

Review

A direct formulation of implied volatility in the Black-Scholes model

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The inverse problem of option pricing, also known as market calibration, attracted the attention of a large number of practitioners and academics, from the moment that Black-Scholes formulated their model. The search for an explicit expression of volatility as a function of the observable variables has generated a vast body of literature, forming a specific branch of quantitative finance. But up to now, no exact expression of implied volatility has been obtained. The main result of this paper is such an exact expression. Firstly, a formula was deduced analytically. Secondly, it was shown that this expression is actually an exact inversion, using simulated data. Thirdly, it was shown that the methodology can be used to express implied volatility in more sophisticated models, such as the Blenman and Clark model. In the conclusion, discussion of the results was made.

Key words: Black-Scholes model, inverse problem, implied volatility, conservation law.

INTRODUCTION

Over the last fifteen years, economic and financial theorists have borrowed several methodological tools from physics. The transpositions have been made possible by the many similarities between the subjects of study. There are analogies, for example, between the behaviour over time of the value of certain financial instruments and modes of particle diffusion. The works of Mantegna and Stanley (1999), Dragulescu and Yakovenko (2000), Sornette (2002) and Bouchaud and Potters (2003) bear witness to the increasing importance of what some have named "econophysics". Theorists' interest in the concept of conservation law, originally developed in physics, is constantly growing. Samuelson (1970), Sato and Ramachandran (1990) and Kataoka and Hashimoto (1995) took a very early interest in the subject, but most of the works have been more recent. Academic research has been published by Samuelson (2004), Mitchell (2004) and Sato (2004). The practical applications presented by these authors concern, for example, the evaluation of corporate performance using characteristics whose values do not change as the firm evolves.

The methodology of conservation laws has also shown

promise in the field of market finance. Knowledge of invariant relations between the derivatives of an unknown function, in dynamical models, makes it possible to resolve a certain number of hitherto unsolvable problems. Description of the symmetries to which the conservation laws correspond implies the careful choice of analytical methods. Thus, the application of Lie's theory to the Black-Scholes equation study, by Gazizov and Ibragimov (1996), Lo and Hui (2001), Pooe et al. (2003) and Silberberg (2004) has produced results that are varied, but of limited use from a practical point of view. Use of the new approach to the theory of the separation of variables, first proposed in quantum mechanics by Fris et al. (1965), Bagrov et al. (1973) and Shapovalov and Sukhomlin (1974), has proved to be more fruitful. By applying this local approach, which is not limited to first-order differential symmetry operators, to the Black-Scholes model, Sukhomlin (2004) constructed a number of conservation laws and defined several classes of new solutions. In particular, he studied the conservation law of option value elasticity. Then Sukhomlin (2006) discovered the symmetry of the classic of Black-Scholes solution. This article shows that one of the most important prospects opened up by the study of conservation laws in the Black-Scholes model concerns the exact expression of volatility as a function of the parameters that can be observed in the market. This is an important theoretical

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advance, because there is a widely-held view (See, for example, Hull (2006).) that it is not possible to invert the Black-Scholes formula. Recent studies have led to formulas that are only approximate (See, for example, Cont (2008)).

Estrella (1996) studied the application of the Taylor series to the Black-Scholes model, and particularly problems of convergence. He concentrated on the “Greeks” *delta* and *gamma*, noting in one particular case that the third derivative of the option value could be expressed as a function of the second derivative. Our study shows that this property is also true in the general case. Moreover, this property of the model proves to be crucial, because its use makes it possible to reveal the very particular symmetry of the classic Black-Scholes solution and to express the implied volatility directly as a function of the other observable parameters.

The mathematical complexity encountered in the approaches to the inverse problem of option pricing appears, for example, in the publications of Bouchouev and Isakov (1997), Chiarella et al. (2003) and Egger et al. (2006). One of the chief difficulties lies in the fact that, in reality, volatility is not constant. However, even under the simplifying hypothesis of constant volatility that constitutes the principal assumption of the classic Black-Scholes model, the inverse problem of option pricing has not yet been solved (In addition to the wide variety of approximate formulas, the literature also contains Dupire’s formula for local volatility (Dupire, 1993). However, this is deduced from Kolmogorov’s direct equation, corresponding to that of Black and Scholes, and not from the classic solution that is the essence of the Black-Scholes model. It is not, therefore, a solution to the inverse problem of option pricing within the context of the Black-Scholes model.). In this article, the solution for this latter case is given.

The structure of this article is as follows:

The introduction gives an auxiliary function of *Ksi* which is defined from the classic solution of the Black-Scholes model. The volatility is then expressed as a function of the option price and its derivatives, the underlying, the risk-free interest rate, the strike price and the time to maturity. Using simulated data, strong congruence between the analytical result and the numerical result obtained by data simulation was shown. Finally illustration of the possibility of transposing these results to a more sophisticated dynamical system than the classic Black-Scholes model was shown.

ANALYTICAL EXPRESSION OF VOLATILITY IN THE BLACK-SCHOLES MODEL

The Black-Scholes partial differential Equation is expressed as follows:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0, \tag{1}$$

Where *V* is the option value, *S* is the value of the underlying at time *t*; *r* is the interest rate and σ is the volatility (with the last two parameters assumed to be constant). The limit condition is: $V = \max(S - K; 0)$ at expiration date for a European call option (Only the study of call options is presented, as all the results can easily be extended to put options) on an underlying asset with no dividends. The constants *K* and *T* represent the strike price and expiration date respectively ($t \in [0, T]$).

The classic Black-Scholes solution is written as:

$$V = S N(d_1) - F(t) N(d_2) \tag{2}$$

With:

$$F(t) \equiv K e^{-r(T-t)} \tag{3}$$

$$N(d) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^d e^{-u^2/2} du \tag{4}$$

$$d_1 \equiv \frac{\ln S - \ln K}{\tau} + (1 - \beta)\tau \tag{5}$$

$$d_2 \equiv \frac{\ln S - \ln K}{\tau} - \beta \tau \tag{6}$$

$$\tau \equiv \sigma \sqrt{T - t} \tag{7}$$

$$\beta \equiv \frac{1}{2} - \frac{r}{\sigma^2} \tag{8}$$

$$\text{Given that } d_1 = d_2 + \tau \tag{9}$$

It is possible to write (See, for example, Wilmott et al. (1995)) that:

$$S N'(d_1) = F(t) N'(d_2) \tag{10}$$

Let there also be the auxiliary function *Ksi*, only used for calculation purpose:

$$\xi \equiv V^{(2)} - V^{(1)} \tag{11}$$

With:

$$V^{(1)} \equiv \partial V / \partial (\ln S) \tag{12}$$

$$\text{And } V^{(2)} \equiv \partial^2 V / \partial (\ln S)^2 \quad (13)$$

Then

$$\xi = \frac{1}{\tau} S \left[N'(d_1) + \frac{1}{\tau} SN''(d_1) \right] + \frac{1}{\tau} F(t) \left[N'(d_2) - \frac{1}{\tau} N''(d_2) \right] \quad (14)$$

Taking into account the normal law's property:

$$N''(d) = -d N'(d) \quad (15)$$

We can deduce that:

$$\xi = \frac{1}{\tau} SN'(d_1) \left[1 - \frac{d_1}{\tau} \right] + \frac{1}{\tau} F(t) N'(d_2) \left[\frac{d_2}{\tau} + \right] \quad (16)$$

Then

$$\xi = \frac{1}{\tau} S N'(d_1) \quad (17)$$

Or again:

$$\xi = \frac{1}{\tau} F(t) N'(d_2) \quad (18)$$

Let E_ξ be the elasticity of the auxiliary function Ksi in relation to S (The concept of price elasticity used in the rest of this article refers to the elasticity of the new characteristic Ksi in relation to the underlying or that of the $gamma$ in relation to the underlying, depending on the context)

$$E_\xi \equiv \frac{\partial \ln |\xi|}{\partial \ln S} = -\frac{1}{\tau^2} (\ln S - \ln K) + \beta \quad (19)$$

And

$$\tau^2 (E_\xi - \beta) = \ln K - \ln S \quad (20)$$

From Equation (17), it is seen that the well-known characteristic $gamma$ is linked to the auxiliary function Ksi by the Equation:

$$\xi = S^2 \Gamma \Leftrightarrow E_\xi = 2 + E_\Gamma, \quad E_\Gamma \equiv \Gamma^{(1)} / \Gamma$$

This enables us to write (20) in another way:

$$\tau^2 (E_\Gamma + 2 - \beta) = \ln K - \ln S \quad (21)$$

By replacing τ and β in the Equation (20) by their

expressions in (7) and (8), the volatility can then be expressed directly. The inverse problem of the classic Black-Scholes model can then be solved exactly, in the form of the following expression of the volatility as a function of four variables: the ratio of the strike price to the underlying, the risk-free interest rate, the time to maturity and the elasticity of the "Greek" characteristic $Gamma$:

$$\sigma = \sqrt{\frac{\ln(K/S) - r(T-t)}{(T-t)(E_\Gamma + \frac{3}{2})}} \quad (22)$$

The volatility is thus exactly defined as a function of the directly observable variables of the model and one other variable, $gamma$ that can be calculated from the option value, which is itself observable.

NUMERICAL VERIFICATION

The aim of numerical verification is to verify the expression of volatility given in Equation (22) and to show it is actually an inversion. The methodology is based on data simulation. The value of a call option V_i was calculated with the help of the Black-Scholes formula, using a given time to maturity ($T-t$), strike price K , risk-free interest rate r , volatility σ and underlying S_i . The operation is repeated with different successive values for the underlying.

Comparison of the volatility calculated using formula (22) was done with that used to simulate the data. Numerically, the values given to the parameters are consistent with the orders of magnitude encountered in real life.

The values of the underlying range from 900 to 1000 by increment of 0.1, the strike price is equal to 950 (mid-point of the range of values of the underlying), the interest rate is equal to 2%, the time to maturity is equal to 0.2, and the volatility is equal to 20%. For each value S_i of the underlying, the value of the $gamma$ is calculated using the following formula:

$$\Gamma_i = \frac{1}{S_i} \frac{1}{\tau \sqrt{2\pi}} e^{-\frac{\left(\frac{\ln S_i - \ln K}{\tau} + (1-\beta)\tau\right)^2}{2}} \quad (23)$$

(This formula is the discrete expression of the well-known equality: $\Gamma = N'(d_1) / S\tau$)

Where $i \in [0;1000]$, $i \in \mathbb{N}$, $S_0 = 900$; $S_{1000} = 1000$ and $S_{i+1} - S_i = 0,1$.

Formula (20) requires the calculation of the elasticity of

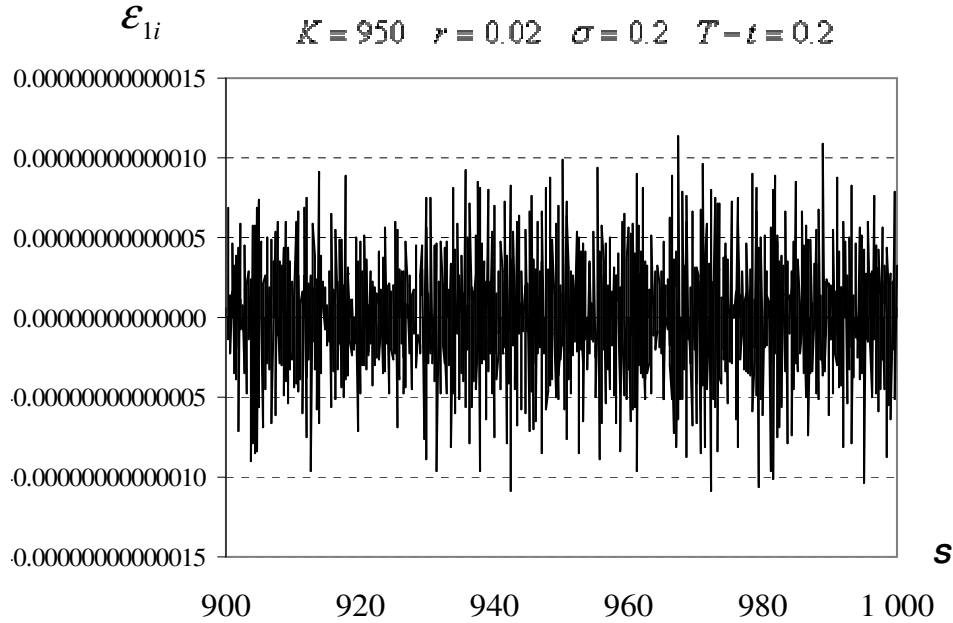


Figure 1. Expression (25) - calculatory residuals.

the *gamma*. This can be performed, on discrete data, using the following formula:

$$E_{\Gamma_i} = \frac{\ln|\Gamma_i| - \ln|\Gamma_{i-1}|}{\ln(S_i) - \ln(S_{i-1})} \quad \forall i \in [1;1000] \quad (24)$$

There is another way of appraising the experimental validity of Equation (18), by rewriting it as follows:

$$\tau^2(E_{\Gamma} + 2 - \beta) - (\ln K - \ln S) = 0 \quad (25)$$

By using the simulated data, the residual ϵ_{1_i} as defined below can be calculated:

$$\epsilon_{1_i} = \tau^2(E_{\Gamma} + 2 - \beta) - \left(\ln K - \frac{\ln(S_i) + \ln(S_{i-1})}{2} \right) \quad (26)$$

Visually, there is no apparent trend in the graphic representation of ϵ_{1_i} (Figure 1). The values of ϵ_{1_i} are uniformly distributed on either side of the *x* axis, with no particular evolution in the dispersion. When the *gamma* and its elasticity have been calculated, the volatility can be determined.

Relation (22) can be transcribed as follows:

$$\sigma_{calc_i} = \sqrt{\frac{\ln K - \frac{\ln(S_i) + \ln(S_{i-1})}{2} - r(T - t)}{(T - t)(E_{\Gamma} + \frac{3}{2})}} \quad \forall i \in [1;1000] \quad (27)$$

We can then determine the deviation between the calculated volatility and the volatility used for data simulation:

$$\epsilon_{2_i} = \sigma - \sigma_{calc_i} \quad (28)$$

This deviation is represented in Figure 2. The values are evenly distributed on either side of the *x* axis, with stronger dispersion around the underlying value of 946.3, corresponding to *gamma* elasticity close to -1.5. Formula (22) shows the high sensitivity around this value. The effects of the approximation linked to the calculation of discrete data are therefore amplified around this point.

By a change in scale, we can observe this phenomenon more precisely (Figure 3). The maximum deviation between the volatility value calculated using formula (22) and the volatility value used for data simulation is of the order of one thousand millionth of the latter (This result is appreciably the same for different interest rates and volatility values). Thus, the analytically-deduced formula for volatility appears to be corroborated, to acceptable levels, by the numbers. The calculation shows that we have only used the other parameters of the Black-Scholes model to obtain implied volatility. Then our formula is really an inversion.

EXTENSION TO OTHER MODELS

The symmetries existing in more sophisticated models can be exploited in the same way as in the classic Black-Scholes model. For example, in a generalisation of the type Constant Underlying Elasticity in Strikes (CUES)

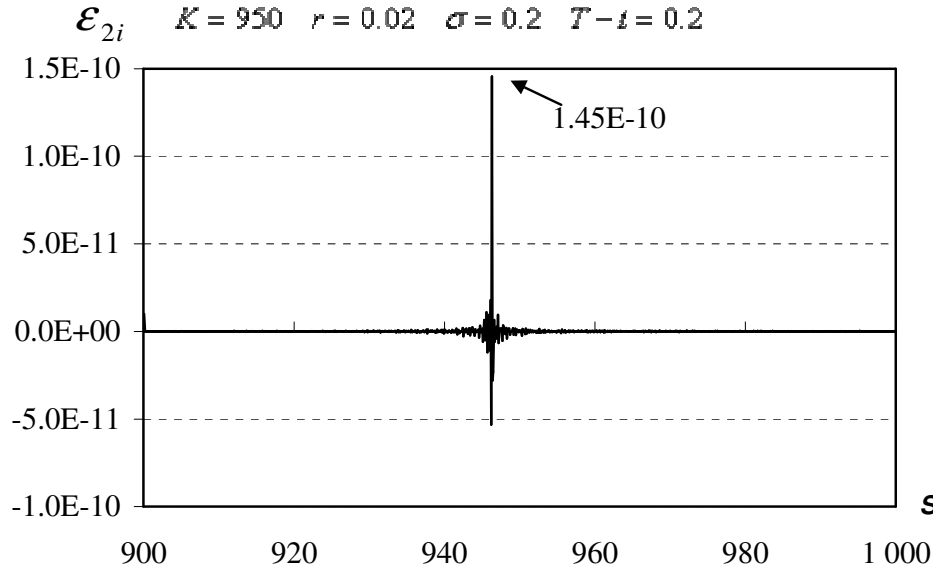


Figure 2. Volatility - calculatory residuals.

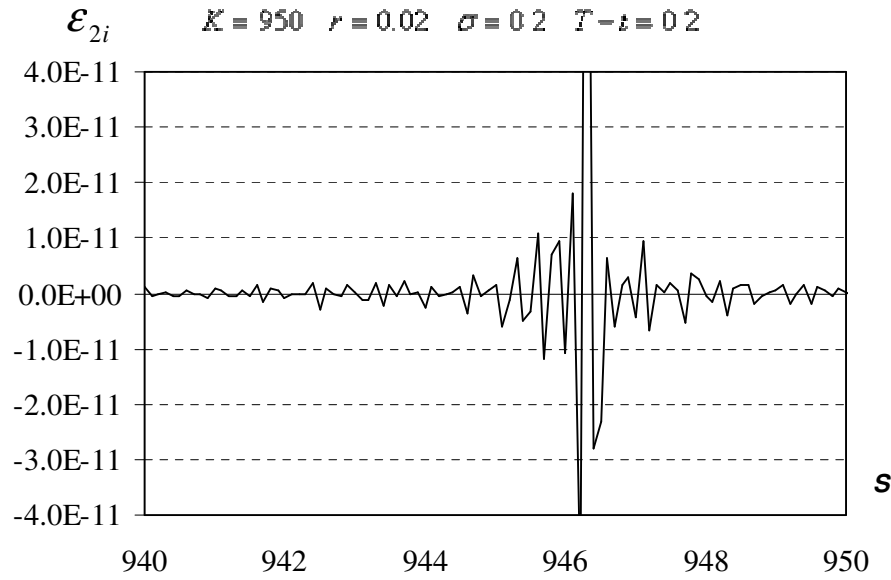


Figure 3. Volatility - calculatory residuals – zoom.

presented by Blenman and Clark (2005), the value of a European call option is the following:

$$C(t, S) = e^{-\varepsilon \tau^2} S N(d_1) - F(t, S) N(d_2) \quad (29)$$

Where we note:

$$S_0 \equiv q^{1/(1-\alpha)}, \varepsilon \equiv \delta / \sigma^2,$$

$$d_1 \equiv \frac{1}{\tau} \ln\left(\frac{S}{S_0}\right) + (1-\beta-\varepsilon)\tau$$

$$d_2 \equiv \frac{1}{\tau} \ln\left(\frac{S}{S_0}\right) + (\alpha-\beta-\varepsilon)\tau, \beta \equiv \frac{1}{2} - \frac{r}{\sigma^2},$$

$$\tau \equiv \sigma \sqrt{T-t}$$

$$F(t, S) \equiv S_0 (S/S_0)^\alpha \exp\left\{\left[-\alpha\varepsilon + (\alpha-1)\left(\frac{\alpha+1}{2} - \beta\right)\right]\tau^2\right\}$$

$N(\cdot)$ is the cumulative normal distribution function. It is

obvious that: $d_2 = d_1 + (\alpha - 1)\tau$. After derivation and simple calculation, we obtain the equivalent of formula (10):

$$e^{-\varepsilon\tau^2} S N'(d_1) = F(t, S) N'(d_2). \quad (30)$$

The equivalent of the characteristic *Ksi* can then be calculated in the case of CUES:

$$\xi = C^{(2)} - (1 + \alpha)C^{(1)} + \alpha C = \frac{1 - \alpha}{\tau} e^{-\varepsilon\tau^2} S N'(d_1) \quad (31)$$

Its price elasticity is a linear function of $\ln(S/S_0)$:

$$E_\xi = \xi^{(1)} / \xi = \beta + \varepsilon - \tau^{-2} \ln(S/S_0). \quad (32)$$

The conservation law in the CUES case then has a similar appearance to that of the classic Black-Scholes solution (27):

$$\tau^2 [E_\xi - (\beta + \varepsilon)] + \ln S = \ln S_0 = \text{const} \quad (33)$$

The volatility is:

$$\sigma = \sqrt{\frac{\ln q / (1 - \alpha) - \ln S - r(T - t)}{(T - t)(E_\xi - \varepsilon - 1/2)}} \quad (34)$$

Or again:

$$\sigma = \sqrt{\frac{\ln q / (1 - \alpha) - \ln S - r(T - t)}{(T - t)(E_\Gamma - \varepsilon + 3/2)}} \quad (35)$$

The Black-Scholes model represents a particular case of the Blenman and Clark model. The formulas (35) becomes the formula (22) when $\alpha \rightarrow 0$ and $\varepsilon \rightarrow 0$.

CONCLUSION

This article, have presented an exact solution of the inverse problem of option pricing within the context of the classic Black-Scholes model. Thus, the implied volatility is expressed as a function of the other observable market parameters. The result is formulated as follows; In the classic Black-Scholes model, the volatility can be expressed exactly in the form of a function of the ratio of the strike price to the underlying, the risk-free interest rate, the maturity and the elasticity of the "Greek" characteristic *Gamma*. This expression of the volatility is given by formula (22).

Therefore, volatility have been expressed directly and solely as a function of known or observable, or calculable

variables. The known variables correspond to the option's specifications: its time-to-maturity and the strike price. The spot price and the interest rate are directly observable. It is possible to calculate the *gamma* elasticity on the basis of the spot price and the option's value, which are both observable. The expression of the implied volatility of the Black-Scholes model has been verified numerically with the use of data simulation. Comparison of the volatility calculated by means of formula (22) with that used to simulate the data leads us to observe a more than satisfactory congruence (of the order of one thousand millionths). The main result of this article is theoretical. It has been shown that, contrary to a common thought, an exact expression of implied volatility can be calculated. The formula was given. The main limit of the result is practical. But it is due to the difficulty of the traditional Black Scholes model to reflect reality, and not this calculation. As have been shown in this study, in order to calculate implied volatility, the elasticity of the *gamma* has to be calculated. So four observations are needed. If these observations are fully compliant with the Black Scholes frame, like in this simulation, the calculated volatility is exact. But, often, the real data do not fit. For example there may be a "smile" effect. In this case the practitioner has to use the usual iterative method to calculate implied volatility. This is the reason why it might be useful to use the methodology used in this paper to express directly implied volatility to more sophisticated models that might better reflect reality. It has been shown that it is possible to apply this methodology to the Blenman and Clark model. Future studies might apply this methodology to much more complicated models so that direct expressions of implied volatility may be used on real data.

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