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Full Length Research Paper

Commodity futures market mechanism: Mathematical formulation and some analytical properties

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This paper deals with the internal structure of a commodity futures market. We proposed a mathematical model representing the mechanism of this market. The model shows the links between market components (transactional prices, transactional quantities, open interest) and traders' states (position, position's average price, potential wealth andrealized wealth). Later, we stated and demonstrated some analytical properties of thismodel. This paper is not dealing with classical economic concepts like arbitrages, market equilibrium etc., rather it focuses on exact mathematical relationships between market platform components. Amongst our main findings is an exact relationship between open interest variation and transactional quantities. Indeed, this result indicates that if transactional quantity is tiny then open interest has 50% chance to not change and 25% chance to either increase or decrease, whereas when transactional quantity is big enough, then there is 75% chance for open interest to increase and 25% chance to decrease.

Key words: Commodity futures market, market microstructure, trader's position, open interest, market average price, market analytical properties.

INTRODUCTION

Our motivations in carrying out this work are two folds. First, we noticed that in technical literature of futures markets, a large array of results are found by empirical methods (Karpoff, 1987; Erband Harvey, 2006; Miffreand Rallis, 2007; Wang and Yu, 2004), whereas we wanted to provide exact results with full mathematical proofs. Secondly, to automatize price negotiation in the futures market, it is necessary to have a detailed mathematical model of this platform, therefore our model can be used in the automatization process.

The Santa Fe stock market simulator was designed in the 1990's (Palmer et al., 1994; LeBaron, 2001; LeBaron et al., 1999). It is a computer model intended to reproduce the behavior of a stock market platform and its empirically some properties of the stock market and explain some related phenomena like financial bubbles, crashes and band wagon practices observed in real markets (Arthur et al., 1997; Boer-Sorban, 2008; Johnson, 2002). In a sense, our work can be viewed as an extension of the Santa Fe simulator, with two exceptions: (1) We are dealing with the *futures* market whereas the Santa Fe was focusing on the *stock* market, and (2) As a starting step, we managed to model only the futures market platform functioning and study some analytical properties of its components, without including the behavior of its agents, which is the subject of forthcoming worksagents. This simulator was used extensively to study.

Our market model was used as an underlying structure for applications in automatizing price negotiation of a futures market in case of one-producer-one-consumer (Laib and Radjef, 2011) and many-producers-many-consumers (Laib and Radjef, 2010).

Nowadays, futures market is a major part of commercial exchanges like CBOT, ICE and LIFFE. The essential instrument traded in this market is a *futures contract*. This is a binding agreement between the exchange, a buyer and a seller. Main terms of the transaction (technical specifications of the commodity, delivery time and locations, etc.) are fixed in the contract,

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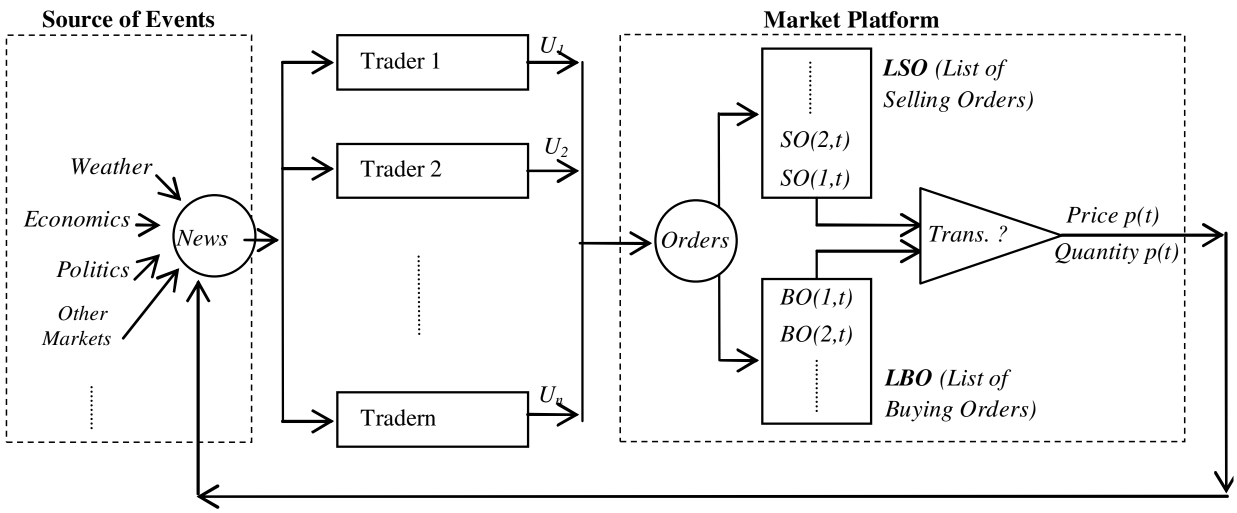


Figure 1. Flowchart of traders' orders.

only price remains negotiable. The price is fixed at present time, but delivery of the commodity occurs after contract expiry (months or years later).The buyer of the contract has a *long* position, whereas the seller has a *short* position. The buyer can sell her contract before its expiry, otherwise she must deliver the underlying commodity; the seller can buy back her contract before its expiry, otherwise she must take delivery of the underlying commodity (Catania and Alonzi, 1998; Hull, 2002; Teweles and Jones, 1999).

The majority of studies on futures markets were conducted from astochastic perspective where time series were analyzed in order to discover empirical relationships between market phenomena (Bodie and Rosansky, 1980; Chanand, 2006; Erband, 2006; Karpov, 1987; Levy et al., 1994; Miffre and Rallis, 2007; Wang and Yu, 2004). In practice, market analysts and traders use extensively *technical analysis* and *fundamental analysis* to forecast price moves and monitoring market trends (Murphy, 1999; Szakmary et al., 2010). Other works, like Shelton (1997) and Howard (1999), used game theory tools in an attempt to suggest wining trading strategies for traders.

The remainder of this study is organized into mathematical formulation and main findings. In mathematical formulation of the futures market platform, we outline the most important variables like transactional prices and quantities, traders' states and the updating process. At instant $t_j \in \mathbf{T}$, each trader $i \in N$ is characterized by a position $y_i(t_j)$, position'saverage price $x_i(t_j)$, potential wealth $w_i(t_j)$, realized wealth $W_i(t_j)$, and total wealth $J_i(t_j)$. The market as a whole is characterized by the instantaneous transactional price and quantity, $[p(t_j), q(t_j)]$, the open interest $y(t_j)$ and the market average price $\bar{p}(t_j)$ measures, with t_j belonging to the time horizon $\mathbf{T}$. The last section provides our main findings which are analytical relationships between the above mathematical measures of the market model. One of the properties on the open interest change seems to have an interesting practical interpretation for market analysis purposes. Indeed, this result indicates that if transactional quantity is tiny then open interest has 50% chance not to change and 25% chance to either increase or decrease, whereas when transactional quantity is big enough then there is 75% chance for open interest to increase and 25% chance to decrease.

MARKET STRUCTURE

Here, we present a mathematical model formulating the functioning of the futures market platform. As shown on Figure 1, a set $N = \{1, \dots, n\}$of traders are trading a specific commodity futures contract, with a life duration $\mathcal{T}$. Traders get information from different sources, then they establish their market orders U_i, with $i \in N$. The orders are sent either to the list of selling orders (LSO) or the list of buying orders (LBO) depending on their type. Sale orders of the LSO are sorted in the ascending order of *ask*prices, and buy orders of the LBO are ordered in the descending order of *bid*prices; this guaranties the fact that the best saleorder is always at the top of the LSO and the best buy order is at the top of the LBO.

We assume that daily marke t sessions of the futures contract, since the first trading day until last business day, are grouped into a compact interval $[0,T]$ which is discretized into a set of instants:

$$\mathbf{T} = \{t_0, \cdots, t_m\} \text{with} t_0 = 0,\ t_m = T,\ t_j = t_{j-1} + h,\ j = 1, \cdots, m,$$

h is the discretization pace (h should tend to zero to reflect reality). At instant t_j, at mostone order can be received and processed. Explicitly, if an order is received at t_j, then it will be directed to the corresponding listof orders, sorted in that list, then an attempt to generate a transaction follows; all these four elementary events take place duringthe same period $[t_j, t_j + h[$.

Price fixation

Order $u_i(t_j)$,issued by trader $i \in N$ at instant $t_j \in \mathbf{T}$, has the following structure:

$$U_i(t_j) = [u_{i1}(t_j), u_{i2}(t_j)],$$

where u_{i1} is the ask price in case of a sale order (respectively the *bid* price in case of a buy order).

Thus $u_{i1} \in \mathbf{R}^+$; and u_{i2} is the number of contracts to sell incase of a sale order (respectively the number of contracts to buy in case of a buyorder). In case of a sale order, we add conventionally a minus sign to u_{i2} to distinguish it from a buy order, therefore $u_{i2} \in \mathbf{Z}$in general.

At each instant, an attempt is made to generate a transaction between the best sale order with the best buy order which are respectively at the top of LSO and LBO. At t_j, the first elements of LSO and LBO are respectively:

$$SO(1,t_j) \equiv U_s(\tau_s) = [u_{s1}(\tau_s), u_{s2}(\tau_s)],$$

$$BO(1,t_j) \equiv U_b(\tau_b) = [u_{b1}(\tau_b), u_{b2}(\tau_b)],$$

at instant $\tau_s \le t_j$; and the best buy order is $U_b(\tau_b)$ issued by trader b at instant $\tau_b \le t_j$. A transaction will occur at instant t_j if $u_{s1}(\tau_s) \le u_{b1}(\tau_b)$, and $u_{s2}(\tau_s) \neq 0$ and $u_{b2}(\vartheta_b) \neq 0$ simultaneously. In this case, the transactionalprice $p(t_j)$ will be:

$$p(t_j) = \begin{cases} u_{b1}(\tau_b), & \text{if } \tau_s < \tau_b, \\ u_{s1}(\tau_s), & \text{if } \tau_b < \tau_s. \end{cases} \quad (1)$$

Transactional price is determined in this way in order to favor the trader who issued her order first. The number of contracts $q(t_j)$ sold by trader s to trader b in this transaction will be:

$$) = min\{u_{b2}(\tau_b);\ |u_{s2}(\tau_s)|\}. \quad (2)$$

Otherwise, no transaction will take place at instant t_j, and we set:

$$p(t_j) = p(t_{j-1}) \text{and} q(t_j) = 0. \quad (3)$$

If a transaction has occurred at instant t_j, then $t_j$is a *transactional time*, otherwise it is a *non-transactional time*.

Traders' states

The trading activity of futures contracts starts at $t_0$and finishes at t_m. At each instant, $t_j \in \mathbf{T}$, thestate of each trader can be described by the following components:

1. $y_i(t_j)$: Is the *position* of trader i, representing the number of contracts she has bought or sold.
2. $x_i(t_j)$: Is the average price of the position $y_i(t_j)$of trader i.
3. $w_i(t_j)$: Is the potential wealth (profit or loss) of trader at t_j. It represents the amount of moneythat she would gain or loss if she closes her position at the current instant t_j. This amount is the difference between the real worth of her position and its current worth value, that is:

$$w_i(t_j) = y_i(t_j)[p(t_j) - x(t_j)]. \quad (4)$$

4. $W_i(t_j)$: Is the realized, or closed, wealth (profit orloss) of trader i since the beginning of the game at $t_0$until t_j. Component $W_i(t_j)$ is updated only when trader $i$closes entirely, or partly, her position. If, at t_j, she closes

$|d(t_j)|$ contracts from her old position then her accumulated realized wealth will be:

$$W_i(t_j) = W_i(t_{j-1}) + d(t_j)[p(t_j) - x_i(t_{j-1})].$$

5. $J_i(t_j)$: Is the total wealth of trader i at t_j, defined by:

$$J_i(t_j) = J_i^0 + W_i(t_j) + w_i(t_j), \tag{6}$$

where J_i^0 is the initial wealth of trader i, that is, the amount of cash shown at the beginning of the game.

We set J^0 as the global wealth of all traders:

$$J^0 = \sum_{i=1}^{n} J_i^0. \tag{7}$$

At the starting time t_0, all components of each trader are flat, that is:

$$_0) = W_i(t_0) = 0, \quad i = 1, \dots, n. \tag{8}$$

Note 2.1

In order to simplify further our notations and avoid lengthy expressions, we drop the letter t_j when no confusion is possible, hence we set:

$$(t_j),\ y_i \equiv y_i(t_j),\ W_i \equiv W_i(t_j),\ w_i \equiv w_i(t_j),\ J_i \equiv J_i(t_j)$$

To make reference to the state of any dynamical variable at the prior instant t_{j-1} we use instead the apostrophe notation ('), that is:

$$p' \equiv p(t_{j-1}), \quad x_i' \equiv x_i(t_{j-1}), \quad y_i' \equiv y_i(t_{j-1}),$$

These notations will be used interchangeably.

Updating traders' states

Consider a step forward in the trading process passing from t_{j-1} to t_j, and let us examine as follows the two possible cases.

Case of no transaction

If no transaction has occurred at t_j, then relation (3) will hold, and all the components of each trader will remain unchanged, that is for every $i \in N$ we have the following:

$$y_i = y_i', \qquad x_i = x_i', \tag{8a}$$

$$w_i = w_i', \qquad J_i = J_i'. \tag{8b}$$

Case where a transaction has occurred

If instant t_j is a transactional time, then a transaction has occurred between a buyer b and a seller s, exchanging q contracts. In this event, an update of the price and traders'components is necessary. The transactional price p and quantity q are given by (1) and (2) respectively.

All traders except the buyer b and the seller s, will only update their potential wealth, in other words, for traders $i \in N \backslash \{b, s\}$, formulas (8a) will apply, but their potential wealthcomponent w_i will evolve with time because the price has changed, that is

$$w_i = y_i(p - x_i), \ \ i \in N \backslash \{b, s\}. \tag{9}$$

Obviously, for these traders, their total wealth component J_i,given by (6), should also be recalculated since itdepends on w_i.

States of the buyer b and seller s are updated according to the following two lemmas. Mathematical proofs of these lemmas are given in appendix A, and $1_{[\cdot]}$ is the *conditional function* defined by $1_{[C]} = 1$ if condition C is satisfied, otherwise $1_{[C]} = 0$.

Lemma 2.1

If t_j is a transactional time, then components of the buyer b are updated as follows:

$$y_b = y_b' + q,$$

$$x_b = \frac{y_b' x_b' + qp}{y_b' + q} 1_{[y_b' \geq 0]} + p 1_{[-q < y_b' < 0]} + x_b' 1_{[y_b' < -q]}, \tag{10}$$

$$W_b = W_b' + (p - x_b')\left(y_b' 1_{[-q < y_b' < 0]} - q 1_{[y_b' \leq -q]}\right), \tag{11}$$

$$W_b = W_b' + (p - x_b')\left(y_b' 1_{[-q < y_b' < 0]} - q 1_{[y_b' \leq -q]}\right), \tag{12}$$

$$w_b = (p - x_b')\left(y_b' 1_{[y_b' \geq 0]} + (y_b' + q) 1_{[y_b' < -q]}\right). \tag{13}$$

Lemma 2.2

If t_j is a transactional time, then components of the seller s are updated as follows:

$$y_s = y_s' - q, \tag{14}$$

$$x_s = \frac{y_s' x_s' - qp}{y_s' - q} 1_{[y_s' \leq 0]} + p 1_{[0 < y_s' < q]} + x_s' 1_{[y_s' > q]}, \tag{15}$$

$$W_s = W_s' + (p - x_s')\left(y_s' 1_{[0 < y_s' < -} \right. \tag{16}$$

$$w_s = (p - x_s')\left(y_s' 1_{[y_s' \leq 0]} + (y_s' - q) 1_{[y_s' > q]}\right). \tag{17}$$

Remark 2.1

Components J_b and J_s are calculated using formula (6) by substituting i with b and s, respectively.

MATHEMATICAL PROPERTIES

In the literature, it is well-known that a futures market is a zero-sum game (Teweles and Jones, 1999), though this was stated in plain words, we did not find any mathematical formula on this issue. Using the model suggested previously in 'market structure', we can describe this property by:

$$\sum_{i=1}^{n} y_i(t_j) = 0 \text{ and } \sum_{i=1}^{n} J_i(t_j) = \tag{18}$$

The first relation follows directly from (10) and (14) since for every transaction there is a buyer and a seller. The second result reflects the fact that total wealth of all traders is constant and that what was lost by some traders is gained by others (Remark 3.2).

Hereafter, we present three new classes of properties. Mathematical proofs are given in appendix A.

Traders' components properties

We show herein that the state variables w_i, W_i and J_i, of trader i at instant t_j, can be identified by knowing only their values at the prior instant t_{j-1}, the market price p and the transactional quantity q, of the current transaction, if any.

Property 3.1

$\forall\, t_j \in \mathbf{T}$, potential wealth w_i of trader i, defined by relation (4), can be obtained by the following formula:

$$w_i = w_i' + y_i'(p - p'), \tag{19}$$

forany trader $i \in N$, except if $i = b$ and $y_b < 0$, or if $i = s$ and $y_s > 0$.

Property 3.2

$\forall\, t_j \in \mathbf{T}$ and $\forall\, i \in N$, total wealth J_i given by relation (6), can be written in terms of J_i' in the following way

$$J_i = J_i' + y_i'(p - p'). \tag{20}$$

This formula facilitates the calculation of total wealth J_i

This formula facilitates the calculation of total wealth J_i as it necessitates only prior wealth J_i' and position y_i'.

Remark 3.1

Consider the summation of (20) over all traders:

$$\sum_{i=1}^{n} J_i = \sum_{i=1}^{n} J_i' + (p - p') \sum_{i=1}^{n} y_i.$$

Since $\sum_{i=1}^{n} y_i = 0$, then $\sum_{i=1}^{n} J_i = \sum_{i=1}^{n} J_i'$ for all $t_j \in \mathbf{T}$, that is, the sum of the wealth of all traders is constant in time. This confirms that *Property* 3.2 is not in disagreement with earlier established results on futures markets second term of (18)).

Remark 3.2

If time t was continuous over the interval $[0,T]$,v then total wealth of trader i can be described by the following differential equation:

$$\dot{J}_i(t) = y_i(t)\dot{p}(t), \qquad i = 1, \dots, n.$$

This result is obtained by the integration of relation (20) over $[0,T]$.

Property 3.3

$\forall\, t_j \in \mathbf{T}$ and $\forall\, i \in N$, realized wealth W_i can be written as

$$W_i = W_i' + w_i' + y_i'(p - p') - w_i. \tag{21}$$

Open interest properties

Open interest measure $y(t_j)$ is a popular concept in futures markets. Stated in simple terms, it represents the numberof contracts held by all traders having *long* positions at instant t_j, which is also equal to the absolute number of contractsheld by traders with *short* positions.

Definition 3.1

Open interest $y(t_j)$ is calculated as follows:

$$y(t_j) = \sum_{i=1}^{n} y_i(t_j) 1_{[y_i(t_j)>0]} = -\sum_{i=1}^{n} y_i(t_j) 1_{[y_i(t_j)<0]}.$$

The value and sign of the change in open interest are monitored continuously by traders and analysts as it helps them assessing the behavior of the market and forecasting its future move.

Property 3.4

At instant $t_j \in \mathbf{T}$, open interest y can be calculated in the following way:

$$y = y' + A - B, \tag{22}$$

where

$$A \equiv A(t_j) = q\, 1_{[y_b' > -q]} + y_b' 1_{[-q < y_b' \le 0]}, \tag{23}$$

$$B \equiv B(t_j) = q\, 1_{[y_s' > q]} + y_s' 1_{[0 < y_s' \le q]}, \tag{24}$$

That is y depends only on transactional quantity q and the state of the system at the previous instant t_{j-1}. The term $A(t_j)$ represents the number of contracts added by the buyer to the open interest, and $B(t_j)$ indicates the number of contracts deducted by the seller from the open interest.

Property 3.5

$\forall\, t_j \in \mathbf{T}$, open interest $y(t_j)$ can be calculated by

$$y(t_j) = \sum_{k=0}^{j} [A(t_k) - B(t_k)]. \tag{25}$$

Property 3.6

Let $\Delta y(t_j)$ be the change in open interest at a transactional time t_j, defined by:

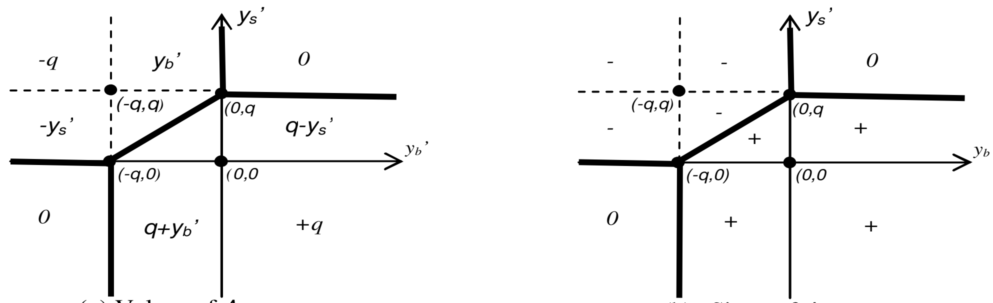
$$\Delta y = y - y'.$$

For a specified value of q, and allowing to the values of y_b' and y_s' to vary over the set of integer numbers, then

Table 1. Values and signs of Δy.

Values of Δy			
	$y_b' \leq -q$	$-q < y_b' \leq 0$	$y_b' > 0$
$y_s' > q$	$-q$	y_b'	0
$0 < y_s' \leq q$	$-y_s'$	$q + y_b' - y_s'$	$q - y_s'$
$y_s' \leq 0$	0	$q + y_b'$	Q

Signs of Δy			
	$y_b' \leq -q$	$-q < y_b' \leq 0$	
$y_s' > q$	< 0	≤ 0	0
$0 < y_s' \leq q$	< 0	*Any*	≥ 0
$y_s' \leq 0$	0	> 0	> 0



(a) Values of Δy (b) Signs of Δy

Figure 2. Values and signs of Δy.

the values and signs of Δy in each possible case are given in Tables 1

Remark 3.4

The results of Table 1 can be further displayed graphically on a 2-dimension space with $(0,0)$ as origin, the horizontalX-axis representing y_b' versus the vertical Y-axis for y_s'. This is shown on Figures 2a and b.

Figure 2a shows the values of Δy for each point $(y_b', y_s') \in \boldsymbol{Z}^2$. Inside the square delimited by the points $(0,0)$, $(-q,0)$, $(-q,q)$ and $(0,q)$, the value of Δy is calculated by Formula (40); this square corresponds to case 5 of Table 1 (Appendix A). In addition to the two zones where $\Delta y = 0$, all the points belonging to the thick lines correspond also to $\Delta y = 0$.

On the other hand, Figure 2b shows the signs of Δy for each point $(y_b', y_s') \in \boldsymbol{Z}^2$. Inside the triangle delimited by the points $(0,0)$, $(-q,0)$ and $(0,q)$, the sign of Δy is positive. All the points of the triangle $(-q,0)$, $(-q,q)$, $(0,q)$ correspond to a negative Δy. The points belonging to the common segment $(-q,0)$, $(0,q)$ of these two triangles satisfy $\Delta y = 0$.

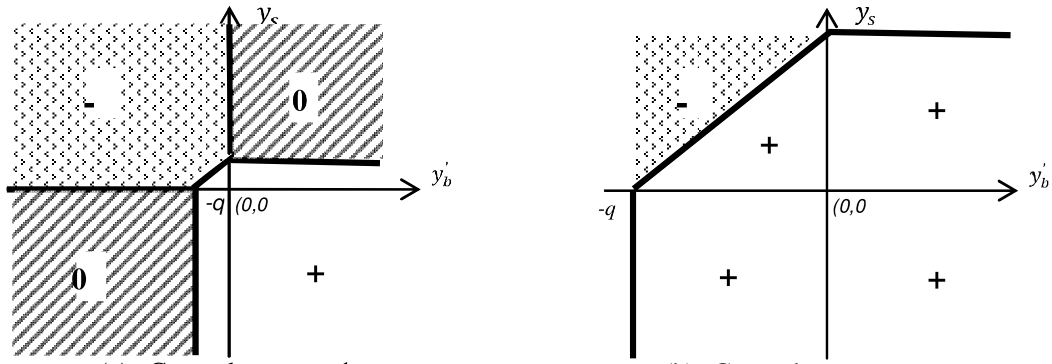
Property 3.7

Assume that M is the biggest number in the set of positive integer numbers (in theory, M stands for $+\infty$). At a transactional time, the probability $\pi(\cdot)$ of the events are:

$$\begin{aligned}
\pi(\Delta y = 0) &= \frac{1}{2} - \frac{q}{2M}, \\
\pi(\Delta y > 0) &= \frac{1}{4} + \frac{q}{2M}\left(1 + \frac{q}{4M}\right), \\
\pi(\Delta y < 0) &= \frac{1}{4} - \frac{1}{8}\left(\frac{q}{M}\right)^2.
\end{aligned}$$

Property 3.8

At a transactional time, assuming that q can vary from 1 to M, then we have the following limits on the probabilities of each event:



(a) Case where $q \to 1$ (b) Case where$q \to$

Figure 3. Limits of $\pi(\Delta y)$.

$$\lim_{q \to 1} \pi(\Delta y = 0) = \frac{1}{2}, \quad \lim_{q \to M} \pi(\Delta y = 0) = 0$$
$$\lim_{q \to 1} \pi(\Delta y > 0) = \frac{1}{4}, \quad \lim_{q \to M} \pi(\Delta y > 0) = \frac{7}{8}$$
$$\lim_{q \to 1} \pi(\Delta y < 0) = \frac{1}{4}, \quad \lim_{q \to M} \pi(\Delta y < 0) = \frac{1}{8}$$

Graphical visualization: The results of *Property* 3.8 are illustrated graphically on Figures 3a and b.

Figure 3a illustrates the case where q is small enough ($q \to 1$). The dotted area represents the zone where $\Delta y < 0$, the blank area corresponds to $\Delta y > 0$, and the two symmetrical dashed areas materialize the points (y_b', y_s') for which $\Delta y = 0$. From a rough observation, we note that the two dashed zones occupy almost half of the plane, confirming the fact $\lim_{q \to 1} \pi(\Delta y = 0) = \frac{1}{2}$. Whereas, the blank and dotted zones, each of them fill approximately one quarter of the plane, hence confirmingthat limits $\lim_{q \to 1} \pi(\Delta y > 0) = \frac{1}{4}$and $\lim_{q \to 1} \pi(\Delta y < 0) = \frac{1}{4}$respectively. We observe also that the blank zone is slightlylarger than the dotted zone, showing that:

$$\pi(\Delta y < 0) < \frac{1}{4} < \pi(\Delta y > 0).$$

Figure 3b illustrates the case where $q \to \infty$. We readily observe that the two dashed zones are no longer visible on this plane, hence confirming that $\lim_{q \to M} \pi(\Delta y = 0) = 0$. On the other hand, the blank zone spreads over a greater space, approximately equal to $7/8$, proving that $\lim_{q \to M} \pi(\Delta y > 0) = \frac{7}{8}$, and inversely, the dotted zone is smaller than before and occupies only $1/8$ confirming that $\lim_{q \to M} \pi(\Delta y < 0) = \frac{1}{8}$.

Contribution to market analysis: *Property* 3.8 can bring further insight to market analysts. Indeed, after a transaction has occurred, open interest y could either increase, or decrease or stagnate; this is reflected by the sign of Δy. This change depends on the transactional quantity q, and the buyer's prior position $y_b'$and the seller's prior position y_s'; all possible cases aregiven in Table 1. For instance, if the buyer was long or flat before the transaction, that is, $y_b' \geq 0$.,and the seller was short or flat, that is, $y_s' \leq 0$, then for any value of q, open interest will increase as a result of this transaction.

If the transactional quantity is small enough ($q \to 1$), then it is more likely that open interest will stagnate after the transaction rather than increase or decrease, since the event $\Delta y = 0$has about 50% chance to occur, whereas the events $\Delta y > 0$ and $\Delta y < 0$ have only about 25% chance for each to occur.

By contrast, if transactional quantity is big enough, then it is more likely that open interest will increase; in fact, this should happen in 75% of cases, and the possibility to see open interest decreases is only 25% in this case. Noticeably, in this case, open interest should not stagnate as the probability of the event $\Delta y = 0$is almost stagnate as the probability of the event $\Delta y = 0$ is almost zero.

Market average price

Definition 3.2

We define *market average price* $\bar{p}$ at instant t_j by

$$\bar{p}(t_j) = \frac{\sum_{k=0}^{j} p(t_k)q(t_k)}{\sum_{k=0}^{j} q(t_k)}, \tag{26}$$

which is simply the weighted average price of all the transactions since the starting time t_0 until t_j.

Property 3.9

In the particular case where

$$y(t_k) = y(t_{k-1}) + q(t_k), \quad \forall\, k = 0 \dots j, \tag{27}$$

Then

$$\bar{p}(t_j) = \frac{\sum_{i=1}^{n} x_i(t_j) y_i(t_j) 1_{|y_i(t_j) > 0|}}{y(t_j)}. \tag{28}$$

The aforementioned property links market average price to open interest and traders' components. That is, Formula (28) allows computing market average price at instant t_j using only the knowledge available at this instant.

CONCLUSION AND PERSPECTIVES

Our study showed that a futures market platform has rich analytical properties. We derived the most basic of them, and we believe that many other features remain to be explored and stated in a mathematical framework. A more important issue is to bring practical interpretation of these properties as it was done with *Property* 3.8. In addition, some results has to be generalized, this is the case of *Property* 3.9 on the market average price that needs to be extended to the case where condition (27) is no more satisfied.

On the other hand, the mathematical model of the futures market platform as stated herein has already a theoretical game format, though a discussion over the game equilibrium is lacking. This could be achieved by introducing trading strategies for trading agents as it was done by Arthur et al. (1997) for the stock market. Additionally, the continuous-time version of the model can be considered as pointed out in *Remark* 3.2.

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APPENDIX

Appendix A: Mathematical proofs

Proof of lemma2.1 (Updating buyer's components)

Buyer b has bought q new contracts during transactional time t_j, her current position y_b will become:

$$y_b = y'_b + q. \tag{29}$$

Since she had added new contracts to her old position, the average price x_b of her new position should be updated. However, this update will depend on the value of her previous position y'_b. Below, we examine the four possible cases, 1-i to 1-iv, corresponding respectively to (i) $y'_b \geq 0$, (ii) $-q < y'_b < 0$, (iii) $y'_b = -q$, and (iv) $y'_b < -q$. In each case, we determine the analytical expressions of x_b, w_b and W_b.

Case i: When $y'_b \geq 0$. In this case, buyer's new average price x_b on her new position will be:

$$x_b = \frac{y'_b x'_b + qp}{y'_b + q}. \tag{30}$$

In this case, her realized wealth will remain unchanged because she has not closed any contract of her old position, thus:

$$W_b = W'_b. \tag{31}$$

Her potential wealth w_b should be updated because the price has moved from p' to *p*, that is:

$$w_b = y_b(p - x_b). \tag{32}$$

Substituting (29) and (30) in (32), we obtain:

$$w_b = y'_b(p - x'_b). \tag{33}$$

Case ii: When $-q < y'_b < 0$. In this case, at instant t_j, buyer bought q new contracts with a price p. This buying operation can be viewed as two consecutive buying operations:

a. she had bought $|y'_b|$ contracts with a price p, then
b. she bought $q - |y'_b|$ contracts with a price p.

When she executed operation (a) she had closed her short position y'_b that she had sold before with a price x'_b, and realized a net profit or loss equal to $|y'_b|(x'_b - p)$. Adding this amount to the old realized wealth W'_b, the new realized wealth will become:

$$W_b = W'_b + |y'_b|(x'_b - p) = W'_b + y'_b(p - x'_b).$$

When she executed operation (b), she had acquired a long position $y_b = q - |y'_b| = q + y'_b$ with a price $x_b = p$ and the potential wealth of this position is $w_b = y_b(p - x_b) = 0$. This is true because the new position $y_b = q - |y'_b|$ was established at the current price p, therefore it has not yet any potential wealth.

Case iii: When $y'_b = -q$. In this case, when the buyer bought the q new contracts, she had closed entirely her short position, hence she realized a net profit or loss equal to $|y'_b|(x'_b - p)$. Adding this amount to her previous realized wealth, will yield:

$$W_b = W'_b + |y'_b|(x'_b - p) = W'_b + q\,(p - x'_b).$$

In this case, her new position is $y_b = y'_b + q = 0$, thus we consider its average price as $x_b = 0$, having a zero potential wealth, $w_b = y_b(p - x_b) = 0$.

Case iv: When $y'_b < -q$. In this case, when the buyer bought the q new contracts, she had closed q contracts in her old short position, hence she realized a net profit or loss equal to $q(x'_b - p)$. Adding this amount to her previous realized wealth W'_b will result in:

$$W_b = W'_b - q(p - x'_b).$$

After this operation, there will remain $y_b = y'_b + q < 0$ contracts in the possession of the buyer. This is a part of her old position that she had sold with an average price x'_b. As these contracts are still in her hand at instant t_j, hence $x_b = x'_b$, and the potential wealth of this position is $w_b = y_b(p - x_b) = (y'_b + q\ )\,(p - x'_b)$.

Summary: In order to write in a single line the functions x_b, W_b, and w_b for the four cases 1-i to 1-iv, we will use the conditional function formulation shown as follows:

$$x_b = \frac{y'_b x'_b + qp}{y'_b + q} 1_{[y'_b \geq 0]} + p 1_{[-q < y'_b < 0]} + x'_b 1_{[y'_b < -q]},$$

$$W_b = W'_b + (p - x'_b)\left(y'_b 1_{[-q < y'_b < 0]} - q 1_{[y'_b \leq -q]}\right)$$

However, we have showed that in both cases 1-ii and 1-iii that potential wealth $w_b = 0$. In the remaining cases 1-i and 1-iv, we know that $w_b \neq 0$, hence we can assert that potential wealth w_b can be written as:

$$w_b = (p - x'_b)\left(y'_b 1_{[y'_b \geq 0]} + (y'_b + q) 1_{[y'_b < -q]}\right).$$

Proof of lemma 2.2 (updating seller's components)

After selling q contracts, the position of the seller s will be:

$$y_s = y_s' - q. \qquad (34)$$

We examine the four possible cases subsequently, 2-i to 2-iv, corresponding respectively to (i) $y_s' \leq 0$, ii) $0 < y_s' < q$, (iii) $y_s' = q$, and (iv) $y_s' > q$. In each case, we determine the analytical expressions of x_s, w_s and W_s.

Case i: When $y_s' \leq 0$. In this case, seller's new average price x_s on her new position, $y_s = y_s' - q$, will be:

$$x_s = \frac{y_s' x_s' - qp}{y_s' - q}. \qquad (35)$$

Her realized wealth will remain unchanged because she has not closed any contract from her old position, thus:

$$W_s = W_s' \qquad (36)$$

Her potential wealth, w_s, should be updated due to the price move from p' to p, that is:

$$w_s = y_s(p - x_s). \qquad (37)$$

Substituting (34) and (35) in (37), we obtain:

$$w_s = y_s'(p - x_s'). \qquad (38)$$

Case ii: When $0 < y_s' < q$. In this case, the action of the seller can be viewed as two consecutive selling operations:

a. she had sold y_s' contracts with a price p, then

b. she sold $q - y_s'$ contracts with a price p.

When she executed operation (a) she had closed her long position y_s' that she had bought before with a price x_s', and realized a net profit or loss equal to $y_s'(p - x_s')$. Adding this amount to the old realized wealth W_s', will yield the new realized wealth:

$$W_s = W_s' + y_s'(p - x_s').$$

When she executed operation (b), she had acquired a short position $y_s = -(q - y_s') = y_s' - q$, with a price $x_s = p$, and the potential wealth of this position is $w_s = y_s(p - x_s) = 0$.

Case iii: When $y_s' = q$. In this case, she had closed entirely her long position, hence she realized a net profit or loss equal to $y_s'(p - x_s')$. Adding this amount to her previous realized wealth will yield:

$$W_s = W_s' + y_s'(p - x_s') = W_s' + q(p - x_s').$$

In this case, her new position $y_s = y_s' - q = 0$, thus we consider its average price as $x_s = 0$, and $w_s = y_s(p - x_s) = 0$.

Case iv: When $y'_s > q$. In this case, she had closed q contracts in her old long position, hence she realized a net profit or loss equal to $q(p - x'_s)$. Adding this amount to her previous realized wealth W'_b will result in

$$W_s = W'_s + q(p - x'_s).$$

After this operation, there will remain $y_s = y'_s - q > 0$ contracts in the possession of the seller. This is a part of her old position that she had bought with an average price x'_s. As these contracts are still in her hand at instant t_j, hence $x_s = x'_s$, and the potential wealth of this position is $w_s = y_s(p - x_s) = (y'_s - q)(p - x'_s)$.

Summary: In order to write on a single line, the functions x_s, W_s, and w_s of the four cases 2-i to 2-iv, we will use the conditional function formulation shown as follows:

$$x_s = \frac{y'_s x'_s - qp}{y'_s - q} 1_{[y'_s \le 0]} + p 1_{[0 < y'_s < q]} + x'_s 1_{[y'_s > q]}.$$

$$W_s = W'_s + (p - x'_s)\left(y'_s 1_{[0 < y'_s < -q]} - q 1_{[y'_s \ge q]}\right).$$

However, we have shown that in both cases 2-ii and 2-iii, we have $w_s = 0$. In the remaining cases 2-i and 2-iv, we know that $w_s \neq 0$, hence we can assert that potential wealth w_s can be written as

$$w_s = (p - x'_s)\left(y'_s 1_{[y'_s \le 0]} + (y'_s - q) 1_{[y'_s > q]}\right).$$

Proof of property 3.1

We will prove this property case by case.

(a) Case where $i \in N\{b, s\}$. At instant t_j, we know that $x_i = x'_i$ and $y_i = y'_i$, therefore,

$$\begin{aligned} w_i &= y_i(p - x_i) = y'_i(p - x'_i) \\ &= y'_i(p - p') + y'_i(p' - x'_i) = y'_i(p - p') + w'_i. \end{aligned}$$

(b) Case where $i = b$. If $y'_b \ge 0$, hence we should be in case 1-i of lemma 1's proof, then from (33) and following the same reasoning as case (a), we show readily this result.
(c) Case where $i = s$. If $y'_s \le 0$, hence we should be in case 2-i of lemma 2's proof, then from (38) and following the same reasoning as case (a), we show this result.

Proof of property 3.2

We will prove this property case by case.

(a) For every $i \in N\backslash\{b,s\}$, we know that relations (8a) and (19) apply, therefore we can write total wealth J_i defined by (6) as follows:

$$J_i = J_i^0 + W_i + w_i = J_i^0 + W_i' + w_i' + y_i'(p-p') = J_i' + y_i'(p-p').$$

(b) If $i = b$, we make use of Formulas (12) and (13) in the following development:

$$\begin{aligned}
J_b &= J_b^0 + W_b + w_b \\
&= J_b^0 + W_b' + (p-x_b')\left(y_b' 1_{[-q<y_b'<0]} - q 1_{[y_b' \le -q]}\right) + (p-x_b')\left(y_b' 1_{[y_b' \ge 0]} + y_b 1_{[y_b'<-q]}\right) \\
&= J_b^0 + W_b' + (p-x_b')\left(y_b' 1_{[-q<y_b'<0]} - q 1_{[y_b' \le -q]} + y_b' 1_{[y_b' \ge 0]} + y_b 1_{[y_b'<-q]}\right) \\
&= J_b^0 + W_b' + (p-x_b')\left(y_b' 1_{[y_b'>-q]} - q 1_{[y_b' \le -q]} + y_b 1_{[y_b'<-q]}\right) \\
&= J_b^0 + W_b' + (p-x_b')\left(y_b' 1_{[y_b'>-q]} - q 1_{[y_b' \le -q]} + y_b \left[1_{[y_b' \le -q]} - 1_{[y_b'=-q]}\right]\right) \\
&= J_b^0 + W_b' + (p-x_b')\left(y_b' 1_{[y_b'>-q]} + (y_b - q) 1_{[y_b' \le -q]} - y_b 1_{[y_b'=-q]}\right),
\end{aligned}$$

but we already know that $1_{[y_b'=-q]} = 1$ if only if $y_b' = -q$, in this event, $y_b = y_b' + q = 0$, hence $y_b 1_{[y_b'=-q]} = 0$, is always true. Now we resume the last expression of J_b, after erasing this zero term, we obtain:

$$\begin{aligned}
J_b &= J_b^0 + W_b' + (p - x_b')\left(y_b' 1_{[y_b'>-q]} + (y_b - q) 1_{[y_b' \le -q]}\right) \\
&= J_b^0 + W_b' + (p - x_b')\left(y_b' 1_{[y_b'>-q]} + (y_b' + q - q) 1_{[y_b' \le -q]}\right) \\
&= J_b^0 + W_b' + (p - x_b')\left(y_b' 1_{[y_b'>-q]} + y_b' 1_{[y_b' \le -q]}\right) \\
&= J_b^0 + W_b' + y_b'(p - x_b') \\
&= J_b^0 + W_b' + y_b'[(p - p') + (p' - x_b')] \\
&= J_b^0 + W_b' + y_b'(p' - x_b') + y_b'(p - p') \\
&= J_b^0 + W_b' + w_b' + y_b'(p - p') \\
&= J_b' + y_b'(p - p').
\end{aligned}$$

(c) If $i = s$, then following the same approach then case (b), we can show easily that:

$$J_s = J_s' + y_s'(p - p').$$

Hence, relation (20) holds true for all traders and in all cases.

Proof of property 3.3

If relation (6) was applied at instant t_{j-1}, it would yield to

$$J_i' = J_i^0 + W_i' + w_i'.$$

On the other hand, from (6) we can extract the expression of W_i shown as follows:

$$W_i = J_i - J_i^0 - w_i$$

Now substituting the term J_i by its expression given in (20) will result in:

$$\begin{aligned} W_i &= [J_i' + y_i'(p - p')] - J_i^0 - w_i \\ &= [(J_i^0 + W_i' + w_i') + y_i'(p - p')] - J_i^0 - w_i \\ &= W_i' + w_i' + y_i'(p - p') - w_i. \end{aligned}$$

Proof of property 3.4

We have:

$$\begin{aligned} y &= \sum_{i \in N} y_i \, 1_{[y_i > 0]} \\ &= y_b 1_{[y_b > 0]} + y_s 1_{[y_s > 0]} + \sum_{i \in N \setminus \{b,s\}} y_i \, 1_{[y_i > 0]} \\ &= y_b 1_{[y_b > 0]} + y_s 1_{[y_s > 0]} + \sum_{i \in N \setminus \{b,s\}} y_i' \, 1_{[y_i' > 0]}, \end{aligned}$$

we write this as:

$$y = Q_1 + Q_2 + \sum_{i \in N \setminus \{b,s\}} y_i' 1_{[y_i' > 0]}, \tag{39}$$

where

$$\begin{aligned} Q_1 &= y_b 1_{[y_b > 0]} = (y_b' + q) 1_{[y_b' + q > 0]} = (y_b' + q) 1_{[y_b' > -q]} \\ &= y_b' 1_{[y_b' > -q]} + q \, 1_{[y_b' > -q]} = y_b' \left(1_{[y_b' > 0]} + 1_{[-q < y_b' \leq 0]}\right) + q \, 1_{[y_b' > -q]} \\ &= y_b' 1_{[y_b' > 0]} + A, \end{aligned}$$

and

$$\begin{aligned} Q_2 &= y_s 1_{[y_s > 0]} = (y_s' - q) 1_{[y_s' - q > 0]} = (y_s' - q) 1_{[y_s' > q]} \\ &= (y_s' - q)\left(1_{[y_s' > q]} + 1_{[0 < y_s' \leq q]} - 1_{[0 < y_s' \leq q]}\right) = (y_s' - q)\left(1_{[y_s' > 0]} - 1_{[0 < y_s' \leq q]}\right) \\ &= y_s' 1_{[y_s' > 0]} - q \, 1_{[y_s' > 0]} - y_s' 1_{[0 < y_s' \leq q]} + q \, 1_{[0 < y_s' \leq q]} \\ &= y_s' 1_{[y_s' > 0]} - q \left(1_{[y_s' > 0]} - 1_{[0 < y_s' \leq q]}\right) - y_s' 1_{[0 < y_s' \leq q]} \\ &= y_s' 1_{[y_s' > 0]} - q 1_{[y_s' > q]} - y_s' 1_{[0 < y_s' \leq q]} \\ &= y_s' 1_{[y_s' > 0]} - B. \end{aligned}$$

Substituting Q_1 and Q_2 in Formula (39), we obtain:

$$
\begin{aligned}
y &= y_b' \mathbf{1}_{[y_b'>0\,]} + A + y_s' \mathbf{1}_{[y_s'>0\,]} - B + \sum_{i\,\in N\setminus\{b,s\}} y_i'\, \mathbf{1}_{[y_i'>0]} \\
&= A - B + \sum_{i\,\in N} y_i'\, \mathbf{1}_{[y_i'>0]} \\
&= y' + A - B.
\end{aligned}
$$

Proof of property 3.5

By definition, we know that $y(t_0) = 0$ because $y_i(t_0) = 0$, $\forall\, i \in N$, thus (25) holds true for t_0. Now, assuming that at instant t_{j-1} relation (25) holds, that is:

$$
y(t_{j-1}) = \sum_{k=0}^{j-1} [\,A(t_k) - B(t_k)],
$$

Hence

$$
\begin{aligned}
y(t_j) &= y(t_{j-1}) + A(t_j) - B(t_j) \\
&= \sum_{k=0}^{j-1} [\,A(t_k) - B(t_k)] + A(t_j) - B(t_j) \\
&= \sum_{k=0}^{j} [A(t_k) - B(t_k)]\,.
\end{aligned}
$$

Proof of property 3.6

Note that if t_j is a non-transactional time, then $y = y'$, therefore $\Delta y = 0$. Thereafter, we are dealing with transactional times only. From (22), we deduce that:

$$
\begin{aligned}
\Delta y &= y - y' = A - B \\
&= \left(q\mathbf{1}_{[y_b' > -q]} + y_b' \mathbf{1}_{[-q<y_b'\leq 0\,]}\right) - \left(q\mathbf{1}_{[y_s'>q]} + y_s' \mathbf{1}_{[0\,<y_s'\leq q]}\right)
\end{aligned}
$$

Table 1 summarizes the calculation for each case: Case (1) corresponds to $y_b' > 0$ and $y_s' > q$, case (2) corresponds to $y_b' > 0$ and $0 < y_s' \leq q$, and so on. For each case, we compute the values of A and B, then we calculate the difference $\Delta y = A - B$, and the last column of the table shows the sign of Δy in each case.

Case (5) of Table 1, where $-q < y_b' \leq 0$ and $0 < y_s' \leq q$, necessitates further analysis to determine the sign of Δy. In this case, we know that

$$
\Delta y = q + y_b' - y_s'. \qquad (40)
$$

Table 1. Calculation of *Δy*.

Case	y_b'	y_s'	$A =$	$B =$	$\Delta y = A - B$	Sign of Δy
1		$y_s' > q$	q	q	0	0
2	$y_b' > 0$	$0 < y_s' \leq q$	q	y_s'	$q - y_s'$	≥ 0
3		$y_s' \leq 0$	q	0	q	> 0
4		$y_s' > q$	$q + y_b'$	q	y_b'	≤ 0
5	$-q < y_b' \leq 0$	$0 < y_s' \leq q$	$q + y_b'$	y_s'	$q + y_b' - y_s'$	Any
6		$y_s' \leq 0$	$q + y_b'$	0	$q + y_b'$	> 0
7		$y_s' > q$	0	q	$-q$	< 0
8	$y_b' \leq -q$	$0 < y_s' \leq q$	0	y_s'	$-y_s'$	< 0
9		$y_s' \leq 0$	0	0	0	0

For this case (5), we can show easily that $-2q < y_b' - y_s' < 0$, hence $-q < q + y_b' - y_s' < q$, therefore Δy could be positive, negative or nil, depending on the values of y_b' and y_s'; we have the following

1. $\Delta y > 0 \Rightarrow q + y_b' > y_s'$,
2. $\Delta y < 0 \Rightarrow q + y_b' < y_s'$,
3. $\Delta y = 0 \Rightarrow q + y_b' = y_s'$.

This completes the proof of this property.

Proof of property 3.7

Assuming that M is the biggest positive number, then from Figure 1 (Appendix A), we observe that any couple (y_b', y_s') belonging to the square delimited by the points $(-M,-M)$, $(-M,M)$, (M,M), and $(M,-M)$; this square has an area of $4\ m^2$ -units.

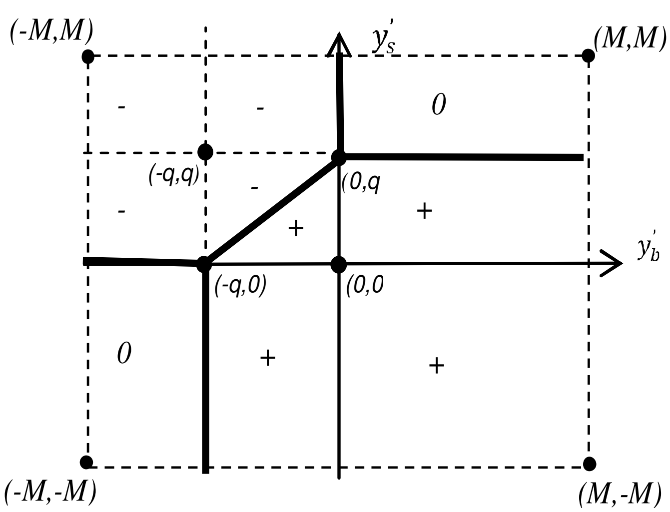

Figure 1. Calculation of $\pi(\Delta y)$.

In this square, we have:

1. Two symmetrical zones where $\Delta y = 0$, with a total area of $2M(M-q)$ square-units, hence:

$$\pi(\Delta y = 0) = \frac{2M(M-q)}{4M^2} = \frac{1}{2} - \frac{q}{2M}$$

2. One zone where $\Delta y > 0$ formed by four sub-zones: One square of M^2 -units, two symmetrical rectangles of $2Mq$ square-units, and a triangle of $\frac{q^2}{2}$ square-units; therefore:

$$\pi(\Delta y > 0) = \frac{M^2 + 2Mq + \frac{q^2}{2}}{4M^2} = \frac{1}{4} + \frac{q}{2M} + \frac{q^2}{8M^2} = \frac{1}{4} + \frac{q}{2M}\left(1 + \frac{q}{4M}\right)$$

3. One zone where $\Delta y < 0$ formed by four sub-zones: One square with $(M-q)^2$ -units, two symmetrical rectangles with $2Mq$ square-units, and a triangle of $\frac{q^2}{2}$ square-units. We can also consider this zone as being formed by a bigger square $(0,0)$, $(-M,0)$, $(-M,-M)$ and $(0,M)$, having an area of M^2 square-units, from which we deducted the triangle $(0,0)$, $(-q,0)$, $(0,q)$ having an area of $\frac{q^2}{2}$ square-units, thus:

$$\pi(\Delta y < 0) = \frac{M^2 - \frac{q^2}{2}}{4M^2} = \frac{1}{4} - \frac{1}{8}\left(\frac{q}{M}\right)^2$$

Proof of property 3.8

Assuming that M is bigger enough ($M \equiv +\infty$), then:

$$\lim_{q \to 1} \frac{q}{M} = 0, \quad \text{and} \quad \lim_{q \to M} \frac{q}{M} = 1$$

Applying these two limits we show easily *Property* 3.8.

Proof of property 3.9

Condition (27) means that, since instant t_0 till t_j, no trader is closing a part of her old position, that is, any trader who bought before continues to buy and any trader who sold before continues to sell. In a mathematical form, if s_k and b_k are respectively the seller and buyer at a transactional instant t_k, then:

$$y_{s_k}(t_{k-1}) \leq 0, \text{ and } y_{b_k}(t_{k-1}) \geq 0, \qquad \forall\, k = 0 \ldots j.$$

In this case, open interest $y(t_k)$ at any instant t_k is growing by the amount of the transactional quantity $q(t_k)$, therefore:

$$y(t_j) = y(t_{j-1}) + q(t_j) = \sum_{k=0}^{j} q(t_k).$$

Assuming that (28) holds true at t_{j-1}, that is:

$$\bar{p}(t_{j-1}) = \frac{\sum_{k=0}^{j-1} p(t_k)q(t_k)}{\sum_{k=0}^{j-1} q(t_k)} = \frac{\sum_{i=1}^{n} x_i(t_{j-1})y_i(t_{j-1})\, 1_{[y_i(t_{j-1})>0]}}{y(t_{j-1})}, \quad (41)$$

And let us show this remains true at t_j. We already know that:

$$\bar{p}(t_j) = \frac{\sum_{k=0}^{j} p(t_k)q(t_k)}{\sum_{k=0}^{j} q(t_k)} = \frac{\sum_{k=0}^{j-1} p(t_k)q(t_k) + p(t_j)q(t_j)}{\sum_{k=0}^{j-1} q(t_k) + q(t_j)} \quad (42)$$

$$= \frac{\sum_{i=1}^{n} x_i(t_{j-1})y_i(t_{j-1})\, 1_{[y_i(t_{j-1})>0]} + p(t_j)q(t_j)}{y(t_{j-1}) + q(t_j)} \quad (43)$$

We know that when passing from instant t_{j-1} to t_j, the components of all traders will remain the same, except the components of the buyer b and the seller s need to be updated, that is $x_i(t_j) = x_i(t_{j-1})$ and $y_i(t_j) = y_i(t_{j-1})$ for all $i \in N \setminus \{s, b\}$. This is true in case of a transactional time. In case of a non-transactional time, components of all traders will remain the same. Now resuming the apostrophe notation, we obtain:

$$\bar{p} = \frac{\sum_{i \in N \setminus \{b\}} x_i y_i\, 1_{[y_i>0]} + x_b' y_b' + p\, q}{y' + q} \quad (44)$$

Note that the components of the seller s do not appear above because we are in the case where $y_s' < 0$, therefore $1_{[y_s>0]} = 0$.

1. If t_j is not a transactional time, that is, $q = 0$, $y = y'$, $y_b = y_b'$ and $x_b = x_b'$, then from (44) will result

$$\bar{p} = \frac{\sum_{i \in N} x_i y_i 1_{[y_i>0]}}{y}, \quad (45)$$

which completes the proof.

2. If t_j is a transactional time, and sinc $y_b' \geq 0$ e then $y_b = y_b' + q > 0 \Rightarrow 1_{[y_b>0]} = 1$ and

$$x_b = \frac{x_b' y_b' + p\, q}{y_b} \Rightarrow pq = x_b y_b - x_b' y_b'. \quad (46)$$

Substituting pq by $x_b y_b - x_b' y_b'$ in (44) will readily complete the proof.