\ q=q,\ r=r,\ c=\frac{(p^2+8qr+6)}{B_1-a_0A_1},\ a_0=a_0,\ a_1=\frac{-12pq(B_1-a_0A_1)}{A_1(p^2+8qr+6)},$$

$$a_2=\frac{-12q^2(B_1-a_0A_1)}{A_1(p^2+8qr+6)},\ a_{-1}=0,\ a_{-2}=0$$

Case 2

$$p=0,\ q=q,\ r=r,\ c=\frac{8qr+6}{B_1-a_0A_1},\ a_0=a_0,\ a_1=0, a_{-2}=\frac{-12r^2(B_1-a_0A_1)}{A_1(8qr+6)},\ a_{-1}=0,\ a_2=0$$

Case 3

$$p=0,\ q=q,\ r=\frac{3}{4a_2q}(2a_0q^2-a_2-2q^2(\frac{B_1}{A_1})),\ c=\frac{-12q^2}{a_2A_1},\ a_0=a_0,\ a_1=0,$$

$$a_{-2}=\frac{9}{16a_2q^4}\left[2a_0q^2-a_2-2q^2(\frac{B_1}{A_1})\right]^2,\ a_{-1}=0,\ a_2\neq 0$$

Exact traveling wave solutions of Equation (1) for Case 1

By using the case 1 and according to the values of solutions of type 1, we obtain the following exact traveling wave solutions for Equation (1):

$$u_1(x,t)=a_0+\frac{3(B_1-a_0A_1)}{A_1(p^2+8qr+6)}[p^2\operatorname{sech}^2(\frac{\sqrt{\Delta}}{2}\xi)+4qr\tanh^2(\frac{\sqrt{\Delta}}{2}\xi)],$$

$$u_2(x,t)=a_0+\frac{3(B_1-a_0A_1)}{A_1(p^2+8qr+6)}[-p^2\operatorname{csch}^2(\frac{\sqrt{\Delta}}{2}\xi)+4qr\coth^2(\frac{\sqrt{\Delta}}{2}\xi)],$$

$$u_3(x,t)=a_0+\frac{3(B_1-a_0A_1)}{A_1(p^2+8qr+6)}[p^2-\Delta(\tanh(\sqrt{\Delta}\xi)\pm i\operatorname{sech}(\sqrt{\Delta}\xi))^2],$$

$$u_4(x,t)=a_0+\frac{3(B_1-a_0A_1)}{A_1(p^2+8qr+6)}[p^2-\Delta(\coth(\sqrt{\Delta}\xi)\pm\operatorname{csch}(\sqrt{\Delta}\xi))^2],$$

$$u_5(x,t)=a_0+\frac{3(B_1-a_0A_1)}{4A_1(p^2+8qr+6)}[4p^2-\Delta(\tanh(\frac{\sqrt{\Delta}}{4}\xi)\pm\coth(\frac{\sqrt{\Delta}}{4}\xi))^2],$$

$$u_6(x,t)=a_0+\frac{3(B_1-a_0A_1)}{A_1(p^2+8qr+6)}[p^2-\Delta\frac{(\sqrt{A^2+B^2}-A\cosh(\sqrt{\Delta}\xi))^2}{(A\sinh(\sqrt{\Delta}\xi)+B)^2}],$$

$$u_7(x,t)=a_0+\frac{3(B_1-a_0A_1)}{A_1(p^2+8qr+6)}[p^2-\Delta\frac{(\sqrt{B^2-A^2}+A\cosh(\sqrt{\Delta}\xi))^2}{(A\sinh(\sqrt{\Delta}\xi)+B)^2}],$$

where A and B are two non-zero real constants satisfying $B^2 - A^2 > 0$,

$$u_8(x,t)=a_0-\frac{24qr(B_1-a_0A_1)}{A_1(p^2+8qr+6)}\left(\frac{\cosh(\frac{\sqrt{\Delta}}{2}\xi)}{(\sqrt{\Delta}\sinh(\frac{\sqrt{\Delta}}{2}\xi)-p\cosh(\frac{\sqrt{\Delta}}{2}\xi))}\right)\times\left(p+2qr\frac{\cosh(\frac{\sqrt{\Delta}}{2}\xi)}{(\sqrt{\Delta}\sinh(\frac{\sqrt{\Delta}}{2}\xi)-p\cosh(\frac{\sqrt{\Delta}}{2}\xi))}\right),$$

$$u_9(x,t)=a_0+\frac{24qr(B_1-a_0A_1)}{A_1(p^2+8qr+6)}\left(\frac{\sinh(\frac{\sqrt{\Delta}}{2}\xi)}{(p\sinh(\frac{\sqrt{\Delta}}{2}\xi)-\sqrt{\Delta}\cosh(\frac{\sqrt{\Delta}}{2}\xi))}\right)\times\left(p-2qr\frac{\sinh(\frac{\sqrt{\Delta}}{2}\xi)}{(p\sinh(\frac{\sqrt{\Delta}}{2}\xi)-\sqrt{\Delta}\cosh(\frac{\sqrt{\Delta}}{2}\xi))}\right),$$

$$u_{10}(x,t)=a_0-\frac{24qr(B_1-a_0A_1)}{A_1(p^2+8qr+6)}\left(\frac{\cosh(\frac{\sqrt{\Delta}}{2}\xi)}{(\sqrt{\Delta}\sinh(\sqrt{\Delta}\xi)-p\cosh(\sqrt{\Delta}\xi)\pm i\sqrt{\Delta})}\right)\times\left(p+2qr\frac{\cosh(\frac{\sqrt{\Delta}}{2}\xi)}{(\sqrt{\Delta}\sinh(\sqrt{\Delta}\xi)-p\cosh(\sqrt{\Delta}\xi)\pm i\sqrt{\Delta})}\right),$$

$$u_{11}(x,t)=a_0-\frac{24qr(B_1-a_0A_1)}{A_1(p^2+8qr+6)}\left(\frac{\sinh(\frac{\sqrt{\Delta}}{2}\xi)}{(-p\sinh(\sqrt{\Delta}\xi)+\sqrt{\Delta}\cosh(\sqrt{\Delta}\xi)\pm\sqrt{\Delta})}\right)\times\left(p+2qr\frac{\sinh(\frac{\sqrt{\Delta}}{2}\xi)}{(-p\sinh(\sqrt{\Delta}\xi)+\sqrt{\Delta}\cosh(\sqrt{\Delta}\xi)\pm\sqrt{\Delta})}\right),$$

$$u_{12}(x,t)=a_0-\frac{48qr(B_1-a_0A_1)}{A_1(p^2+8qr+6)}\left(\frac{\sinh(\frac{\sqrt{\Delta}}{4}\xi)\cosh(\frac{\sqrt{\Delta}}{4}\xi)}{(-2p\sinh(\frac{\sqrt{\Delta}}{4}\xi)\cosh(\frac{\sqrt{\Delta}}{4}\xi)+2\sqrt{\Delta}\cosh^2(\frac{\sqrt{\Delta}}{4}\xi)-\sqrt{\Delta})}\right)\times\left(p+4qr\frac{\sinh(\frac{\sqrt{\Delta}}{4}\xi)\cosh(\frac{\sqrt{\Delta}}{4}\xi)}{(-2p\sinh(\frac{\sqrt{\Delta}}{4}\xi)\cosh(\frac{\sqrt{\Delta}}{4}\xi)+2\sqrt{\Delta}\cosh^2(\frac{\sqrt{\Delta}}{4}\xi)-\sqrt{\Delta})}\right).$$

According to the values of solutions of Type 2, we obtain the following exact traveling wave solutions for Equation (1):

$$u_{13}(x,t)=a_0+\frac{3(B_1-a_0A_1)}{A_1(p^2+8qr+6)}[p^2\sec^2(\frac{\sqrt{-\Delta}}{2}\xi)-4qr\tan^2(\frac{\sqrt{-\Delta}}{2}\xi)],$$

$$u_{14}(x,t)=a_0+\frac{3(B_1-a_0A_1)}{A_1(p^2+8qr+6)}[p^2\csc^2(\frac{\sqrt{-\Delta}}{2}\xi)-4qr\cot^2(\frac{\sqrt{-\Delta}}{2}\xi)],$$

$$u_{15}(x,t)=a_0+\frac{3(B_1-a_0A_1)}{A_1(p^2+8qr+6)}[p^2+\Delta(\tan(\sqrt{-\Delta}\xi)\pm\sec(\sqrt{-\Delta}\xi))^2],$$

$$u_{16}(x,t)=a_0+\frac{3(B_1-a_0A_1)}{A_1(p^2+8qr+6)}[p^2+\Delta(\cot(\sqrt{-\Delta}\xi)\pm\csc(\sqrt{-\Delta}\xi))^2],$$

$$u_{17}(x,t)=a_0+\frac{3(B_1-a_0A_1)}{4A_1(p^2+8qr+6)}[4p^2+\Delta(\tan(\frac{\sqrt{-\Delta}}{4}\xi)-\cot(\frac{\sqrt{-\Delta}}{4}\xi))^2],$$

$$u_{18}(x,t)=a_0+\frac{3(B_1-a_0A_1)}{A_1(p^2+8qr+6)}[p^2+\Delta\left(\frac{\sqrt{A^2-B^2}-A\cos(\sqrt{-\Delta}\xi)}{A\sinh(\sqrt{-\Delta}\xi)+B}\right)^2,$$

$$u_{19}(x,t)=a_0+\frac{3(B_1-a_0A_1)}{A_1(p^2+8qr+6)}[p^2+\Delta\left(\frac{\pm\sqrt{A^2-B^2}+A\cos(\sqrt{-\Delta}\xi)}{A\sin(\sqrt{-\Delta}\xi)+B}\right)^2,$$

$$u_{20}(x,t)=a_0+\frac{24qr(B_1-a_0A_1)}{A_1(p^2+8qr+6)}\left(\frac{\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{\sqrt{-\Delta}\sin(\frac{\sqrt{-\Delta}}{2}\xi)+p\cos(\frac{\sqrt{-\Delta}}{2}\xi)}\right)\times\left(p-qr\frac{\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{\sqrt{-\Delta}\sin(\frac{\sqrt{-\Delta}}{2}\xi)+p\cos(\frac{\sqrt{-\Delta}}{2}\xi)}\right),$$

$$u_{21}(x,t)=a_0-\frac{24qr(B_1-a_0A_1)}{A_1(p^2+8qr+6)}\left(\frac{\sin(\frac{\sqrt{-\Delta}}{2}\xi)}{-p\sin(\frac{\sqrt{-\Delta}}{2}\xi)+\sqrt{-\Delta}\cos(\frac{\sqrt{-\Delta}}{2}\xi)}\right)\times\left(p+2qr\frac{\sin(\frac{\sqrt{-\Delta}}{2}\xi)}{-p\sin(\frac{\sqrt{-\Delta}}{2}\xi)+\sqrt{-\Delta}\cos(\frac{\sqrt{-\Delta}}{2}\xi)}\right),$$

$$u_{22}(x,t)=a_0+\frac{24qr(B_1-a_0A_1)}{A_1(p^2+8qr+6)}\left(\frac{\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{\sqrt{-\Delta}\sin(\frac{\sqrt{-\Delta}}{2}\xi)+p\cos(\frac{\sqrt{-\Delta}}{2}\xi)\pm\sqrt{-\Delta}}\right)$$
$$\times\left(p-2qr\frac{\sin(\frac{\sqrt{-\Delta}}{2}\xi)}{\sqrt{-\Delta}\sin(\frac{\sqrt{-\Delta}}{2}\xi)-p\cos(\frac{\sqrt{-\Delta}}{2}\xi)\pm\sqrt{-\Delta}}\right),$$

$$u_{23}(x,t)=a_0-\frac{24qr(B_1-a_0A_1)}{A_1(p^2+8qr+6)}\left(\frac{\sin(\frac{\sqrt{-\Delta}}{2}\xi)}{-p\sin(\sqrt{-\Delta}\xi)+\sqrt{-\Delta}\cos(\sqrt{-\Delta}\xi)\pm\sqrt{-\Delta}}\right)$$
$$\times\left(p+2qr\frac{\sin(\sqrt{-\Delta}\xi)}{-p\sin(\sqrt{-\Delta}\xi)+\sqrt{-\Delta}\cos(\sqrt{-\Delta}\xi)\pm\sqrt{-\Delta}}\right),$$

$$u_{24}(x,t)=a_0-\frac{48qr(B_1-a_0A_1)}{A_1(p^2+8qr+6)}\left(\frac{\sin(\frac{\sqrt{-\Delta}}{4}\xi)\cos(\frac{\sqrt{-\Delta}}{4}\xi)}{-2p\sin(\frac{\sqrt{-\Delta}}{4}\xi)\cos(\frac{\sqrt{-\Delta}}{4}\xi)+2\sqrt{-\Delta}\cos^2(\frac{\sqrt{-\Delta}}{4}\xi)-\sqrt{-\Delta}}\right)$$
$$\times\left(p+4qr\frac{\sin(\frac{\sqrt{-\Delta}}{4}\xi)\cos(\frac{\sqrt{-\Delta}}{4}\xi)}{-2p\sin(\frac{\sqrt{-\Delta}}{4}\xi)\cos(\frac{\sqrt{-\Delta}}{4}\xi)+2\sqrt{-\Delta}\cos^2(\frac{\sqrt{-\Delta}}{4}\xi)-\sqrt{-\Delta}}\right),$$

where $\xi=\frac{1}{L}x-\frac{(p^2+8qr+6)}{(B_1-a_0A_1)}\frac{t}{\tau}$, and a_0 is an arbitrary constant.

According to the values of solutions of the type, we obtain the following exact traveling wave solutions for Equation (1):

$$u_{25}(x,t)=a_0+\frac{12p^2d(B_1-a_0A_1)}{A_1(p^2+6)}\left(\frac{\cosh(p\xi)-\sinh(p\xi)}{(d+\cosh(p\xi)-\sinh(p\xi))^2}\right),$$

$$u_{26}(x,t)=a_0+\frac{12p^2d(B_1-a_0A_1)}{A_1(p^2+6)}\left(\frac{\cosh(p\xi)+\sinh(p\xi)}{(d+\cosh(p\xi)+\sinh(p\xi))^2}\right).$$

where $\xi=\frac{1}{L}x-\frac{(p^2+6)}{(B_1-a_0A_1)}\frac{t}{\tau}$, a_0 is an arbitrary constant.

According to the values of the type 4, we obtain the following exact traveling wave solution for (1):

$$u_{27}(x,t)=a_0-\frac{2q(B_1-a_0A_1)}{A_1}\left(\frac{q}{(q\xi+C_1)^2}\right),$$

where $\xi=\frac{1}{L}x-\frac{6}{(B_1-a_0A_1)}\frac{t}{\tau}$.

Exact traveling wave solutions of Equation (1) for Case 2

By using the Case 2 and according to the values of solutions of type 1, we obtain the following exact traveling wave solutions for Equation(1) :

$$u_1(x,t)=a_0+\frac{12qr(B_1-a_0A_1)}{A_1(8qr+6)}\coth^2(\sqrt{-qr}\xi),$$
$$u_2(x,t)=a_0+\frac{12qr(B_1-a_0A_1)}{A_1(8qr+6)}\tanh^2(\sqrt{-qr}\xi),$$

$$u_3(x,t)=a_0+\frac{12qr(B_1-a_0A_1)}{A_1(8qr+6)}\frac{1}{[\tanh(\sqrt{-4qr}\xi)\pm i\ \mathrm{sec}h(\sqrt{-4qr}\xi)]^2},$$
$$u_4(x,t)=a_0+\frac{12qr(B_1-a_0A_1)}{A_1(8qr+6)}\frac{1}{[\coth(\sqrt{-4qr}\xi)\pm \mathrm{csc}h(\sqrt{-4qr}\xi)]^2},$$
$$u_5(x,t)=a_0+\frac{48qr(B_1-a_0A_1)}{A_1(8qr+6)}\frac{1}{[\tanh(\frac{\sqrt{-qr}}{2}\xi)\pm\coth(\frac{\sqrt{-qr}}{2}\xi)]^2},$$
$$u_6(x,t)=a_0+\frac{12qr(B_1-a_0A_1)}{A_1(8qr+6)}\left[\frac{A\sinh(\sqrt{-4qr}\xi)+B}{-A\cosh(\sqrt{-4qr}\xi)+\sqrt{A^2+B^2}}\right]^2,$$
$$u_7(x,t)=a_0+\frac{12qr(B_1-a_0A_1)}{A_1(8qr+6)}\left[\frac{A\sinh(\sqrt{-4qr}\xi)+B}{A\cosh(\sqrt{-4qr}\xi)+\sqrt{B^2-A^2}}\right]^2.$$

where *A* and *B* are two non-zero real constants satisfying $B^2-A^2>0$,

$$u_8(x,t)=u_2(x,t)$$
$$u_9(x,t)=u_1(x,t)$$
$$u_{10}(x,t)=a_0+\frac{12qr(B_1-a_0A_1)}{A_1(8qr+6)}\left[\frac{\sinh(\sqrt{-4qr}\xi)\pm i}{\cosh(\sqrt{-qr}\xi)}\right]^2,$$
$$u_{11}(x,t)=a_0+\frac{12qr(B_1-a_0A_1)}{A_1(8qr+6)}\left[\frac{\cosh(\sqrt{-4qr}\xi)\pm 1}{\sinh(\sqrt{-qr}\xi)}\right]^2,$$
$$u_{12}(x,t)=a_0+\frac{48qr(B_1-a_0A_1)}{A_1(8qr+6)}\left(\frac{\cosh(\sqrt{-4qr}\xi)}{\sinh(\sqrt{-qr}\xi)}\right)^2.$$

According to the values of the solutions of the type 2, we obtain the exact traveling wave solutions for Equation (1):

$$u_{13}(x,t)=a_0-\frac{12qr(B_1-a_0A_1)}{A_1(8qr+6)}\cot^2(\sqrt{qr}\xi),$$
$$u_{14}(x,t)=a_0-\frac{12qr(B_1-a_0A_1)}{A_1(8qr+6)}\tan^2(\sqrt{qr}\xi),$$
$$u_{15}(x,t)=a_0-\frac{12qr(B_1-a_0A_1)}{A_1(8qr+6)}\left[\tan(\sqrt{4qr}\xi)\pm\sec(\sqrt{4qr}\xi)\right]^{-2},$$
$$u_{16}(x,t)=a_0-\frac{12qr(B_1-a_0A_1)}{A_1(8qr+6)}\left[\cot(\sqrt{4qr}\xi)\pm\csc(\sqrt{4qr}\xi)\right]^{-2}$$

$$u_{17}(x,t)=a_0-\frac{48qr(B_1-a_0A_1)}{A_1(8qr+6)}\left[\tan(\frac{\sqrt{qr}}{2}\xi)-\cot(\frac{\sqrt{qr}}{2}\xi)\right]^{-2},$$

$$u_{18}(x,t)=a_0-\frac{12qr(B_1-a_0A_1)}{A_1(8qr+6)}\left[\frac{\pm\sqrt{A^2-B^2}-A\cos(\sqrt{4qr}\xi)}{B+A\sin(\sqrt{4qr}\xi)}\right]^{-2},$$

$$u_{19}(x,t)=a_0-\frac{12qr(B_1-a_0A_1)}{A_1(8qr+6)}\left[\frac{\pm\sqrt{A^2-B^2}+A\cos(\sqrt{4qr}\xi)}{B+A\sin(\sqrt{4qr}\xi)}\right]^{-2},$$

where A and B are two non-zero real constants satisfying $A^2 - B^2 > 0$,

$$u_{20}(x,t)=u_{14}(x,t),$$
$$u_{21}(x,t)=u_{13}(x,t),$$
$$u_{22}(x,t)=a_0-\frac{12qr(B_1-a_0A_1)}{A_1(8qr+6)}\left[\frac{\sin(\sqrt{4qr}\xi)\pm1}{\cos(\sqrt{qr}\xi)}\right]^{2},$$
$$u_{23}(x,t)=a_0-\frac{12qr(B_1-a_0A_1)}{A_1(8qr+6)}\left[\frac{\cos(\sqrt{4qr}\xi)\pm1}{\sin(\sqrt{qr}\xi)}\right]^{2},$$
$$u_{24}(x,t)=u_{13}(x,t).$$

$$\xi=\frac{1}{L}x-\frac{(8qr+6)}{(B_1-a_0A_1)}\frac{t}{\tau}.$$

Exact traveling wave solutions of Equation (1) for Case 3

By using the Case 3 and according to the values of the solutions of the Type 1, we obtain the following exact traveling wave solutions for Equation (1):

$$u_1(x,t)=u_2(x,t)=a_0-\frac{a_2r}{q}\left[\tanh^2(\sqrt{-qr}\xi)+\coth^2(\sqrt{-qr}\xi)\right],$$

$$u_3(x,t)=a_0-\frac{a_2r}{q}\left[\begin{array}{l}\left(\tanh(\sqrt{-4qr}\xi)\pm i\,\mathrm{sec}h(\sqrt{-4qr}\xi)\right)^2\\+\left(\tanh(\sqrt{-4qr}\xi)\pm i\,\mathrm{sec}h(\sqrt{-4qr}\xi)\right)^{-2}\end{array}\right],$$

$$u_4(x,t)=a_0-\frac{a_2r}{q}\left[\begin{array}{l}\left(\coth(\sqrt{-4qr}\xi)\pm \mathrm{csc}h(\sqrt{-4qr}\xi)\right)^2\\+\left(\coth(\sqrt{-4qr}\xi)\pm \mathrm{csc}h(\sqrt{-4qr}\xi)\right)^{-2}\end{array}\right],$$

$$u_5(x,t)=a_0-\frac{a_2r}{4q}\left[\left(\tanh(\frac{\sqrt{-qr}}{2}\xi)\pm\coth(\frac{\sqrt{-qr}}{2}\xi)\right)^2\right]-\frac{4a_2r}{q}\left[\left(\tanh(\frac{\sqrt{-qr}}{2}\xi)\pm\coth(\frac{\sqrt{-qr}}{2}\xi)\right)^{-2}\right],$$

$$u_6(x,t)=a_0-\frac{a_2r}{q}\left[\left[\frac{\sqrt{A^2+B^2}-A\cosh(\sqrt{-4qr}\xi)}{B+A\sinh(\sqrt{-4qr}\xi)}\right]^{-2}+\left[\frac{B+A\sinh(\sqrt{-4qr}\xi)}{\sqrt{A^2+B^2}-A\cosh(\sqrt{-4qr}\xi)}\right]^{-2}\right],$$

$$u_7(x,t)=a_0-\frac{a_2r}{q}\left[\left[\frac{\sqrt{B^2-A^2}+A\cosh(\sqrt{-4qr}\xi)}{B+A\sinh(\sqrt{-4qr}\xi)}\right]^{-2}+\left[\frac{B+A\sinh(\sqrt{-4qr}\xi)}{\sqrt{B^2-A^2}+A\cosh(\sqrt{-4qr}\xi)}\right]^{-2}\right].$$

where A and B are two non-zero real constants satisfying $B^2 - A^2 > 0$,

$$u_8(x,t)=u_9(x,t)=u_1(x,t),$$

$$u_{10}(x,t)=a_0-\frac{a_2r}{q}\left[\left(\frac{\cosh(\sqrt{-qr}\xi)}{\sinh(\sqrt{-4qr}\xi)\pm i}\right)^2+\left(\frac{\sinh(\sqrt{-4qr}\xi)\pm i}{\cosh(\sqrt{-qr}\xi)}\right)^2\right],$$

$$u_{11}(x,t)=a_0-\frac{a_2r}{q}\left[\left(\frac{\sinh(\sqrt{-qr}\xi)}{\cosh(\sqrt{-4qr}\xi)\pm1}\right)^2+\left(\frac{\cosh(\sqrt{-4qr}\xi)\pm1}{\sinh(\sqrt{-qr}\xi)}\right)^2\right],$$

$$u_{12}(x,t)=a_0-\frac{a_2r}{q}\left[\left(\frac{\sinh(\sqrt{-qr}\xi)}{\cosh(\sqrt{-4qr}\xi)}\right)^2+\left(\frac{\cosh(\sqrt{-4qr}\xi)}{\sinh(\sqrt{-qr}\xi)}\right)^2\right].$$

According to the value of the solutions of the type, we obtain the exact traveling wave solutions for Equation (1):

$$u_{13}(x,t)=u_{14}(x,t)=a_0+\frac{a_2r}{q}\left(\tan^2(\sqrt{qr}\xi)+\cot^2(\sqrt{qr}\xi)\right),$$

$$u_{15}(x,t)=a_0+\frac{a_2r}{q}\left[\left(\tan(\sqrt{4qr}\xi)\pm\sec(\sqrt{4qr}\xi)\right)^2+\left(\tan(\sqrt{4qr}\xi)\pm\sec(\sqrt{4qr}\xi)\right)^{-2}\right],$$

$$u_{16}(x,t)=a_0+\frac{a_2r}{q}\left[\left(\cot(\sqrt{4qr}\xi)\pm\csc(\sqrt{4qr}\xi)\right)^2+\left(\cot(\sqrt{4qr}\xi)\pm\csc(\sqrt{4qr}\xi)\right)^{-2}\right],$$

$$u_{17}(x,t)=a_0+\frac{a_2r}{4q}\left(\tan\left(\frac{\sqrt{qr}}{2}\xi\right)-\cot\left(\frac{\sqrt{qr}}{2}\xi\right)\right)^2+\frac{4a_2r}{q}\left(\tan\left(\frac{\sqrt{qr}}{2}\xi\right)-\cot\left(\frac{\sqrt{qr}}{2}\xi\right)\right)^{-2},$$

$$u_{18}(x,t)=a_0+\frac{a_2r}{q}\left[\left(\frac{\pm\sqrt{A^2-B^2}-A\cos(\sqrt{4qr}\xi)}{B+A\sin(\sqrt{4qr}\xi)}\right)^2+\left(\frac{B+A\sin(\sqrt{4qr}\xi)}{\pm\sqrt{A^2-B^2}-A\cos(\sqrt{4qr}\xi)}\right)^2\right],$$

$$u_{19}(x,t)=a_0+\frac{a_2r}{q}\left[\left(\frac{\pm\sqrt{A^2-B^2}+A\cos(\sqrt{4qr}\xi)}{B+A\sin(\sqrt{4qr}\xi)}\right)^2+\left(\frac{B+A\sin(\sqrt{4qr}\xi)}{\pm\sqrt{A^2-B^2}+A\cos(\sqrt{4qr}\xi)}\right)^2\right].$$

where A and B are two non-zero real constants satisfying $A^2-B^2>0$,

$$u_{20}(x,t)=u_{21}(x,t)=u_{13}(x,t),$$

$$u_{22}(x,t)=a_0+\frac{a_2r}{q}\left[\left(\frac{\cos(\sqrt{qr}\xi)}{\sin(\sqrt{4qr}\xi)\pm1}\right)^2+\left(\frac{\sin(\sqrt{4qr}\xi)\pm1}{\cos(\sqrt{qr}\xi)}\right)^2\right],$$

$$u_{23}(x,t)=a_0+\frac{a_2r}{q}\left[\left(\frac{\sin(\sqrt{qr}\xi)}{\cos(\sqrt{4qr}\xi)\pm1}\right)^2+\left(\frac{\cos(\sqrt{4qr}\xi)\pm1}{\sin(\sqrt{qr}\xi)}\right)^2\right],$$

$$u_{24}(x,t)=u_{13}(x,t).$$

$$where\, \xi = \frac{1}{L}x - \left(\frac{-12q^2}{a_2A_1}\right)\frac{t}{\tau},\ r = \frac{3}{4a_2q}(2a_0q^2 - a_2 - 2q^2(\frac{B}{A}))$$

DISCUSSION AND CONCLUSION

In this section, we will compare some of our results obtained in this article with the well-known results obtained in (Sekulic et al., 2011) with the interchanges $r \leftrightarrow b,\ A_1 \leftrightarrow A, B_1 \leftrightarrow B$ as follows:

1. If we set $p = 0$ and $q = 1$ in our results $u_1(x,t)$ and $u_{13}(x,t)$, we have respectively the special forms:

$$u_1(x,t) = a_0 - a_2 \tanh^2(\sqrt{-r}\xi), \tag{10}$$

And

$$u_{13}(x,t) = a_0 + a_2 \tan^2(\sqrt{r}\xi), \tag{11}$$

Where

$$r = \frac{3}{4a_2}(2a_0 - a_2 - 2\frac{B_1}{A_1}),\quad \xi = \frac{1}{L}x + (\frac{12}{a_2A_1})\frac{t}{\tau}. \tag{12}$$

We have noted that the results (10) and (11) are equivalent to the well-known results (10) and (11) obtained in (Sekulic et al., 2011) respectively.

2. If we set $q = 1$ in our results $u_1(x,t)$ and $u_{13}(x,t)$, we have respectively the special forms:

$$u_1(x,t) = a_0 + \frac{12r}{A_1}\left(\frac{B_1 - a_0A_1}{8r+6}\right)\coth^2(\sqrt{-r}\xi), \tag{13}$$

and

$$u_{13}(x,t) = a_0 - \frac{12r}{A_1}\left(\frac{B_1 - a_0A_1}{8r+6}\right)\cot^2(\sqrt{r}\xi), \tag{14}$$

where r is an arbitrary constant, while

$$\xi = \frac{1}{L}x - (\frac{8r+6}{B_1 - a_0A_1})\frac{t}{\tau}.$$

After our careful revisions for the article (Sekulic et al, 2011), we found a minor error in the results (12) and (13) of this reference, which can be corrected here to become respectively as:

$$u(x,t) = a_0 + \frac{12b}{A}\left(\frac{B - a_0A}{8b+6}\right)\coth^2(\sqrt{-b}\xi), \tag{15}$$

and

$$u(x,t) = a_0 - \frac{12b}{A}\left(\frac{B - a_0A}{8b+6}\right)\cot^2(\sqrt{b}\xi), \tag{16}$$

In this case our above results (13) and (14) are equivalent to the corrected results (15) and (16) respectively.

3- If we set $q = 1$ in our results $u_1(x,t)$ and $u_{13}(x,t)$, we have respectively the special forms:

$$u_1(x,t) = a_0 - \frac{3}{4}(2a_0 - a_2 - 2\frac{B_1}{A_1})\left\{\tanh^2(\sqrt{-r}\xi) + \coth^2(\sqrt{-r}\xi)\right\}, \tag{17}$$

And

$$u_{13}(x,t) = a_0 + \frac{3}{4}(2a_0 - a_2 - 2\frac{B_1}{A_1})\left\{\tan^2(\sqrt{r}\xi) + \cot^2(\sqrt{r}\xi)\right\}, \tag{18}$$

where r and ξ have the same forms of (12). We have noted that the results (17) and (18) are equivalent to the results (14) and (15) obtained in (Sekulic et al., 2011) respectively.

From these discussions, we deduce that the exact solutions of Equation (1) obtained in (Sekulic et al., 2011) using the modified extended tanh-function method are special cases of some of our results obtained in the present article using the improved Riccati equation mapping method. Furthermore, the proposed method in this article has been played an important role in obtaining many families of exact solutions of Equation (1) which look new and recover the well-known exact solutions obtained in (Sekulic et al., 2011) using the modified extended tanh-function method. Finally, all the solutions obtained in the present article have been checked with Maple or Mathematica by putting them back into the original Equation (1).

Numerical examples

In this section, we will give some numerical examples to illustrate some of our obtained results in this article. To this end, we select some special values of the parameters obtained in the exact traveling wave solutions of Equation (1) for case 1. So, we choose the following values:

$$C_0 = 4,\ Z = 3,\ G_0 = 1,\ \delta = 6,\ R = 2,\ L = \frac{1}{2}, \tau = \frac{1}{2}, p = 8, r = 3,$$

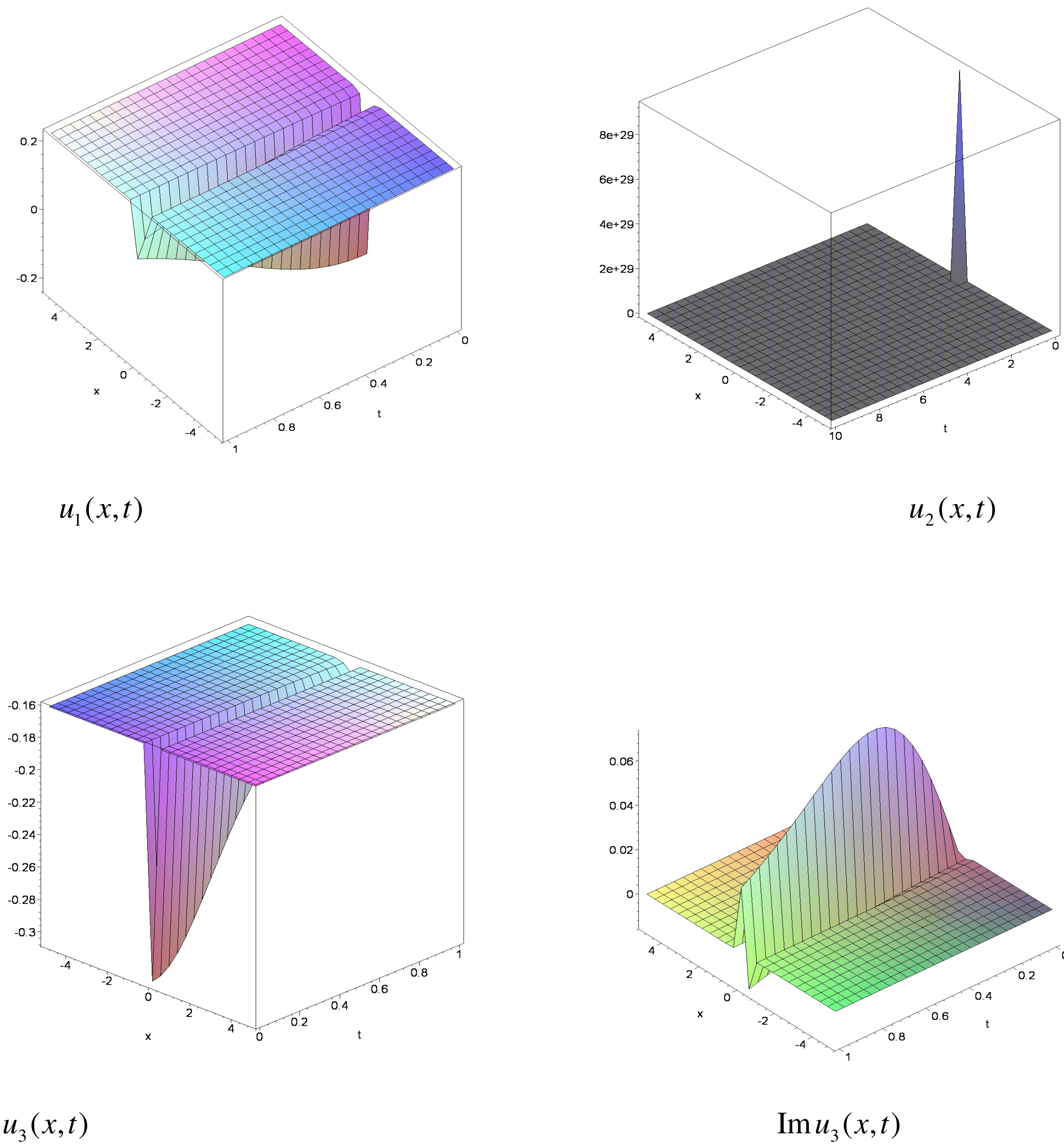

Figure 1. Some numerical solutions of Equation (1).

a_0 = 0.5, and using the computer programs, such as the Matlab, to draw diagrams for some of our results (Figure 1)

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