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Full Length Research Paper

Modified exponentially weighted moving average (EWMA) control chart for an analytical process data

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This article introduces modified exponentially weighted moving average (modified EWMA) control chart. The modified EWMA control chart is very effective in detecting small and abrupt shifts in monitoring process mean. This chart is based on modified EWMA control chart statistic, which is a correction of EWMA chart statistic and it is free from inertia problem. The advantage of using modified EWMA chart is its good performance for observations that are autocorrelated or independently normal distributed. The performances of Modified EWMA chart are illustrated along with EWMA chart for the two types of analytical process data. The average run length (ARL) properties of modified EWMA scheme are derived using Markov Chain approach.

Key words: Abrupt change, autocorrelated, average run length, exponentially weighted moving average (EWMA), modified.

INTRODUCTION

The control charts namely, Shewart chart (Shewhart, 1924), EWMA chart (Roberts, 1959) and CUSUM chart (Page, 1954) are often used for detecting shifts in a sequence of independent normal observations with common variance coming from a particular process. The usefulness of a chart is determined from the timely detection of shifts as they occur. The literature on evaluation of properties of these three charts indicates that Shewhart chart detect all single large shifts, CUSUM chart detects shifts by accumulating changes in a direction, while EWMA chart detects shifts through changes accumulated under exponential smoothing. In general, these three control charts will fail to detect shift in one case, either the large shift or the small shift in the process. Moreover, these charts cannot be used directly in chemical and pharmaceutical industries because the observations from processes in these industries are frequently autocorrelated. This autocorrelation has a large impact on the control charts developed under the independence assumption. A typical effect of autocorrelation is to decrease the in-control average run length (ARL) leading to a higher false alarm rate than for an independent process. Moreover, the autocorrelated processes may also have abrupt changes. Both the situations are bothersome for these industries.

In particular, in most of the industrial processes, temperature is an essential process variable and the process is sensitive to the cumulative small changes as well as some abrupt changes in temperature values. An abrupt change is the unpredictable large shift occurring in the process being monitored for small shifts. Recently, research efforts have been put for the process subject to abrupt changes. EWMA chart has received more attention as it also provides an estimate of forecast of process mean. Yaschin (1995) have discussed estimation of current process mean in a process subject to small changes, as well as abrupt changes on the basis of EWMA. Capizzi and Masarotto (2003) have provided a class of new control charts, AEWMA charts on the basis of EWMA to detect shifts of all types for independent data. Reynolds and Stoumbos (2006) suggested using two EWMA controls simultaneously, one for process mean and another for process variance to tackle detection of small shifts with abrupt changes. Woodall and Mahmoud (2005) have proposed a measure of inertia, the signal resistance, and shown that EWMA suffers from inertia problem more than the CUSUM. On the other hand, several authors have discussed process control methods for autocorrelated observations. Montgomery and Mastrangelo (1991) presented method

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that consists of modeling the autocorrelative structure in the original data using EWMA and applying Shewhart control charts to the independent normal residuals. Their paper led to discussion, and one of the reviewers: Ryan (1991) stated that, residual chart for AR(1) will perform poorly unless autoregressive (AR) parameter is negative or extremely close to 1 and that the derivation of ARL properties was unclear because most of the time, the true state of autocorrelation in the process is not known. Faltin and Woodall (1991) appreciated their method for the virtue of simplicity in the interest of operators and engineers. Cox (1961) have shown that the EWMA is not a useful predictor for an AR(1) process having an autoregressive parameter less than 1/3. It is found that for ARMA (1,1) processes, the EWMA is useful as a predictor provided that, the AR term sufficiently dominates the moving average (MA) component. Harris and Ross (1991) studied the impact of traditional control charts based on residuals under several correlation structures of process and concluded that the traditional EWMA, CUSUM charting techniques perform poorly in the presence of serial correlation and the use of residuals in Shewhart chart is not effective for detecting small shifts when the observations are highly positively correlated. Lu and Reynolds (1999) considered the problem of monitoring the mean of a process in which the observations can be modeled as an AR(1) process plus a random error. They have suggested EWMA control chart based on the residuals from the forecast values of the model using an integral equation method. They showed that when the level of autocorrelation is low or moderate, the EWMA chart for residuals and the EWMA chart for original observations require about the same amount of time to detect various shifts; but for high levels of autocorrelation and large shifts, the EWMA chart of the residuals is faster a little. In all the efforts, an easy solution to control charting is missing. Control charts are effective tools for improving process quality and productivity and simplicity is always demanded by the users.

In this article, we introduce a Modified EWMA (Modified EWMA) chart that combines the features of a Shewhart chart and an EWMA chart in a simple way and has the ability to detect small, as well as large shift as soon as possible as required by some industrial processes with high level of first order autocorrelation. Modified EWMA control statistic gives weight to the past observations in slightly different way than EWMA and each current change is considered with full weight. This corrects EWMA statistic for suffering from inertia problem. The underlying idea is to adapt the weight of the past observations, past changes, the current observation and the current change. Unlike AEWMA control chart which corrects EWMA statistic by considering the difference between process value and control statistic value as the "error"=Y_n-X_{n-1} to detect shifts of different sizes, Modified EWMA control chart does it by measuring the "current fluctuation" contained in EWMA control statistics, as Y_n - Y_{n-1}, the difference between two consecutive process values to detect abrupt changes. This is a new approach and does not require consideration of score function (error function) and separate detection limits; hence it is simpler to operate in comparison to AEWMA chart. We feel that just like EWMA, AEWMA should be applicable to autocorrelated processes, whenever EWMA is apt. The Modified EWMA control statistic is defined as:

$$X_n = (1-\lambda)X_{n-1} + \lambda Y_n + (Y_n - Y_{n-1}),\ 0< \lambda \leq 1.$$

Note that X_n is the geometric sum of past observations, past changes, current observation and the current change in the process. This make Modified EWMA scheme perform like an EWMA scheme for small shifts and at the same time detect abrupt changes alike Shewhart scheme.

EXPONENTIAL WEIGHT BASED QUALITY CONTROL CHARTS

Let Y_n, n =1, 2,… be a sequence of first order autocorrelated or independent normal observations with the process target mean as μ_0 and a common variance σ^2, such as single measurements of the process.

EWMA control chart

An EWMA control chart is based on the statistic:

$$X_n = (1-\lambda)X_{n-1} + \lambda Y_n\ , \qquad (1)$$

where λ is a suitable constant, such that $0< \lambda \leq 1$, X_n is n^{th} statistic and Y_n is n^{th} observation . The quantity X_0 represents the starting value, often the target value μ_0.

This schemes signals when X_n value exceeds specified control limit. In general upper and lower control limits of EWMA Chart are:

$$UCL = \mu_0 + L\,\sigma \sqrt{\frac{\lambda}{(2-\lambda)}}$$

$$CL = \mu_0$$

$$LCL = \mu_0 - L\,\sigma \sqrt{\frac{\lambda}{(2-\lambda)}}\ . \qquad (2)$$

L is suitable in control width limit, and σ is the process standard deviation. The detailed discussion on EWMA control schemes can be found in Montgomery (1996).

Adaptive EWMA control chart

EWMA control chart cannot detect abrupt change/s, and thus fail in worst-case situation. To overcome this problem, Capizzi and Masarotto (2003) designed Adaptive EWMA (AEWMA) control charts. Their class of control charts considers the following statistic:

$$X_n = X_{n-1} + \phi(e_n),\ X_0 = \mu_0 \quad (3)$$

Where $e_n = Y_n - X_{n-1}$ and $\phi(e_n)$ is a "score" function. An alarm is raised when $|X_n - \mu_0| > h$, where μ_0 denotes the target value of the process mean and h is a suitable threshold. The h value is mainly determined by ensuring that the desired mean time between false alarms is large. They have designed AEWMA schemes for three score functions, each based on exponential weight parameter λ and specified values of one or two constants. The AEWMA scheme is effective in detecting shifts of different sizes, but the monitoring scheme is not simple.

Modified EWMA (Modified EWMA) control chart

The desirable properties of a control chart are that it is easy to implement and is effective for detecting shifts of all sizes as per technical specifications. The Modified EWMA chart that we introduce considers past observations similar to EWMA scheme and additionally considers past changes, as well as latest change in the process. A Modified EWMA control chart is based on the statistic:

$$X_n = (1-\lambda)X_{n-1} + \lambda Y_n + (Y_n - Y_{n-1}), \quad (4)$$

where, n = 0,1,… $0 < \lambda \le 1$ is a constant and the starting value is, $X_0 = \mu_0 = Y_0$.

The mean and variance of control statistic X_n are respectively, the process mean μ and $\left[\frac{\lambda}{(2-\lambda)} + \frac{2\lambda(1-\lambda)}{(2-\lambda)}\right]\sigma^2$ for autocorrelated process (Appendix 1), based on $\rho \rightarrow 1$, the first order autocorrelation coefficient, and correlation structure of the process observations and process fluctuations.

Therefore, the upper and lower control limits of Modified EWMA chart are:

$$UCL = \mu_0 + L\sigma\sqrt{\frac{\lambda}{2-\lambda} + \frac{2\lambda(1-\lambda)}{2-\lambda}}$$

$$CL = \mu_0$$

$$LCL = \mu_0 - L\sigma\sqrt{\frac{\lambda}{2-\lambda} + \frac{2\lambda(1-\lambda)}{2-\lambda}} \quad (5)$$

Where μ_0 is the target mean, σ^2 is the process variance, λ and L are suitable Modified EWMA scheme constants representing exponential weight and in control width limit. This chart reduces to Shewhart chart for highly autocorrelated process for λ=1, L=3. Modified EWMA chart is capable for capturing signals of small shift in the process, as well as abrupt changes in an autocorrelated process. The process is in control, as long as the values are plotted within the control limits. A point plots outside the control limits is interpreted as evidence that the process is out of control, and action are required to find and eliminate the assignable causes responsible for this behavior.

We know that EWMA chart suffer from inertia problem because of 'error' in EWMA statistic. Modified EWMA statistic is a corrected EWMA statistic and EWMA is the best predictor in the class of linear predictors (Yashchin, 1993). Hence Modified EWMA is the best predictor of process mean; its mean square error (MSE) is nil for autocorrelated process with nearly one autocorrelation (Appendix 1). It forecasts the process mean state accurately. This makes Modified EWMA chart free from inertia problem. Now we apply Modified EWMA control scheme monitoring along with EWMA scheme on temperature data and capsule weight data from an analytical process.

Application 1: Modified EWMA chart for highly autocorrelated normal process

Table 1 displays the part of measurements on temperature column taken every minute from a chemical process that is working in control and out of control situations, abrupt changes and small shifts occur. The target mean is 110.468°C, and the process standard deviation is σ = 1.131°C (σ^2 =1.279) (Table 1).

Here, EWMA (λ=0.1, L=2.814, UCL= 111.198, LCL=109.738, CL=110.468) control scheme could not detect two abrupt changes at 21st and 28th observations, and the small shifts were detected from 71st observation. Modified EWMA (λ=0.1, L=1.683, UCL = 111.199, LCL=109.737) detects those two abrupt changes and desired small shifts early, timely in 61st run. In Modified EWMA statistic, the smaller the value of λ, the larger is the effect of past history of the process. Therefore, if we select λ=0.1, then L=1.683 is automatically fixed through ARL.

Application 2: Modified EWMA chart for independent normal process

The data in Table 2 are part of the measurements (in grams) taken every 30 s from a manufacturing process that is working in control and an out of control situation,

Table 1. Monitoring performance of EWMA and modified EWMA for the chemical process temperature data.

No. n	Y_n AR(1)≈1	EWMA	Modified EWMA
		λ = 0.1	
		L = 2.814	L = 1.683
0		110.468	110.468
1	110.004	110.422	109.958
…	…	…	…
20	110.379	110.282	110.390
21	108.750	110.123	108.597*
22	110.419	110.129	110.448
…	…	…	…
27	110.518	110.291	110.543
28	107.800	110.042	107.551*
29	110.596	110.097	110.651
…	…	…	…
60	111.169	110.969	111.191
61	111.189	110.991	111.211*
62	111.208	11.013	111.230*
…	…	…	…
70	111.366	111.181	111.387*
71	111.386	111.201*	111.407*
…	…	…	…
780	113.610	113.628*	113.608*

*Shift.

Table 2. Monitoring performance of EWMA and Modified EWMA for the pharmaceutical process capsule weight data.

No. n	Y_n	EWMA	Modified EWMA
		λ = 0.04	
		L = 2.477	L = 1.423
0	5	5	5
1	5.22	5.009	5.229*
2	4.95	5.007	4.948
3	5.2	5.014	5.208*
4	5.41	5.030	5.426*
5	5.2	5.037	5.207*
6	5.02	5.036	5.019
7	5.11	5.039	5.113*
8	5.26	5.048	5.269*
9	5.27	5.057	5.279*
10	3.83	5.009	3.780*

*Shift.

abrupt change occurs. The target mean is 5 g, and the process standard deviation is σ = 0.3 g. Data were adapted from Capazzi and Masarotto (2003).

Here, EWMA (λ=0.04, L=2.477, UCL= 5.106, LCL=4.894, CL=5) control scheme could not detect any changes in observations. Modified EWMA (λ = 0.04, L=1.423, UCL = 5.104, LCL=4.896) detects upper shifts at 1^{st} , 3^{rd} to 5^{th} , 7^{th} to 9^{th}, and lower shift at 10^{th} observations.

From Figure 1, we can see that EWMA chart could not detect any Shift. On the other hand Modified EWMA chart detects all type of shifts as shown in Figure 2.

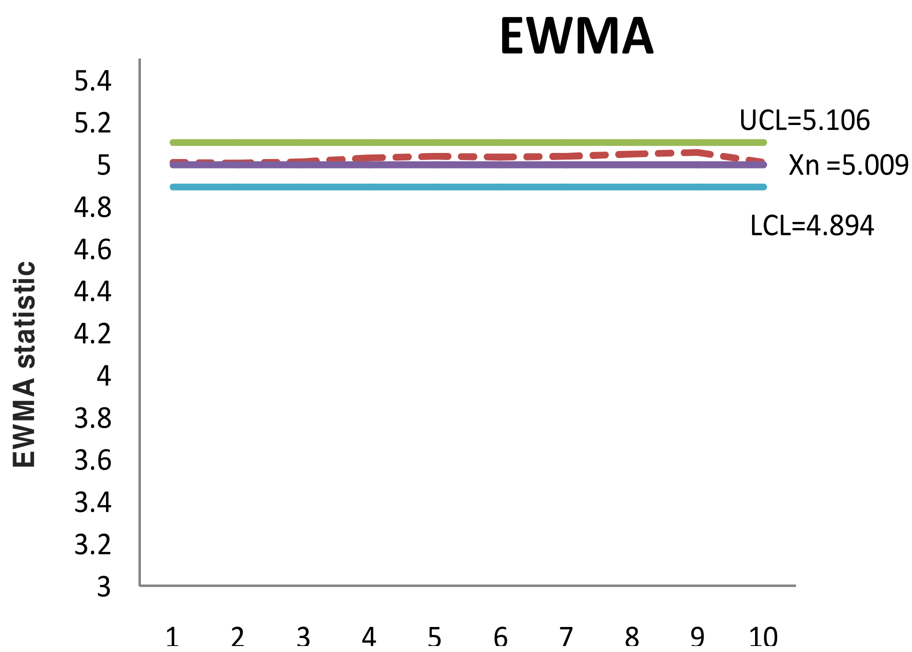

Figure 1. Plot of EWMA control chart.

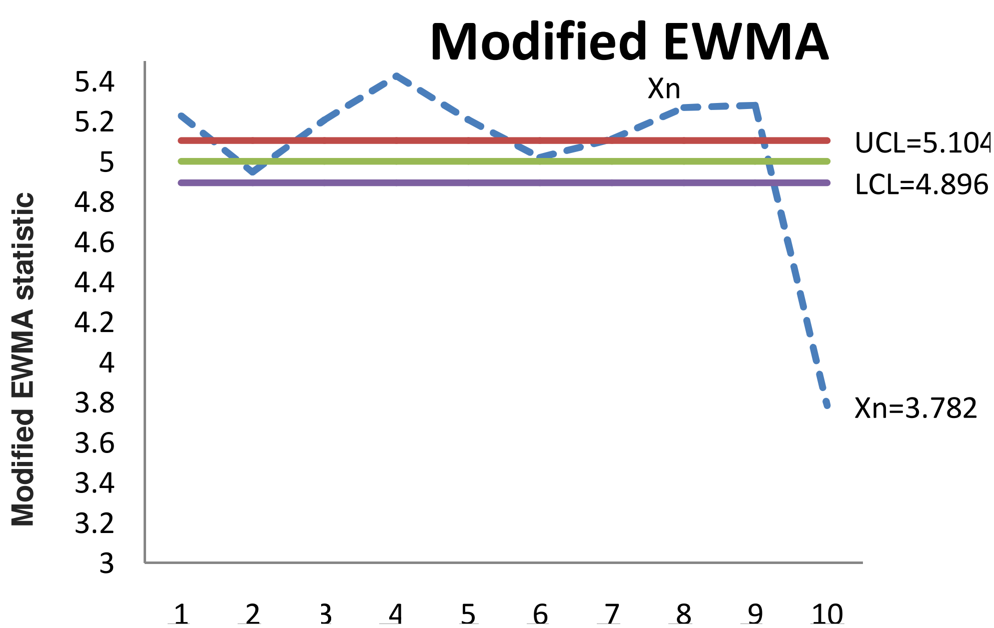

Figure 2. Plot of modified EWMA control chart.

PROPERTIES OF MODIFIED EWMA SCHEME AND COMPARISONS WITH EWMA SCHEME

ARL properties

All the ARL computations were carried out using Markov-chain approach described in Appendix 2. ARL properties of Modified EWMA scheme have been derived for different choices of {λ, L} and in control ARL to be 500 for detection of small shift only, because the abrupt changes are detected as they occur.

The Modified EWMA scheme introduced in this article is a modification of EWMA scheme for detecting small shifts along with abrupt shifts. Here, we have compared the ARL values for Modified EWMA scheme with ARL values of EWMA scheme over a wide range of parameter

Table 3. Average run lengths of modified EWMA schemes.

	Modified EWMA ARL_0 = 500							
L	**2.5**	**2.155**	**2.048**	**1.948**	**1.9**	**1.843**	**1.683**	**1.423**
λ	**0.75**	**0.5**	**0.4**	**0.3**	**0.25**	**0.2**	**0.1**	**0.043**
Shift \ Abrupt	**> 2 σ**	**> 2 σ**	**> 2 σ**	**>2 σ**	**>1 σ**	**>1 σ**	**>0.5σ**	**>0.5σ**
0.0	501	505	502	497	500	500	498	500
0.25	293	232	209	183	170	150	104	73.4
0.50	131	83.1	67.7	53.7	47.7	41.2	30.3	27.3
0.75	58.3	33.3	26.73	21.3	19.24	17.3	14.9	16.6
1.0	28.4	15.87	13.01	10.9	10.2	9.6	9.35	11.6
1.50	8.99	5.52	4.87	4.53	4.49	4.53	5.09	7.08
2.0	3.96	2.8	2.61	2.58	2.64	2.76	3.37	4.99
2.50	2.29	1.84	1.75	1.75	1.8	1.91	2.45	3.8
3.0	1.65	1.47	1.42	1.39	1.4	1.44	1.87	3.03
3.50	1.36	1.34	1.32	1.27	1.24	1.23	1.47	2.33
4.0	1.21	1.28	1.29	1.23	1.23	1.17	1.2	1.98
5.0	1.04	1.07	1.08	1.1	1.13	1.06	1.07	1.15

Table 4. Average run lengths of EWMA scheme (Lucas and Saccucci, 1990).

	EWMA ARL_0 = 500										
Shift	**0**	**0.25**	**0.50**	**0.75**	**1**	**1.50**	**2**	**2.5**	**3**	**3.5**	**4**	**5**
λ = 0.75, L = 3.087	500	321	140	62.4	30.5	9.86	4.52	2.67	1.87	1.46	1.23	1.04
λ = 0.5, L = 3.071	499	254	88.4	35.7	17.3	6.44	3.58	2.47	1.91	1.58	1.36	1.10
λ = 0.1, L = 2.814	492	104	30.6	15.5	10.1	5.99	4.31	3.41	2.85	2.47	2.20	1.83
λ = 0.04, L = 2.477	487	78.4	28.0	16	11.2	7.03	5.18	4.14	3.48	3.02	2.68	2.22

values. In Table 3, the ARL values of Modified EWMA for all choices of λ are smaller than those for EWMA scheme. ARL values for EWMA schemes as per Markov Chain approach are shown in Table 4 for the sake of easy comparisons.

Thus, detection of small, as well as large shifts by a single Modified EWMA chart is possible with proper choice of λ and knowledge of AR(1) parameter and/or correlation structure of the process observations and the process fluctuations. It should be clear that, the process fluctuation variance would be smaller for process being controlled for small shifts and few large abrupt changes.

Inertia (Signal resistance) properties

Woodall and Mahmoud (2005) proposed a measure of inertia, the signal resistance, to be the largest standard deviation from target not leading to an immediate out-of-control signal. They define a simple measure of inertia that they called "signal resistance". In physics, "inertia" refers to the state of resistance an object has to a change in its state of motion. Similarly, in statistical process control, "inertia" can refer to a measure of the resistance of a chart to signaling a particular process shift. For the EWMA chart, the signal resistance is:

$$SR(EWMA) = [\, h - (1-\lambda)\, w \,] / \lambda \qquad (6)$$

Where w is the value of the EWMA statistic, $h = L\,\sigma\sqrt{\frac{\lambda}{(2-\lambda)}}$. The asymptotic control limits for the EWMA chart are $\pm h$. The signal resistance for an EWMA control chart in conjunction with Shewhart control limits is:

$$SR\ (EWMA + Shewhart) = \begin{cases} L & \text{if } -h \le w < (h-\lambda L)/(1-\lambda) \\ [h-(1-\lambda)w]/\lambda & \text{if } (h-\lambda L)/(1-\lambda) \le w \le h \end{cases} \qquad (7)$$

Where L is the value of the multiplier used to obtain the Shewhart limit and w is the value of the EWMA statistic.

Obviously, in this case, the signal resistance cannot exceed the value of the multiplier used to obtain the Shewhart limit, that is, L. The signal resistance for Modified EWMA control chart is:

SR(Modified EWMA) = $[h_1 - (1-\lambda)w_1]/\lambda \quad if - h_1 \le w_1 \le h_1$ (8)

Where w_1 is the value of the Modified EWMA statistic,

$h_1 = L\,\sigma\,\sqrt{\frac{\lambda}{2-\lambda}+\frac{2\lambda(1-\lambda)}{2-\lambda}}$ is the Modified EWMA decision limit.

Conclusion

A simple control chart for monitoring small, as well as large shifts in highly autocorrelated or independent normally distributed observations, such as analytical processes are given. In addition, the Modified EWMA chart can also be used to forecast the observation in the next period, which can help analysts take preventive actions before process departures to the out-of-control state.

REFERENCES

Brook D, Evans DA (1972). An Approach to the Probability Distribution of CUSUM Run Lengths, Biometrika, 59: 539-549.

Capizzi G, Masarotto G (2003). An Adaptive Exponentially Weighted Moving Average Control Chart. Technometrics, 45: 199-207.

Cox DR (1961). Prediction by Exponentially Weighted Moving Averages and Related Methods. J. Roy. Stat. Soc., 23: 414-442.

Crosier RB (1986). A New Two-Sided Cumulative Sum Quality Control Scheme. Technometrics, 28: 187-194.

Harris TJ, Ross WH (1991). Statistical Process Control Procedures for Correlated Observations. The Can. J. Chem. Eng., 69: 48-57.

Lu CW, Reynolds MR Jr. (1999). EWMA Control charts for Monitoring the Mean of Autocorrelated Processes. J. Qual. Tech., 32(2): 166-188.

Lucas JM, Saccucci MS (1990). Exponentially Weighted Moving Average Control Schemes: Properties and Enhancements. Technometrics, 32: 1-12.

Montgomery DC, Mastranglo CM (1991). Some Statistical Process Control Methods for Autocorrelated Data. J. Qual. Tech., 23(3): 179-204.

Montgomery DC (1996). Introduction to Statistical Quality Control., 3rd Ed, Wiley, p. 9

Reynolds MR Jr, Stoumbos ZG (2006). Comparisons of Some Exponentially weighted Moving Average Control Charts for Monitoring the Process Mean and Variance. Technometrics, 48(4): 550-567.

Page ES (1954). Continuous Inspection Schemes. Biometrika, (41): 100-115.

Roberts SW (1959). Control chart tests Based on Geometric Moving Averages. Technometrics, 1: 239-250.

R Language: Copyright 2004, The R Foundation for Statistical Computing Version 2.0.1 (2004-11-15), ISBN 3-900051-07-0

Wardell DG, Moskowitz H, Plante RD (1994). Run Length Distributions of Special-Cause Control Charts for Correlated Processes. Technometrics, (36): 3-17.

Woodall WH, Mahmoud (2005). The Inertial Properties of Quality Control Charts. Technometrics, 47: 4.

Yashchin E (1995). Estimating the Current Mean of a Process Subject to Abrupt Changes. Technometrics, 37: 311-323.

APPENDIX

Appendix 1: Mean and variance of modified EWMA control statistic

Let $\{Y_n, n \geq 1\}$ are first order auto correlated ($\rho \to 1$) process observations following normal distribution with mean μ and variance σ^2. For mean and variance, consider expanding Modified EWMA statistic:

$X_n = \lambda Y_n + (1-\lambda) X_{n-1} + (Y_n - Y_{n-1})$
$= \lambda Y_n + (1-\lambda) [\lambda Y_{n-1} + (1-\lambda) X_{n-2} + (Y_{n-1} - Y_{n-2})] + (Y_n - Y_{n-1})$
$= \lambda Y_n + \lambda(1-\lambda)Y_{n-1} + (1-\lambda)^2 X_{n-2} + (1-\lambda) (Y_{n-1} - Y_{n-2}) + (Y_n - Y_{n1})$
$= \lambda Y_n + \lambda(1-\lambda)Y_{n-1} + (1-\lambda)^2 [\lambda Y_{n-2} + (1-\lambda) X_{n-3} + (Y_{n-2} - Y_{n-3})] + (1-\lambda) (Y_{n-1} - Y_{n-2}) + (Y_n - Y_{n-1})$
$= \lambda Y_n + \lambda(1-\lambda)Y_{n-1} + \lambda (1-\lambda)^2 Y_{n-2} + (1-\lambda)^3 X_{n-3} + (1-\lambda)^2 (Y_{n-2} - Y_{n-3}) + (1-\lambda)^1 (Y_{n-1} - Y_{n-2}) + (1-\lambda)^0 (Y_n - Y_{n-1})$

And continuing like this recursively for X_{n-j}, j = 2, 3, . . . ,n, we obtain:

$$X_n = \lambda \sum_{j=0}^{n-1} (1-\lambda)^j Y_{n-j} + (1-\lambda)^n X_0 + \sum_{j=0}^{n-1} (1-\lambda)^j (Y_{n-j}-Y_{n-j-1}).$$

Here, $\sum_{j=0}^{n-1} (1-\lambda)^j (Y_{n-j}-Y_{n-j-1})$ accounts for sum of the past and latest change/fluctuations in the process; the unaccounted current fluctuations accumulated to time n in EWMA statistic:

Let $X_n = \lambda \sum_{j=0}^{n-1} (1-\lambda)^j Y_{n-j} + (1-\lambda)^n Y_0 + \sum_{j=0}^{n-1} (1-\lambda)^j (Y_{n-j}-Y_{n-j-1})$

Taking expectation on both side:

$$E(X_n) = \lambda \sum_{j=0}^{n-1} (1-\lambda)^j E(Y_{n-j}) + (1-\lambda)^n E(Y_0) + \sum_{j=0}^{n-1} (1-\lambda)^j E(Y_{n-j}-Y_{n-j-1})$$

But $\lambda \sum_{j=0}^{n} (1-\lambda)^j = \dfrac{\lambda\,[1-(1-\lambda)^n]}{[1-(1-\lambda)]} = [1-(1-\lambda)^n]$

$$\therefore\ E(X_n) = [1-(1-\lambda)^n]\mu + (1-\lambda)^n \mu + 0 \qquad \text{(i)}$$
$= \mu$

The starting value of process is, $X_0 = \mu_0 = Y_0$ and $0 < \lambda \leq 1$ is a constant. The mean is:

$E(X_n) = E((1-\lambda)X_{n-1} + \lambda Y_n + (Y_n - Y_{n-1})) = \mu_0$.

The variance of Modified EWMA control statistic X_n is:

$V(X_n) = V(\lambda \sum_{j=0}^{n-1} (1-\lambda)^j Y_{n-j}) + V((1-\lambda)^n Y_0) + V(\sum_{j=0}^{n-1} (1-\lambda)^j (Y_{n-j}-Y_{n-j-1}))$ $V(X_n) = (1-\lambda)^{2n} V(Y_0)$ +

$$\sum_{j=0}^{n-1} \lambda^2 (1-\lambda)^{2j} V(Y_j) + 2\sum_{j=0}^{n-1} \lambda^2 (1-\lambda)^{2j+1} Cov(Y_{n-j}, Y_{n-j-1})$$

+

$$\sum_{j=0}^{n-1} (1-\lambda)^{2j} V(Y_{n-j} - Y_{n-j-1}) + 2\sum_{j=0}^{n-1} (1-\lambda)^{2j+1} Cov[(Y_{n-j} - Y_{n-j-1}), (Y_{n-j-1} - Y_{n-j-2})]$$

+

$$\sum_{j=0}^{n-1} \lambda(1-\lambda)^{2j} Cov(Y_{n-j}, (Y_{n-j} - Y_{n-j-1})) + \sum_{j=0}^{n-1} \lambda(1-\lambda)^{2j+1} Cov(Y_{n-j-1}, (Y_{n-j} - Y_{n-j-1}))$$

Since Y_n's are autocorrelated normal with variance σ^2, the variance of (Y_n-Y_{n-1}) $(n \geq 1)$ is:

$\sigma_1^2 = 2\sigma^2 - 2\rho\sigma^2 = 2(1-\rho)\sigma^2$ (small when $\rho \to 1$).

The weights $\lambda(1-\lambda)^{2j}$ decrease geometrically with the age of sample mean. Suppose Y_n's are correlated to the forward fluctuation (Y_n-Y_{n-1}) $(n \geq 1)$ with common correlation ρ_1 and correlated to the backward fluctuation $(Y_{n+1}-Y_n)$, $(n \geq 0)$ with common correlation ρ_2, and forward fluctuation (Y_n-Y_{n-1}) are correlated to the backward fluctuation $(Y_{n+1}-Y_n)$, $(n \geq 1)$ with common correlation ρ_3, then asymptotic variance for large n is given as:

$$V(X_n) = \frac{\lambda}{(2-\lambda)}\sigma^2 + \frac{2\lambda(1-\lambda)}{(2-\lambda)}\rho\sigma^2 + \frac{2(1-\rho)\sigma^2}{\lambda(2-\lambda)} + \frac{4\rho_3(1-\rho)(1-\lambda)\sigma^2}{\lambda(2-\lambda)} + \frac{2\sqrt{2}\rho_1\sqrt{(1-\rho)}\sigma^2}{(2-\lambda)} + \frac{(1-\lambda)2\sqrt{2}\rho_2\sqrt{1-\rho}\,\sigma^2}{(2-\lambda)} \qquad \text{(ii)}$$

In normal autocorrelated process (a) with ρ_3 nearly negative half and ρ_1 , ρ_2 nearly equal and opposite in sign and being monitored for small shifts, (b) with autocorrelation ρ nearly one, the aforementioned expression (ii), reduces to:

$$V(X_n) = \frac{\lambda}{(2-\lambda)}\sigma^2 + \frac{2\lambda(1-\lambda)}{(2-\lambda)}\rho\sigma^2 \qquad \text{(iii)}$$

Let $\dfrac{2\lambda(1-\lambda)}{(2-\lambda)}\rho\sigma^2$ be a small value for high value of $\rho \to 1$ and small λ, sometimes even negligibly small such that

Modified EWMA limits equal EWMA limits.

Therefore:

$$V(X_n) = \left[\frac{\lambda}{(2-\lambda)} + \frac{2\lambda(1-\lambda)}{(2-\lambda)} \right] \sigma^2. \qquad \text{(iv)}$$

The upper-lower control limits for Modified EWMA chart are:

$$\mu_0 \pm L\,\sigma \sqrt{\frac{\lambda}{(2-\lambda)} + \frac{2\lambda(1-\lambda)}{(2-\lambda)}}. \qquad \text{(v)}$$

Appendix 2: Cyclical steady state average run lengths of modified EWMA scheme using Markov-chain approach

The use of Markov chains to approximate ARL of CUSUM schemes was discussed by Brook and Evans (1972). Crosier (1986) modified their Markov chain approach to compute conditional and cyclical steady state ARLs for CUSUM scheme. Further, Lucas and Saccucci (1990) customized their approach for ARL properties of EWMA scheme. They evaluated the properties of the continuous state Markov chain by discretizing the infinite state transition probability matrix. This procedure involves dividing the interval between the upper and lower control limits into t = 2m+1 subintervals of width 2δ. The control statistic, X_n, is said to be in transient state i at time n, if $S_i - \delta < X_n \le S_i + \delta$ for i = -m,-m+1, . . . , m, where S_i represent the midpoint of the i^{th} interval. For example, when the LCL = -0.6455759 to UCL = 0.6455759 distance is divided into 5 sub-intervals each of width 2δ (δ=0.1291), -0.6455759, -0.3873455 ,-0.1291152, 0.1291152 , 0.3873455 , 0.6455759; the S_i 's are -0.5165 ,-0.2582 , 0.0000, 0.2582, 0.5165.

The transition probability matrix (t.p.m.), represented in partitioned matrix form, is given by:

$$P = \begin{pmatrix} \mathbf{R} & (\mathbf{I}-\mathbf{R})\mathbf{1} \\ \mathbf{0}^T & 1 \end{pmatrix},$$

where the sub matrix R contains the probabilities of going from one transient state to another, **I** is the identity matrix, and **1** is a column vector of ones. Hence P_{ij} represents the probability that the control statistic goes from state (i) to state (j) in one step. We have derived algorithm for calculating cyclical steady state ARL of Modified EWMA scheme using the Markov chain approach following Crosier (1986) and Lucas and Saccuscci (1990). The transition probabilities in **R** are approximated by assuming that the control statistic is equal to S_i whenever it is in state (i), i = -m, -m+1, … , m.

Let Y_n, n = 1, 2 … denote the sequence of process observations, which are first order autocorrelated normal with mean μ and variance σ^2. Then the transient change occurring in state of Modified EWMA control statistic is $X_n-(1-\lambda)X_{n-1}$, which is equivalent to $\lambda Y_n + (Y_n - Y_{n-1})$. The transition in Modified EWMA control chart statistic has the following mean and variance:

$$E(S_n|S_{n-1}) = E[\lambda Y_n + (Y_n-Y_{n-1})] = \lambda\mu, \text{ and}$$

$$\begin{aligned} V(S_n|S_{n-1}) &= V[\lambda Y_n + (Y_n-Y_{n-1})] \\ &= \lambda^2 V(Y_n) + V(Y_n-Y_{n-1}) + 2\mathrm{cov}\{Y_n, (Y_n-Y_{n-1})\} \\ &= \lambda^2\sigma^2 + 2(1-\rho)\sigma^2 + 2\sqrt{2(1-\rho)}\,\rho_1\,\sigma^2 \\ &= \lambda^2\sigma^2 \end{aligned}$$

where variance, is a small value for nearly 1 values ρ .

Then the in control transition probabilities are approximated as:

P_{ij} = Pr(going to S_j | in S_i)
≈ Pr $[(\lambda\sigma)^{-1}\{(S_j-\delta)-(1-\lambda)S_i\} < Y_n \le (\lambda\sigma)^{-1}\{(S_j+\delta)-(1-\lambda)S_i\}]$,

i, j = -m, -m+1, …, m.

(m =12 imply t = 25), where Φ represents the standard normal distribution function.

Appendix 3: Algorithm and R program for ARL computation of modified EWMA

Step-1: Choose the parameter constant λ, target mean μ, standard deviation σ, Limit L.

Step -2: Compute $UCL = \mu_0 + L\sigma\sqrt{\frac{\lambda}{(2-\lambda)} + \frac{2\lambda(1-\lambda)}{(2-\lambda)}}$,

$LCL = \mu_0 - L\sigma\sqrt{\frac{\lambda}{(2-\lambda)} + \frac{2\lambda(1-\lambda)}{(2-\lambda)}}$.

Step-3: Choose number of sub-intervals (states), t = 2 m + 1.
Step-4: Compute width w = (UCL - LCL)/t, shift δ = w/2.
Step-5: Compute the values of S_i i = -m, -m+1, …, m for each sub-interval.
Step-6: Compute the transition probability matrix (t. p. m.). Compute R,
$R = \Phi[(\lambda\sigma)^{-1}\{(S_j+\delta) - (1-\lambda)S_i - \lambda\mu\}] - \Phi[(\lambda\sigma)^{-1}\{(S_j-\delta) - (1-\lambda)S_i - \lambda\mu\}]$
Step-7: Adjust the t. p.m. such that row sums are unity.
Step-8: Compute $u = [I - R]^{-1}\,\mathbf{1}$
Step-9: Compute q = R'* I
Step-10: ARL = q' * u, OR ARL = $q'[I - R]^{-1}\,\mathbf{1}$