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Full Length Research Paper

Magneto–thermo-viscoelastic material with a spherical cavity

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The present paper is concerned with a homogeneous isotropic perfect conducting viscoelastic body with a spherical cavity subjected to both ramp-type heating and external constant magnetic field. The model of the equations in the context of generalized thermoelasticity with one relaxation time is introduced. The closed form solution for distributions of displacement; temperature, strain, and stress are obtained by using the Laplace transform and the state-space approach. Numerical results applicable to a material - like copper are presented graphically.

Key words: Thermoelasticity, magneto-thermo-viscoelasticity, Lord-Shulman theory, state space approach.

INTRODUCTION

The classical uncoupled theory of thermoelasticity predicts two phenomena not compatible with physical observations. First, the equation of heat conduction of this theory does not contain any elastic terms; second, the heat equation is of a parabolic type, predicting infinite speeds of propagation for heat waves.

Biot (1956) introduced the theory of coupled thermoelasticity to overcome the first shortcoming. The governing equations for this theory are coupled, eliminating the first paradox of the classical theory. However, both theories share the second shortcoming since the heat equation for the coupled theory is of a mixed parabolic-hyperbolic type.

Three generalizations to the coupled theory are considered here. The first is due to Lord and Shulman (1967), who obtained a wave-type heat equation by postulating a new law of heat conduction (the Maxwell–Cattaneo equation) to replace the classical Fourier law. Because the heat equation of this theory is of the wave-type, it automatically ensures finite speeds of propagation for heat and elastic waves. The remaining governing equations for this theory, namely, the equations of motion and constitutive relations, remain the same as those for the coupled and the uncoupled theories. Joseph and Preziosi (1989, 1990) state that the Maxwell–Cattaneo equation is the most obvious and simple generalization of the Fourier law that gives rise to a finite propagation speed.

The second generalization to the coupled theory of elasticity is what is known as the theory of thermoelasticity with two relaxation times or the theory of temperature-rate-dependent thermoelasticity. Müller (1971), in a review of the thermodynamics of thermoelastic solids, proposed an entropy production inequality, with the help of which he considered restrictions on a class of constitutive equations. A generalization of this inequality was proposed by (Green and Laws (1972). Green and Lindsay (1975) obtained another version of the constitutive equations. These equations were also obtained independently and more explicitly by Şuhubi (1975). This theory contains two constants that act as relaxation times and modify all the equations of the coupled theory, not only the heat equation. The classical Fourier law of heat conduction is not violated if the medium under consideration has a center of symmetry. One can refer to Ignaczak (1991), for a review, presentation of the two theories, and some important results obtained in this field.

The third generalization to the coupled theory is known as the dual-phase-lag thermoelasticity, proposed by Chandrasekhraiah (1998), in which the Fourier law is

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replaced by an approximation to a modification of the Fourier law with two different translations for the heat flux and the temperature gradient. One can refer to Hetnarski and Ignaczak in their survey article (1999) in which they examined five generalizations to the coupled theory and obtained a number of important analytical results.

Viscoelastic materials are those for which the relationship between stress and strain depends on time. All materials exhibit some viscoelastic response. In common metals such as steel, aluminum, copper etc. at room temperature and small strain, the behavior does not deviate much from linear elasticity. Synthetic polymer, wood as well as metals at high temperature display significant viscoelastic effects. With the rapid development of polymer science and plastic industry, as well as the wide use of materials under high temperature in modern technology and application of biology and geology in engineering, the theoretical study and applications in viscoelastic materials has become an important task for solid mechanics.

Linear viscoelastic materials are rheological materials that exhibit time temperature rate-of-loading dependence. When their response is not only a function of the current input, but also of the current and past input history, the characterization of the viscoelastic response can be expressed using the convolution (hereditary) integral. A general overview of time-dependent material properties has been presented by Tschoegl (1997). Additionally, a detailed description of the physical response of linear voiscoelastic materials has been explained by Lee and Knauss (2000), based on ramp tests to determine the relaxation modulus, which is a time-domain linear viscoelastic response function. The mechanical-model representation of linear viscoelastic behavior results was investigated by Gross (1953), Staverman and Schwarzl (1956), Alfery and Gurnee (1956) and Ferry (1977). One can refer to Atkinson and Craster (1995) for a review of fracture mechanics and generalizations to the viscoelastic materials.

The theory of coupled thermo-viscoelasticity, and the solutions of some boundary value problems of thermo-viscoelasticity were investigated by Biot (1954), Morland and Lee (1960), Ilioushin and Pobedria (1970) and Gurtin (1972).

The theory of magneto-thermo-elasticity (MTE) was developed with the possibilities of their extensive practical applications in diverse fields such as geophysics, optics and acoustics and so on. A survey of relevant magneto-thermo–elasticity theories were studied by Wilson (1963) and Paria (1967) in the second half of the last century. Using generalized theory of heat conduction of Lord–Shulman, a large number of research workers made valuable contributions in magneto-thermo-elasticity during the last three decades. Öncü and Moodie (1989, 1990) made an analysis of the thermal transient generated by non-uniform sources applied to circular cavities and circular hole in inhomogeneous conductor. (Sherief and Ezzat, 1996) solved a thermal shock half-space problem using asymptotic expansions. Lately, Sherief and Ezzat (1998) solved a problem for an infinitely long annular cylinder, while Ezzat (1997) and Ezzat and Youssef (2005) solved a two-dimensional problem for perfectly conducting media.

The theory of magneto-thermo-viscoelasticity (MTVE) has aroused much interest in many industrial appliances, particularly in nuclear devices, where there exists a primary magnetic field. Various investigations have been carried out by considering the interaction between magnetic, thermal and strain fields. Analyses of such problems also influence various applications in biomedical engineering as well as in different geomagnetic studies. Misra et al. (1992), has studied a one-dimensional uncoupled magnetic-thermoelastic problem in a viscoelastic medium using Maclaurin's approximation method valid for only a specific range of parameters.

The generalized thermo-viscoelasticity models ignoring the relaxation effects of the volume are established by El-Karamany and Ezzat (2004a, b). Among the theoretical contributions to the subject are the proofs of uniqueness theorems under different conditions by Ezzat and El-Karamany (2002) and the boundary element formulation were presented by (El-Karamany and Ezzat, 2002). A state-space method for the calculation of dynamic response of systems made of viscoelastic materials with exponential type relaxation kernels was introduced by (Menon and Tang, 2004). Extensions of thermo-viscoelastic and magneto-thermo-viscoelastic problems in generalized theory are found to be present in the works of many researchers amongst whom are Mukhopadhyay and Bera (1992) and Rakshit and Mukhopadhyay (2005).

Youssef (2005), studied the problem of generalized thermoelasticity with one relaxation time with variable modulus of elasticity and the thermal conductivity were used to solve a problem of an infinite material with a spherical cavity. The inner surface of the cavity was taken to be traction free and acted upon by a thermal shock to the surface and Youssef and Al-Harby (2007) solved the previous problem in two-temperature theory when elastic parameters are taken as constant values.

This paper introduces a model for generalized magneto- thermo- viscoelasticity with one relaxation time. For this model, we shall formulate the state space approach developed in Ezzat (2008) to problems of magneto-thermoviscoelasticity. The resulting formulation is applied to a thermal stresses problem of an electrically perfect conducting infinite solid with a spherical cavity subjected to ramp-type heating in the presence of a constant magnetic field. Laplace transform technique is used throughout. The inversion of the transforms is carried out using a numerical inversion technique (Honig and Hirdes, 1984).

Formulation of the problem

We shall consider a homogeneous isotropic thermo-viscoelastic medium occupying the region R ≤ r < ∞ of a perfect electrically conductivity permeated by an initial constant magnetic field H_o, where R is the radius of the shell. Due to the effect of this magnetic field there arises in the conducting medium an induced magnetic field *h* and induced electric field *E*. Also, there arises a force F (the Lorentz Force). Due to the effect of this force, points of the medium undergo a displacement u, which gives rise to a temperature.

The linearized equations of electromagnetism for slowly moving media are as described by Ezzat (1997):

$$\text{Curl } \boldsymbol{h} = \boldsymbol{J}, \tag{1}$$

$$\text{curl}\,\mathbf{E} = -\mu_o \frac{\partial \mathbf{h}}{\partial t} \tag{2}$$

$$\boldsymbol{B} = \mu_o \boldsymbol{H}, \tag{3}$$

$$div\,\boldsymbol{B} = 0, \tag{4}$$

where, $\boldsymbol{H} = \boldsymbol{H_o} + \boldsymbol{h}$ is the total magnetic field vector, $\boldsymbol{J}$ is current density vector, $\boldsymbol{B}$ is the magnetic induce vector, μ_0 is the magnetic permeability.

The above field equations are supplemented by constitutive equations which consist first of Ohm's law

$$\boldsymbol{E} = -\mu_0 \frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{H}_0. \tag{5}$$

The second constitutive equation is the one for the Lorenz force which is

$$\boldsymbol{F} = \boldsymbol{J} \times \boldsymbol{B}. \tag{6}$$

The third constitutive equation is the stress – displacement – temperature relation for viscoelastic medium of Kelvin – Voigt type.

$$\tau_{ij} = 2\left(\mu_e + \mu_v \frac{\partial}{\partial t}\right) e_{ij} + \left(\lambda_e + \lambda_v \frac{\partial}{\partial t}\right) e\,\delta_{ij} - \gamma\theta\,\delta_{ij}, \tag{7}$$

where, $u = (u_r, u_\psi, u_\phi)$ are displacement vector, τ_{ij} are the components of the stress tensor, e_{ij} are the components of the strain tensor, $e = e_{ii}$ is the dilatation, λ_e and μ_e are Lame's elastic constants, λ_v and μ_v are Lame's viscoelastic constants for the viscoelastic solid, $\gamma = (2\mu_e + 3\lambda_e)\alpha_t$ is a constant material, α_t being the coefficient of linear thermal expansion, $\theta = T - T_o$ is the temperature increment such that $|\theta / T_o|$ <<1 and δ_{ij} is the Kronecker's delta.

The equation of motion is given by

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \left[(\lambda_e + \mu_e) + (\lambda_v + \mu_v)\frac{\partial}{\partial t}\right] u_{j,ij} + \left[\mu_e + \mu_v \frac{\partial}{\partial t}\right] u_{j,ij} - \gamma\theta_i + \mu_0 (\boldsymbol{J} \times \boldsymbol{H_o})_i \tag{8}$$

where, ρ is the density, the comma denotes material derivatives and the summation convention is used.

The generalized heat conduction equation is given by

$$K\theta_{,ii} = \rho C_E(\dot{\theta} + \tau_0 \ddot{\theta}) + \gamma T_0(\dot{e} + \tau_0 \ddot{e}), \tag{9}$$

where, *K* is the thermal conductivity, C_E is the specific heat at constant strain, τ_0 is the relaxation time.

The strain displacement relation is given by

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{10}$$

Together with the previous equations, constitute a complete system of generalized magneto–thermo–viscoelasticity equations for a medium with a perfect electric conductivity.

Let (*r*, ψ, ϕ) denote the radial coordinates, the co- latitude, and the longitude of a spherical coordinates system, respectively. Due to spherical symmetry, all the considered function will be functions of r and t only.

The components of the displacement vector will be taken the form

$$u_r = u(r,t) \quad , \quad u_\psi = u_\phi = 0. \tag{11}$$

The strain tensor components are thus given by

$$e_{rr} = \frac{\partial u}{\partial r} \quad , \quad e_{\psi\psi} = e_{\phi\phi} = \frac{u}{r} \quad , \quad e_{r\phi} = e_{\phi\psi} = 0. \tag{12}$$

It follow that the cubical dilatation is of the form

$$e = \frac{\partial u}{\partial r} + \frac{2u}{r} = \frac{1}{r^2}\,\frac{\partial (r^2 u)}{\partial r}. \tag{13}$$

From Equation (7) we obtain the components of the stress tensor as

$$\tau_{rr} = 2\left(\mu_e + \mu_v \frac{\partial}{\partial t}\right)\frac{\partial u}{\partial r} + \left(\lambda_e + \lambda_v \frac{\partial}{\partial t}\right) e - \gamma\theta, \tag{14}$$

$$\tau_{\phi\phi}=\tau_{rr}=2\left(\mu_e+\mu_v\frac{\partial}{\partial t}\right)\frac{u}{r}+\left(\lambda_e+\lambda_v\frac{\partial}{\partial t}\right)e-\gamma\theta, \quad (15)$$

$$\tau_{r\phi}=\tau_{r\psi}=\tau_{\psi\phi}=0 \quad (16)$$

Assume now that the initial magnetic field acts in the ϕ–direction and has the components (0, 0, H_0).

The induced magnetic field h will have one component h in the Φ–direction, while the induced electric field ***E*** will have one component *E* in the ψ – direction.

Then, equations (1), (2) and (5) yield

$$J=H_0\frac{\partial e}{\partial r}, \quad (17)$$

$$h=-H_0\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right), \quad (18)$$

$$E=\mu_0 H_0\frac{\partial u}{\partial t}. \quad (19)$$

From Equations (17) and (6), we get that the Lorentz force has only one component F_r in the *r* – direction:

$$F_r=\mu_0 H_0^2\frac{\partial e}{\partial r}. \quad (20)$$

Also, we arrived at

$$\rho\frac{\partial^2 u}{\partial t^2}=\left[(\lambda_e+2\mu_e)+(\lambda_v+2\mu_v)\frac{\partial}{\partial t}+\mu_0 H_0^2\right]\frac{\partial e}{\partial r}-\gamma\frac{\partial\theta}{\partial r}. \quad (21)$$

Equation (21) is to be supplemented by the constitutive Equation (13) and the heat conduction equation

$$K\nabla^2\theta=\left(\frac{\partial}{\partial t}+\tau_0\frac{\partial^2}{\partial t^2}\right)\left(\rho C_E\theta+\gamma T_0 e\right), \quad (22)$$

where ∇^2 is **L**aplace's operator in spherical coordinates which is given by

$$\nabla^2=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2 sin\psi}\frac{\partial}{\partial\psi}\left(sin\psi\frac{\partial}{\partial\psi}\right)+\frac{1}{r^2 sin^2\psi}\frac{\partial^2}{\partial\varphi^2}$$

In case of dependence on r only, this reduce to

$$\nabla^2=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)$$

Now, we shall use the following non dimensional variables

$$r'=C_1\eta r,\ u'=C_1\eta u,\ t'=C_1^2\eta t,\ \tau_0^1=C_1^2\eta\tau_0,$$

$$\tau'_{ij}=\frac{\tau_{ij}}{\mu_e},\ \lambda'_e=\lambda_e,\ \mu'_e=\mu_e,\ \lambda_v=C_1^2\eta\lambda_v,$$

$$\mu'_v=C_1^2\eta\mu_v,\ e'=e,\ \theta=\frac{\theta}{T_0},\ h'=\frac{h}{H_0},$$

$$E'=\frac{E}{\mu_0 H_0 C_1},\ J'=\frac{J}{\eta H_0 C_1}.$$

Equations (14)–(19), (21) and (22) take the following form (droping the primes for convenience).

$$J=\frac{\partial e}{\partial r}, \quad (23)$$

$$h=-\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right), \quad (24)$$

$$E=\frac{\partial u}{\partial t}, \quad (25)$$

$$\tau_{rr}=\left(\beta_e^2+\beta_v^2\frac{\partial}{\partial t}\right)e-\left(4+2a\frac{\partial}{\partial t}\right)\frac{u}{r}-b\,\theta, \quad (26)$$

$$\tau_{\psi\psi}=\left(\beta_e^2-2+a_1\frac{\partial}{\partial t}\right)e+\left(2+a\frac{\partial}{\partial t}\right)\frac{u}{r}-b\theta, \quad (27)$$

$$\tau_{r\phi}=\tau_{r\psi}=\tau_{\phi\psi}=0, \quad (28)$$

$$\frac{\partial^2 u}{\partial t^2}=\left(1+a_2\frac{\partial}{\partial t}+R_H\right)\frac{\partial e}{\partial r}-b_1\frac{\partial\theta}{\partial r}, \quad (29)$$

$$\nabla^2\theta=\left(\frac{\partial}{\partial t}+\tau_0\frac{\partial^2}{\partial t^2}\right)\left(\theta+g\,e\right), \quad (30)$$

where,

$$\eta=\frac{\rho C_E}{K},\ C_1^2=\frac{\lambda_e+2\mu_e}{\rho},\ \beta_e^2=\frac{\lambda_e+2\mu_e}{\mu_e},\ \beta_v^2=a+a_1,$$

$$a=\frac{2\mu_v}{\mu_e},\ a_1=\frac{\lambda_v}{\mu_e},\ a_2=\frac{\lambda_v+2\mu_v}{\lambda_e+2\mu_e},$$

$$b_1=\frac{b}{\beta^2},\ b=\frac{\gamma T_0}{\mu_e},\ g=\frac{\gamma T_0}{\rho C_E},\ R_H=\frac{\mu_0 H_0^2}{\lambda_e+2\mu_e},$$

where the coefficient R_H represent the effect of the applied magnetic field on the thermoelastic process proceeding in the body. Equation (29) could be written in the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial^2 u}{\partial t^2})=\left(1+a_2\frac{\partial}{\partial t}+R_H\right)\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial e}{\partial r}\right)-b_1\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)$$

Using Equation (13), we obtain

$$\left(1+a_2\frac{\partial}{\partial t}+R_H\right)\nabla^2 e-b_1\nabla^2\theta=\frac{\partial^2 e}{\partial t^2}. \tag{31}$$

We shall now define the Laplace transform with respect to a function *f (r,t)* by the relation

$$L[f(r,t)]=\bar{f}(r,p)=\int_0^\infty e^{-pt}f(r,t)\,dt\ ,\ \mathrm{Re(p)}>0$$

Applying the Laplace transform to both sides of Equations (23)–(28), (30) and (31), we get

$$\bar{\tau}_{rr}=\left(\beta_e^2+\beta_v^2\ p\right)\bar{e}-\left(4+2a\ p\right)\frac{\bar{u}}{r}-b\ \bar{\theta}\ , \tag{32}$$

$$\bar{\tau}_{\varphi\varphi}=\bar{\tau}_{\psi\psi}=\left(\beta_e^2-2+a_1 p\right)\bar{e}+\left(2+a_1 p\right)\frac{\bar{u}}{r}-b\bar{\theta}\ , \tag{33}$$

$$\nabla^2\bar{\theta}=\left(p+\tau_0 p^2\right)\left(\bar{\theta}+g\bar{e}\right), \tag{34}$$

$$\left(1+a_2 p+R_H\right)\nabla^2\bar{e}-b_1\nabla^2\bar{\theta}=p^2\bar{e}, \tag{35}$$

$$\bar{J}=\frac{\partial\bar{e}}{\partial r}, \tag{36}$$

$$\bar{h}=-\left(\frac{\partial\bar{u}}{\partial r}+\frac{\bar{u}}{r}\right), \tag{37}$$

$$\bar{E}=P\bar{u}\,. \tag{38}$$

State space formulation

Equations (34) and (35) can be written in the form

$$\nabla^2\bar{e}=\frac{p^2+b_1 g\,(p+\tau_0 p^2)}{1+a_2 p+R_H}\bar{e}+\frac{b_1(p+\tau_0 p^2)}{1+a_2 p+R_H}, \tag{39}$$

$$\nabla^2\bar{\theta}=g\ (p+\tau_0 p^2)\bar{e}+(p+\tau_0 p^2)\,\bar{\theta}\ . \tag{40}$$

Choosing as state variable the temperature increment and the strain component, Equations (39) and (49) can be written in the matrix from

$$\nabla^2\bar{V}(r,p)=A(P)\bar{V}(r,p)\,, \tag{41}$$

where,

$$\bar{V}(r,p)=\begin{bmatrix}\bar{e}(r,p)\\ \bar{\theta}(r,p)\end{bmatrix},\ \text{and}$$

$$A\ (p)=\begin{bmatrix}\dfrac{p^2+b_1 g\,(p+\tau_0 p^2)}{1+a_2 p+R_H} & \dfrac{b_1(p+\tau_0 p^2)}{1+a_2 P+R_H}\\ g\ (p+\tau_0 p^2) & (p+\tau_0 p^2)\end{bmatrix}$$

The formal solution of system (Equation 41) can be written in the form

$$\bar{\mathrm{V}}(\mathrm{r,p})=\frac{\mathrm{e}^{-\sqrt{\mathrm{A(P)}}\ \mathrm{r}}}{\mathrm{r}}\mathrm{C}_1+\frac{\mathrm{e}^{\sqrt{\mathrm{A(p)}}\ \mathrm{r}}}{\mathrm{r}}\mathrm{C}_2\,, \tag{42}$$

where, C_1 and C_2 are constants.

For a bounded solution as $r\longrightarrow\infty$, we have to choose $C_2 = 0$, hence we have

$$\bar{\mathrm{V}}\ (\mathrm{r,\ p})\ =\frac{\mathrm{e}^{-\sqrt{\mathrm{A(P)}}\ \ \mathrm{r}}}{\mathrm{r}}\mathrm{C}_1\,. \tag{43}$$

Since $r = R$ must satisfy the last equation. We can get the constant C_1 in the form

$$C_1=R\,e^{\sqrt{A(p)}R}\ \bar{V}\ (R,p)\,. \tag{44}$$

Hence, Equation (43) will take the form

$$\bar{V}\ (r,p)=\frac{R}{r}e^{-\sqrt{A(p)}(r-R)}\ \bar{V}\ (R,p)\ ,\ r\geq R\ , \tag{45}$$

where,

$$\bar{V}\ (R,p)=\begin{bmatrix}\bar{e}\ (R,p)\\ \bar{\theta}\ (R,p)\end{bmatrix}. \tag{46}$$

We will use the well - known Cayley – Hamilton theorem to find the form of the matrix of exp $\left(-\sqrt{A\ (p\ )}(r-R\ )\right)$.

The characteristic equation of the matrix *A (p)* can be written as

$$\kappa^2 - m_1\kappa + m_2 = 0, \tag{47}$$

where,

$$m_1 = \frac{p^2 + (p + \tau_0 p^2)(1 + \varepsilon + a_2 p + R_H)}{1 + a_2 p + R_H},$$

$$m_2 = \frac{p^2 + (p + \tau_0 p^2)}{1 + a_2 p + R_H},$$

where, $\varepsilon = b_1 g$.

The roots of Equation (47) namely, κ_1 and κ_2 , satisfy the relations

$$\kappa_1 + \kappa_2 = \frac{p^2 + (p + \tau_0 p^2)(1 + \varepsilon + a_2 p + R_H)}{1 + a_2 p + R_H}, \tag{48}$$

$$\kappa_1 \kappa_2 = \frac{p^2 + (p + \tau_0 p^2)}{1 + a_2 p + R_H}. \tag{49}$$

The Taylor series expansion for the matrix exponential in Equation (43) is given by

$$\exp\left(-\sqrt{A(p)}\,(r-R)\right) = \sum_{n=0}^{\infty} \frac{\left[-\sqrt{A(p)}\,(r-R)\right]^n}{n\,!}. \tag{50}$$

Using Cayley–Hamilton theorem, we can express $\sqrt{A}$ and higher order of the matrix *A* in terms of *A* and *I* where *I* is the unit matrix of second order.

Thus, the infinite series in Equation (50) can be reduced to

$$\exp\left(-\sqrt{A(p)}(r-R)\right) = a_0 I + a_1 A, \tag{51}$$

where, a_0 and a_1 are some coefficients depending on *p* and *r* only.

By Cayley–Hamilton theorem, the characteristic roots κ_1 and κ_2 of the matrix *A* must satisfy Equation (51), thus we have

$$\exp\left(-\sqrt{\kappa_1}(r-R)\right) = a_0 + a_1\kappa_1. \tag{52}$$

$$\exp\left(-\sqrt{\kappa_2}(r-R)\right) = a_0 + a_1\kappa_2. \tag{53}$$

Solving the above linear system of equations, we get

$$a_o = \frac{\kappa_1 e^{-\sqrt{\kappa_2}(r-R)} - \kappa_2 e^{-\sqrt{\kappa_1}(r-R)}}{\kappa_1 - \kappa_2}, \tag{54}$$

$$a_1 = \frac{e^{-\sqrt{\kappa_1}(r-R)} - e^{-\sqrt{\kappa_2}(r-R)}}{\kappa_1 - \kappa_2}. \tag{55}$$

From Equations (54) and (55) in (51), we deduce the following matrix

$$\exp\left(-\sqrt{A(p)}(r-R)\right) = [L_{ij}], \quad i, j = 1,2 \tag{56}$$

where,

$$L_{11} = \frac{1}{\kappa_1 - \kappa_2}\left[\left(\frac{p^2 + \varepsilon(p + \tau_0 p^2)}{1 + a_2 p + R_H} - \kappa_2\right) e^{-\sqrt{\kappa_1}(r-R)} - \left(\frac{p^2 + \varepsilon(p + \tau_0 p^2)}{1 + a_2 p + R_H} - \kappa_1\right) e^{-\sqrt{\kappa_2}(r-R)}\right]$$

$$L_{12} = \frac{1}{\kappa_1 - \kappa_2}\left[\frac{b_1(p + \tau_0 p^2)}{1 + a_2 p + R_H} e^{-\sqrt{\kappa_1}(r-R)} - \frac{b_1(p + \tau_0 p^2)}{1 + a_2 p + R_H} e^{-\sqrt{\kappa_2}(r-R)}\right]$$

$$L_{21} = \frac{1}{\kappa_1 - \kappa_2}\left[g\,(p + \tau_0 p^2) e^{-\sqrt{\kappa_1}(r-R)} - g\,(p + \tau_0 p^2) e^{-\sqrt{\kappa_2}(r-R)}\right]$$

$$L_{22} = \frac{1}{\kappa_1 - \kappa_2}\left[(p + \tau_0 p^2 - \kappa_2) e^{-\sqrt{\kappa_1}(r-R)} - (p + \tau_0 p^2 - \kappa_1) e^{-\sqrt{\kappa_2}(r-R)}\right]. \tag{57}$$

We can write the solution in the form

$$\bar{V}(r,p) = \frac{R}{r}[L_{ij}]\bar{V}(R,p). \tag{58}$$

Finally, we have

$$\bar{e}(r,p) = \frac{R}{(\kappa_1 - \kappa_2)\,r}\left[L_1 e^{-\sqrt{\kappa_1}(r-R)} - L_2 e^{-\sqrt{\kappa_2}(r-R)}\right], \tag{59}$$

$$\bar{\theta}(r,p) = \frac{R}{(\kappa_1 - \kappa_2)\,r}\left[M_1 e^{-\sqrt{\kappa_1}(r-R)} - M_2 e^{-\sqrt{\kappa_2}(r-R)}\right], \tag{60}$$

$$\bar{u}=\frac{-R}{(\kappa_1-\kappa_2)r}\left[L_1\left(\frac{1+\sqrt{\kappa_1}\,r}{\kappa_1\,r}\right)e^{-\sqrt{\kappa_1}(r-R)}-L_2\left(\frac{1+\sqrt{\kappa_2}\,r}{\kappa_2\,r}\right)e^{-\sqrt{\kappa_2}(r-R)}\right] \quad (61)$$

$$\bar{\tau}_{rr}=\frac{R}{(\kappa_1-\kappa_2)r}\left\{\left(\left[\beta_e^2+p\beta_v^2+(4+2aP)\left(\frac{1+\sqrt{\kappa_1}\,r}{\kappa_1\,r^2}\right)\right]L_1-bM_1\right)e^{-\sqrt{\kappa_1}(r-R)}-\left(\left[\beta_e^2+p\beta_v^2+(4+2ap)\left(\frac{1+\sqrt{\kappa_2}\,r}{\kappa_2\,r^2}\right)\right]L_2-bM_2\right)e^{-\sqrt{\kappa_2}(r-R)}\right\} \quad (62)$$

$$\bar{\tau}_{\varphi\varphi}=\bar{\tau}_{\psi\psi}=\frac{R}{(\kappa_1-\kappa_2)r}\left\{\left(\left[\beta_e^2-2a_1p+(2+ap)\left(\frac{1+\sqrt{\kappa_1}\,r}{\kappa_1 r^2}\right)\right]L_1-bM_1\right)e^{-\sqrt{\kappa_1}(r-R)}-\left(\left[\beta_e^2-2a_1p+(2+ap)\left(\frac{1+\sqrt{\kappa_2}\,r}{\kappa_2 r^2}\right)\right]L_2-bM_2\right)e^{-\sqrt{\kappa_2}(r-R)}\right\} \quad (63)$$

where,

$$L_1=\left(\frac{p^2+\varepsilon(p+\tau_0p^2)}{1+a_2p+R_H}-\kappa_2\right)\bar{e}(R,p)+\left(\frac{b_1(p+\tau_0p^2)}{1+a_2p+R_H}\right)\bar{\theta}(R,p)$$

$$L_2=\left(\frac{p^2+\varepsilon(p+\tau_0p^2)}{1+a_2p+R_H}-\kappa_1\right)\bar{e}(R,p)+\left(\frac{b_1(p+\tau_0p^2)}{1+a_2p+R_H}\right)\bar{\theta}(R,p)$$

$$M_1=g(p+\tau_0p^2)\bar{e}(R,p)+(p+\tau_0p^2-\kappa_2)\bar{\theta}(R,p)$$

$$M_2=g(p+\tau_0p^2)\bar{e}(R,p)+(p+\tau_0p^2-\kappa_1)\bar{\theta}(R,p)$$

Application

In order to evaluate the unknown parameters L_1, L_2, M_1 and M_2, we will use the boundary conditions on the internal surface of the shell, r = R which is given by:

Thermal boundary condition

The internal surface with $r = R$ is subjected to ramp – type heating in the form

$$\theta(R,t)=\begin{cases}0 & t\le 0\\ \dfrac{\theta_1}{t_o}t & 0<t\le t_o\\ \theta_1 & t>t_o\end{cases}, \quad (64)$$

where, θ_1 is constant and to is called the ramping parameter. Applying the Laplace transform, we get

$$\bar{\theta}(R,P)=\frac{\theta_1(1-e^{-pt_0})}{t_0p^2} \quad (65)$$

Mechanical boundary condition

The internal surface $r = R$ has a rigid foundation, which is rigid enough to prevent any strain, yield
e (R, t) = 0.

Applying the Laplace transform, we get

$$\bar{e}(R,p)=0 \quad (66)$$

Using the conditions (65) and (66) in to Equations (59) – (63), we get

$$\bar{e}(r,p)=\frac{RL}{(\kappa_1-\kappa_2)r}\left[e^{-\sqrt{\kappa_1}(r-R)}-e^{-\sqrt{\kappa_2}(r-R)}\right], \quad (67)$$

$$\bar{\theta}(r,p)=\frac{R}{(\kappa_1-\kappa_2)r}\left[M_1'e^{-\sqrt{\kappa_1}(r-R)}-M_2'e^{-\sqrt{\kappa_2}(r-R)}\right], \quad (68)$$

$$\bar{u}=\frac{-LR}{(\kappa_1-\kappa_2)r}\left[\left(\frac{1+\sqrt{\kappa_1}\,r}{\kappa_1\,r}\right)e^{-\sqrt{\kappa_1}(r-R)}-\left(\frac{1+\sqrt{\kappa_2}\,r}{\kappa_2\,r}\right)e^{-\sqrt{\kappa_2}(r-R)}\right] \quad (69)$$

$$\bar{\tau}_{rr}=\frac{R}{(\kappa_1-\kappa_2)r}\left\{\left(\left[\beta_e^2+p\beta_v^2+(4+2ap)\left(\frac{1+\sqrt{\kappa_1}\,r}{\kappa_1 r^2}\right)\right]L-bM_1'\right)e^{-\sqrt{\kappa_1}(r-R)}-\left(\left[\beta_e^2+p\beta_v^2+(4+2ap)\left(\frac{1+\sqrt{\kappa_2}\,r}{\kappa_2 r^2}\right)\right]L-bM_2'\right)e^{-\sqrt{\kappa_2}(r-R)}\right\}$$

(70)

$$\bar{\tau}_{\varphi\varphi}=\bar{\tau}_{\psi\psi}=\frac{R}{(\kappa_1-\kappa_2)r}\left\{\left(\left[\beta_e^2-2a_1p+(2+ap)\left(\frac{1+\sqrt{\kappa_1}\,r}{\kappa_1 r^2}\right)\right]L_1-bM_1'\right)e^{-\sqrt{\kappa_1}(r-R)}-\left(\left[\beta_e^2-2a_1p+(2+ap)\left(\frac{1+\sqrt{\kappa_2}\,r}{\kappa_2 r^2}\right)\right]L_2-bM_2'\right)e^{-\sqrt{\kappa_2}(r-R)}\right\} \quad (71)$$

where,

$$L = \left(\frac{b_1(p + \tau_0 p^2)}{1 + a_2 p + R_H} \right) \bar{\theta}(R, p) \quad (72)$$

$$M_1' = (p + \tau_0 p^2 - K_2)\bar{\theta}(R, p) \quad (73)$$

$$M_2' = (p + \tau_0 p^2 - K_1)\bar{\theta}(R, p) \quad (74)$$

By obtaining θ_1, the temperature increment θ can be obtained by solving Equation (64) to give

$$\theta = \frac{-1 + \sqrt{1 + 2\kappa_1 \theta}}{\kappa_1} \quad (75)$$

They complete the solution on the Laplace domain.

Inversion of the Laplace transforms

In order to invert the Laplace transforms, we adopt a numerical inversion method based on a Fourier series expansion [42]. In this method, the inverse g (t) of the Laplace transform $\bar{g}(s)$ is approximated by the relation;

$$g(t) = \frac{e^{ct}}{t_1}\left[\frac{1}{2}\bar{g}(c) + \mathrm{Re}\left(\sum_{k=1}^{\infty} e^{ik\pi t/t_1}\,\bar{g}(c + ik\pi/t_1)\right)\right], 0 \le t \le 2t_1,$$

where N is a sufficiently large integer representing the number of terms in the truncated infinite Fourier series. N must chosen such that

$$e^{ct}\mathrm{Re}\left[e^{iN\pi t/t_1}\,\bar{g}(c + iN\pi/t_1)\right] \le \varepsilon_1,$$

where ε_1 is a persecuted small positive number that corresponds to the degree of accuracy to be achieved. The parameter c is a positive free parameter that must be greater than the real parts of all singularities of $\bar{g}(s)$. The optimal choice of c was obtained according to the criteria described by (Honig and Hirdes, 1984).

RESULTS AND DISCUSSION

The copper material was chosen for purpose of numerical evaluations and the constants of the problem were taken as follows (Abd-Alla et al., 2004):

α_t = 1.78 (10^{-5}) K^{-1}, C_E = 383.1 m^2/K , μ_e =3.86(10^{10}) N/m^2, λ_e = 7.76(10^{10}) N/m^2 , $\rho = 8954$ kg/m^3 , $\tau_0 = 0.02$ sec., T_o= 293 K , $\beta_e^2 = 4$, $\beta_v^2 = 8$.

The computations were carried out for $\theta_1 = 1$. The temperature, stresses, displacement and strain distributions are represented graphically at different value of time *t*. The field quantities , temperature , stresses , displacement and strain depend not only on the state and space variable *t* and *r* , but also depend on t_o , R_H , and Lame's viscoelastic constants (λ_v and μ_v) . It has been observed that, t_o, R_H , and Lame's viscoelastic constants have significant effect on the temperature, stresses, displacement and strain distributions quantities. Here all the variables / parameters are taking in the non-dimensional forms. Taking *r* range from 1.0 to 2.2 has carried out in numerical analysis.

Figure 1 exhibits the space variation of temperature distribution for (MTVE) theory and for different value of t_o, and we observe that:

(1) Some difference in the values of temperature is noticed for different values of t_o. We can see that the ramping parameter t_o has a clear effect on the values of temperature; actually, the value of temperature increase when $t \ge t_o$ and decrease when $t < t_o$ where larger t with respect to t_o means larger heating on the boundary. Figure 2 exhibits the space variation of temperature distribution for the two theories, and we observe that:

(1) Lame's viscoelastic constants (λ_v and μ_v) have a small effect on the values of temperature; actually, the value of temperature increases in the case of (MTE) theory but decreases in the case of (MTVE) theory, and on the boundary the values of temperature have the same values for the two theories .

Figure 3 exhibits the space variation of strain distribution for (MTVE) theory and for different values of R_H and t_o , and we observe the following:

(1) Some differences in the values of strain are noticed for different values of R_H .We can see that, the absolute value of the maximum point of the strain increase in the absence of magnetic field, (that is, the magnetic field causes increasing in the values of the strain).
(2) The ramping parameter t_o has a clear effect on the values of the strain; actually, the absolute value of the strain increase when $t \ge t_o$ and decrease when $t < t_o$.

Figure 4 exhibits the space variation of strain distribution for the two theories and we observe that:

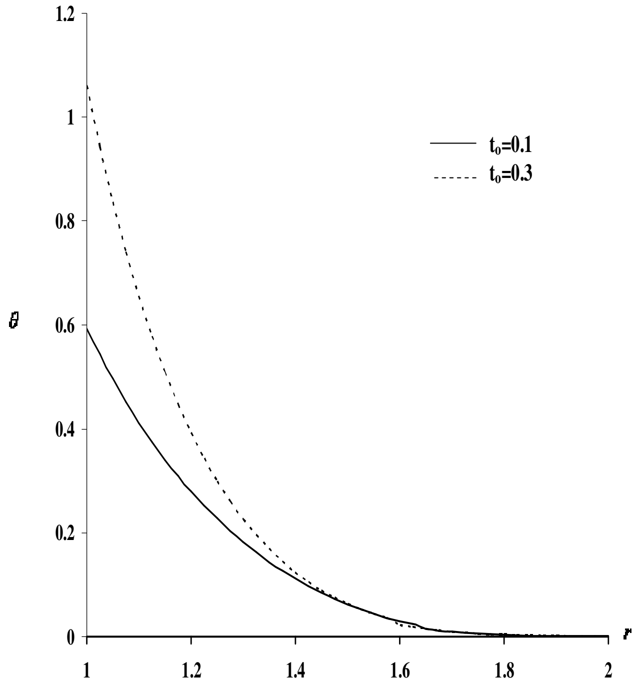

Figure 1. Temperature distribution for MTVE theory at t = 0.2.

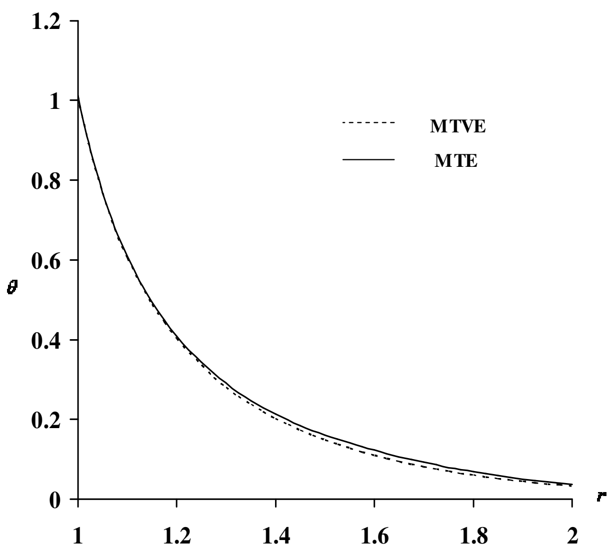

Figure 2. Temperature distribution for two theories at t = 0.3.

(1) Some difference in the value of strain is noticed for the two theories. We can see that the absolute value of the maximum point of the strain decrease in the case of (MTVE) theory and it increase in the case of (MTE) theory.

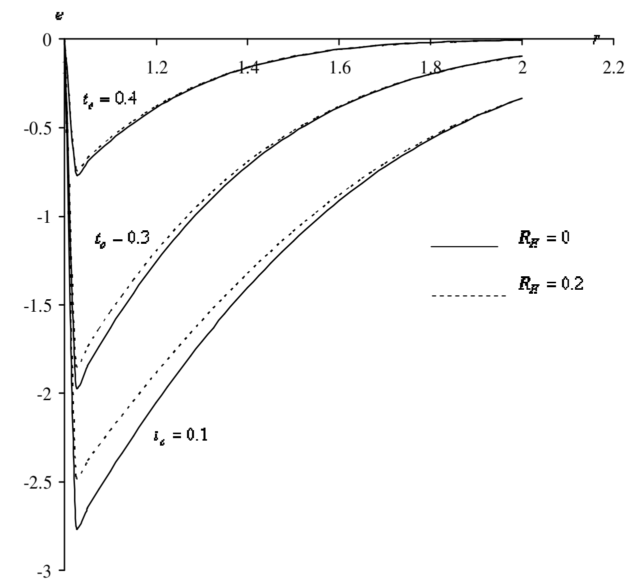

Figure 3. Strain distribution for MTVE theory at t = 0.2.

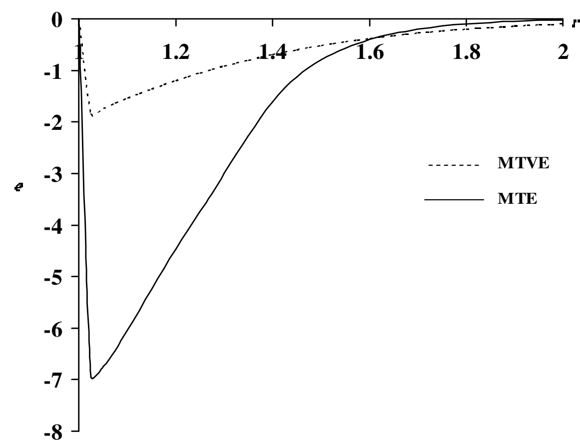

Figure 4. Strain distribution for two theories.

Figure 5 exhibits the space variation of the displacement distribution for (MTVE) theory and for different values of R_H, and we observe the following:

(1) Some difference in the values of displacement is noticed for different value of the parameter R_H.
(2) On the boundary, the values of displacement increase in the absence of magnetic field.

Figure 6 exhibits the space variation of displacement distribution for the two theories, and we observe the following:

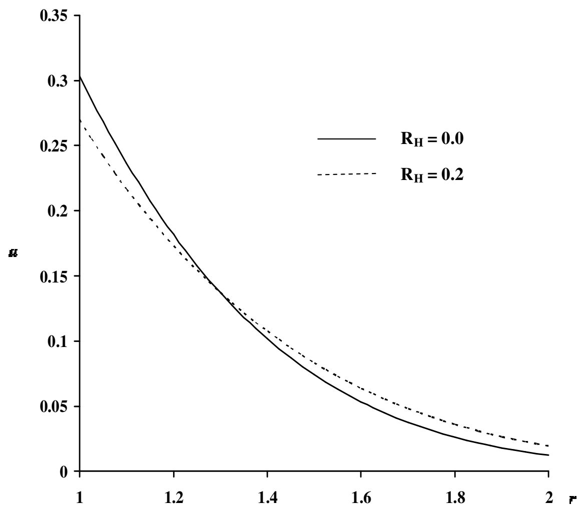

Figure 5. Displacement distribution at t = 0.2.

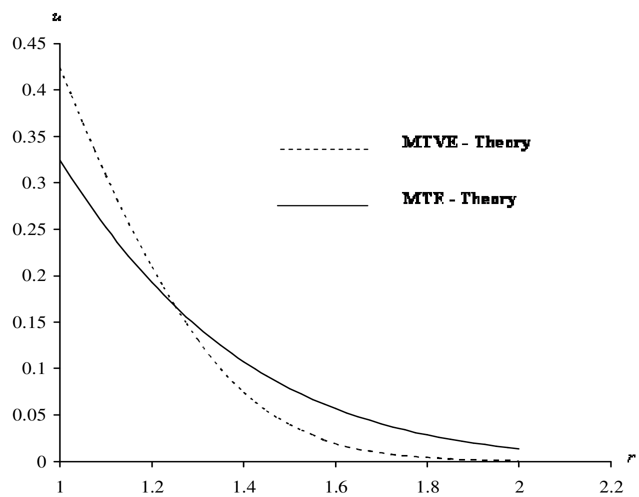

Figure 6. Displacement distribution for two theories.

(1) Some difference in the values of displacement is noticed for the two theories.

(2) On the boundary, the value of displacement increase in the case of (MTVE) theory.

Figure 7 exhibits the space variation of the radial stress distribution for (MTVE) theory and for different values of R_H, and we observe that:

Some differences in the values of the radial stress is noticed for different values of R_H

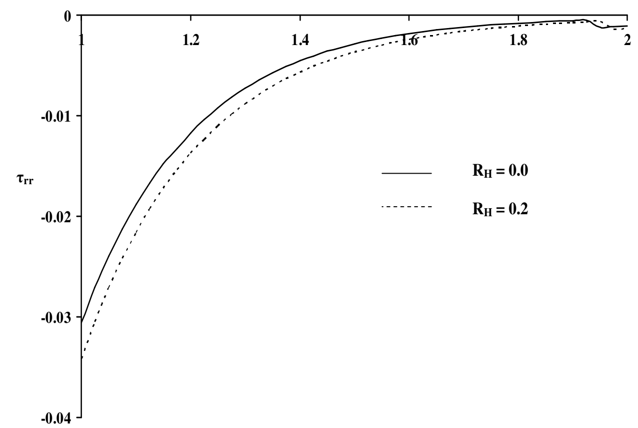

Figure 7. Radial stress distribution for MTVE-theory at t = 0.2.

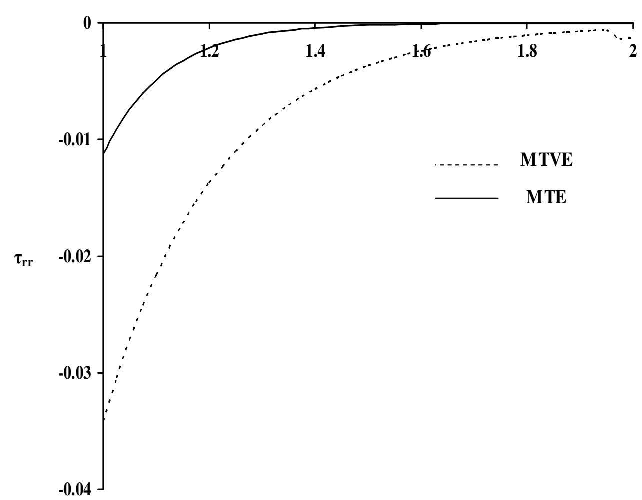

Figure 8. Radial stress distribution for two theories.

Figure 8 exhibits the space variation of the radial stress distribution for the two theories, and we observe the following:

(1) The absolute value of the radial stress increase in the case of (MTVE) theory, and decrease in the case of (MTE) theory, that is, λ_v and μ_v have a clear effect on the values of the radial stress.

(2) On the boundary, the absolute values of the radial stress increase in the case of (MTVE) theory.

One can see that although the curves are smooth and do not have any discontinuous points; they take the same behavior in some range of *r*, where the boundary conditions somewhere are different.

Concluding remarks

The importance of the model used in generalized thermo-viscoelasticity theory is the ability to the separating between the two theories (MTE) and (MTVE). This means that, one able to study the effects of viscosity on the considered functions. Studding the effect of R_H and t_0 on the variables fields, it is found that, all the variables have been affected by the changing of R_H and t_0. Mathematically and physically the ramp-type heating is more realistic than the thermal shock, where the ramping parameter of heating has a clear affect all the fields.

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