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Single basis function method for solving diffusion convection equations

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In this study, we developed a new numerical finite difference method for solving various diffusion convection equations. The method involves reduction of the diffusion convection equations to a system of algebraic equations. By solving the system of algebraic equations we obtain the problem approximate solutions. The study of the numerical accuracy of the method has shown that the method provides similar results to the known explicit finite difference method for solving diffusion convection equations, but with fewer numbers of iterations.

Key words: Lines, multistep collocation, parabolic, Taylor’s polynomial.

INTRODUCTION

In this study, we will deal with diffusion convection equations, where t and x are the time and space coordinates respectively, and the quantities h and k are the mesh sizes in the space and time directions.

We consider,

$$\frac{\partial U(x,t)}{\partial t} = V\frac{\partial^2 U(x,t)}{\partial x^2} + \overline{U}\frac{\partial U(x,t)}{\partial x}, \qquad (1)$$

A special type of diffusion convection equation where $V, \overline{U} > 0$ are constants, defined for $a < x < b$ and positive time with appropriate initial and boundary conditions, and U is the temperature in the medium. We are interested in the development of numerical techniques for solving diffusion convection equations. Recently, there is a growing interest concerning continuous numerical methods of solution for Ordinary Differentials Equations (ODEs) (Adam and David, 2002; Awoyemi, 2002, 2003; Bao et al., 2003; Benner and Mena, 2004; Bensoussan et al., 2007; Motmans et al., 2005; Brown, 1979; Chawla and Katti, 1979; Cook, 1974; Crandall, 1955; Crane and Klopfenstein, 1965; Crank and Nicolson, 1947; Dahlquist and Bjorck, 1974; Dehghan, 2003; Dieci, 1992; Douglas, 1961; D’ Yakonov and Ye, 1963; Eyaya, 2010; Fox, 1962; Penzl, 2000; Pierre, 2008; Richard and Albert, 1981; Richard et al., 2001; Saumaya et al., 2012; Yildiz and Subasi, 2001; Zheyin and Qiang, 2012). We are interested in the extension of a particular

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continuous method to solve the diffusion convection equation. This is done based on the collocation and interpolation of the Partial Differential Equations (PDEs) directly over multi steps along lines but with reduction to a system of ODEs.

THE SOLUTION METHOD

We subdivide the interval $0 \le x \le b$ into N equal subintervals by the grid points $x_m = mh, \quad m = 0,..., N$ where $Nh = b$. We want to obtain an approximation of the form

$$U(x,t) \approx U_K(x,t) = \sum_{r=0}^{k+1} a_r Q_r(x,t), \qquad x_{i-1} \le x \le x_{i+1}, \quad r = 0,1,2,...k+1 \tag{2}$$

Where $Q_r(x,t) = x^r t^r$ are canonical polynomials which are used as basic functions in the approximation, and a_r are parameters to be determined.

From the collocation equations

$$U_K(x_m,t) = \sum_{r=0}^{k} a_r Q_r(x_m,t) \tag{3}$$

where $m = i-1, j \quad i, j \quad i+1, j$

We can generate the following equations from Equation 3 as follows:

$$\left.\begin{aligned} U_k(x_{i-1},t_j) &= a_o + a_1 x_{i-1} t_j + ... + a_k x^k{}_{i-1} t^k{}_j \approx U_{i-1,j} \\ U_k(x_i,t_j) &= a_o + a_1 x_i t_j + ... + a_k x^k{}_i t^k{}_j \approx U_{i,j} \\ U_k(x_{i+1},t_j) &= a_o + a_1 x_{i+1} t_j + ... + a_k x^k{}_{i+1} t^k{}_j \approx U_{i+1,j} \end{aligned}\right\} \tag{4}$$

Writing Equation 4 as matrix in its augmented form, we have

$$\begin{bmatrix} 1 & x_{i-1}t_j & ... & x^k{}_{i-1}t^k{}_j \\ 1 & x_i t_j & ... & x^k{}_i t^k{}_j \\ 1 & x_{i+1}t_j & ... & x^k{}_{i+1}t^k{}_j \end{bmatrix} \begin{bmatrix} a_0 \\ ... \\ ... \\ ... \\ a_k \end{bmatrix} = \begin{bmatrix} U_{i-1,j} \\ U_{i,j} \\ U_{i+1,j} \end{bmatrix} \tag{5}$$

We solve Equation 5 for the value of a_k by Gaussian elimination method to obtain

$$a_k = ... + \frac{U_{i+1,j} + U_{i-1,j} - U_{i,j}}{2ht^k{}_j},$$

From Equation (2) we can generate

$$U(x,y) = a_0 + a_1 xt + ... + a_k x^k t^k \tag{6}$$

Putting the value of a_k in Equation 6, we obtain

$$U(x,t) = a_0 + a_1 xt + ... + x^k t^k \left(\frac{U_{i+1,j} + U_{i-1,j} - 2U_{i,j}}{2ht^k{}_j} \right) \tag{7}$$

We take the first and second derivatives of Equation 7 with respect to x and obtain

$$U'(x,t) = a_1 t + ... + kx^{k-1} t^k \left(\frac{U_{i+1,j} + U_{i-1,j} - 2U_{i,j}}{2ht^k{}_j} \right)$$

$$U_i''(x,t) = ... + k(k-1)x^{k-2} t^k \left(\frac{U_{i+1,j} + U_{i-1,j} - 2U_{i,j}}{2ht^k{}_j} \right) \tag{8}$$

From the collocation Equation 3

$$U_K(x_n,t) = \sum_{r=0}^{k} a_r Q_r(x_n,t)$$

where $n = i-1, j \ \ and \ \ i+1, j$

We generate the following equations as follows:

$$\left.\begin{aligned} U_k(x_{i-1},t_j) &= a_o + a_1 x_{i-1} t_j + ... + a_k x^k{}_{i-1} t^k{}_j \approx U_{i-1,j} \\ U_k(x_{i+1},t_j) &= a_o + a_1 x_{i+1} t_j + ... + a_k x^k{}_{i+1} t^k{}_j \approx U_{i+1,j} \end{aligned}\right\} \tag{9}$$

Writing Equation 9 as matrix in its augmented form, we have

$$\begin{bmatrix} 1 & x_{i-1}t_j & ... & x^k{}_{i-1}t^k{}_j \\ 1 & x_{i+1}t_j & ... & x^k{}_{i+1}t^k{}_j \end{bmatrix} \begin{bmatrix} a_0 \\ ... \\ ... \\ ... \\ a_k \end{bmatrix} = \begin{bmatrix} U_{i-1,j} \\ U_{i+1,j} \end{bmatrix} \tag{10}$$

Solving Equation 10 for a_k we obtain

$$a_k = \frac{U_{i+1,j} - U_{i-1,j}}{2ht^k{}_j},$$

but from Equation 2 we obtain

$$U(x,t) = a_0 + a_1 xt + ... + a_k x^k t^k \quad (11)$$

Putting the value of a_k in Equation 11

$$U(x,t) = a_0 + a_1 xt + ... + x^k t^k \left(\frac{U_{i+1,j} - U_{i-1,j}}{2ht^k{}_j} \right) \quad (12)$$

Taking the first derivative of Equation 12 with respect to x, we obtain

$$U'(x,t) = k(x^{k-1}t^k)\left(\frac{U_{i+1,j} - U_{m-1,j}}{2ht^k{}_j} \right) \quad (13)$$

Similarly, by interchanging the roles of x *and* t we can obtain an approximation of the form

$$U(t,x) \approx U_k(t,x) = \sum_{r=0}^{k+1} a_r Q_r(t,x), \quad t_{j-1} \le t \le t_{j+1} \quad , \quad r = 0,1,2,..,k+1 \quad (14)$$

Again, from the collocation equation

$$U_K(x,t_g) = \sum_{r=0}^{k} a_r Q_r(x,t_g), \quad g = \; i,j \; \text{and} \; i,j+1 \quad (15)$$

We have the following equations:

$$\left.\begin{aligned} U_k(t_j,x_i) &= a_o + a_1 t_j x_i + ... + a_k t^k{}_j x^k{}_i \approx U_{i,j} \\ U_k(t_{j+1},x_i) &= a_o + a_1 t_{j+1} x_i + ... + a_k t^k{}_{j+1} x^k{}_i \approx U_{i,j+1,} \end{aligned}\right\} \quad (16)$$

Writing Equation 16 as matrix in augmented form, we have

$$\begin{bmatrix} 1 & x_i t_j & ... \; x^k{}_i t^k{}_j \\ 1 & x_i t_{j+1} & ... \; x^k{}_i t^k{}_{j+1} \end{bmatrix} \begin{bmatrix} a_0 \\ ... \\ ... \\ ... \\ a_k \end{bmatrix} = \begin{bmatrix} U_{i,j} \\ U_{i,j+1} \end{bmatrix} \quad (17)$$

We solve Equation 17 for a_k by using Gaussian elimination method

$$a_k = ... + \frac{U_{i,j+1} - U_{i,j}}{2hx^k{}_j},$$

Again, from Equation 2 we obtain

$$U(t,x) = a_0 + a_1 xt + ... + a_k x^k t^k \quad (18)$$

Substituting the value of a_k in Equation 18 we have

$$U(t,x) = a_0 + a_1 xt + ... + x^k t^k \left(\frac{U_{i,j+1} - U_{i,j}}{2hx^k{}_i} \right) \quad (19)$$

Taking the first derivative of Equation 19 with respect to t, we obtain

$$U_j{}'(x,t) = ... + kt^{k-1}x^k \left(\frac{U_{i,j+1} - U_{i,j}}{2hx^k{}_i} \right), \quad (20)$$

We collocate Equation (20) at $x = x_i$, Equations 8 and 13 at $t = t_j$; and substituting the resulting equations into Equation 1, we obtain a scheme that solves diffusion convection equations numerically.

ADVANTAGE OF THE NEW SCHEME OVER FINITE DIFFERENCE METHOD

The new scheme requires fewer numbers of iterations than the finite difference method to achieve the same results.

NUMERICAL EXAMPLES

Here, we will test the numerical accuracy of the new method by using the scheme to solve two test examples. We compute an approximate solution of Problems (1) and (2) at each time level (Tables 1 and 2). To achieve this, we truncate the polynomials after second degree.

SPECIFIC PROBLEM

Example 1

Here we use the scheme to approximate the solution to

Table 1. Result of action of the scheme on Problem 1.

x	Exact solution $U(x,t)$	Solution from New Method $U(x,t)$	Solution from Finite Difference Method $U(x,t)$	Error from Finite Difference Method	Error from New Method
0	0	0	0	0	0
0.25	0.380814721	0.380862804	0.380862804	4.8 × E-5	4.8 × E-5
0.50	0.703723471	0.703742699	0.703742699	1.9 × E-5	1.9 × E-5
0.75	0.919471568	0.919484148	0.919484148	1.3 × E-5	1.3 × E-5
1.00	0.995167871	0.99524247	0.99524247	7.5 × E-5	7.5 × E-5
1.25	0.919471568	0.919484148	0.919484148	1.3 × E-5	1.3 × E-5
1.5	0.703723471	0.703742699	0.703742699	1.9 × E-5	1.9 × E-5
1.75	0.380814721	0.380862804	0.380862804	4.8 × E-5	4.8 × E-5
2.00	0	0	0	0	0

Table 2. Results of action of the scheme on Problem 2.

x	Exact Solution $U(x,t)$	Solution from Finite Difference method $U(x,t)$	Solution from New Method $U(x,t)$	Error from finite difference Method	Error from New Method
0	0	0	0	0	0
0.10	0.30806537	0.30807172	0.30807172	6.3 × E-6	6.3 × E-6
0.20	0.585975167	0.58598723	0.58598723	1.20 × E-5	1.20 × E-5
0.30	0.806525626	0.80654224	0.80654224	1.7 × E-5	1.7 × E-5
0.40	0.948127737	0.948147264	0.948147264	2.0 × E-5	2.0 × E-5
0.50	0.9969205	0.996941032	0.996941032	2.1 × E-5	2.1 × E-5
0.60	0.948127737	0.948147264	0.948147264	2.0 × E-5	2.0 × E-5
0.70	0.806525626	0.80654224	0.80654224	1.7 × E-5	1.7 × E-5
0.80	0.585975167	0.58598723	0.58598723	1.20 × E-5	1.20 × E-5
0.90	0.30806537	0.30807172	0.30807172	6.3 × E-6	6.3 × E-6
1.00	0	0	0	0	0

the diffusion convection equation (Table 1)

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} + 2\frac{\partial U}{\partial x} \qquad 0 < x < 20,\ 0 < t < 20$$

$$U(x,0) = |10 - x|,\ 0 \le x \le 20$$

$$U(0,t) = U(20,t) = 0, \qquad 0 \le t \le 20$$

Example 2

Here we use the scheme to approximate the solution to the diffusion convection equation (Table 2)

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} + 2\frac{\partial U}{\partial x} \qquad 0 < x < 20,\ 0 < t < 20$$

$$U(x,0) = \frac{x}{2},\ 0 \le x \le 20$$

$$U(0,t) = 0,\ U(20,t) = 0, \qquad 0 \le t \le 20$$

Conclusion

A numerical method which involves the discretization of the diffusion convection equations to a system of algebraic equations is formulated for solving diffusion convection problems. To check the accuracy of the numerical method, it is applied to solve two different test problems with known finite difference solutions. The overall numerical results obtained showed that the values obtained are the same with the results of the finite difference method. These confirmed the validity of the new numerical scheme and suggested that it is an interesting and viable numerical method.

CONFLICT OF INTERESTS

The author has not declared any conflict of interests.

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