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Full Length Research Paper

Polygon graphs of Girth 6

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This paper introduces a family of simple bipartite graphs denoted by $G_P(m,p)$, named and constructed as polygon graphs. These polygon graphs are bicubic simple graphs possessing Hamiltonian cycles. Some important results are proved for these graphs. The girth of these graphs is counted as 6($m \geq 3$). $G_P(3,1)$ Polygon graph is isomorphic to a famous Pappus graph. Since polygon graphs are bipartite, therefore they can be used as Tanner graphs for low density parity check codes. This family of graphs may also be used in the design of efficient computer networks.

Key words: Polygon graph, girth, Hamiltonian cycle, bicubic graph.

INTRODUCTION

A graph G is a triple, consisting of a vertex set $V= V(G)$, an edge set $E=E(G)$ and a map that associates to each edge two vertices (not necessarily distinct) called its end points. A loop is an edge whose end points are equal. Multiple edges are edges having same end points. A simple graph is one having no loops or multiple edges. To any graph, we can associate the adjacency matrix A which is an $n \times n$ matrix ($n=|v|$) with rows and columns indexed by the elements of vertex set and the (x,y)-th entry is the number of edges connecting x and y. A path is a non-empty graph

$P = (V, E)$

where,

$V= \{x_0, x_1,\ldots\ldots\ldots\ldots x_k\}$, $E = \{x_0x_1,\ldots\ldots x_{k-1}\, x_k\}$

01Where the x_i's are all distinct.

The vertices x_0 and x_k are linked by P and are called its ends; the vertices $x_1,\ldots\ x_{k-1}$ are the inner vertices of P. The number of edges of a path is its length. If $P = x_0, \ldots x_{k-1}$ is a path with $k \geq 3$, then the graph $C{:} = P + x_k\, x_0$ is known as cycle. As with paths, we often denote a cycle by its (cyclic) sequence of vertices, the above cycle C might be written as $x_0 \ldots x_{k-1}\, x_0$. The length of a cycle is the number of its edges (or vertices). The cycle of length k is called *k-cycle* and denoted by C^k. The length of the shortest cycle (contained) in a graph G is the girth g (G) of G, the maximum length of a cycle is G and is its circumference. If G does not contain a cycle, we set the former to ∞ the latter to zero (Reinhard, 2000). Cubic graphs also known as trivalent graphs are intensively studied in graph theory. The bipartite cubic graphs widely known as bicubic graphs took special interest. Tait (1888), conjectured that every planer cubic graph contains Hamiltonian cycle, it was challenged by Tutte (1946), by a counter example, a 46-vertex graph now named for him (Tutte, 1946). In 1971, Tutte conjectured that all bicubic graphs are Hamiltonian; however, Horton provided a 96-vertex counter example (Horton, 1982).

Study of cubic graphs is one of the favorite topics for graph theorists. One main reason is that they have wide application. For example, it follows from the result of Malle et al. (1994) that, almost every finite simple group has a cubic Cayley graph. Moreover, the generalization to the graphs of degree k>3 does not appear to be more difficult than the case k=3. Finally, the cubic case is the only one where we have specific examples that improve significantly on the best general results currently available incorrect format (Biggs, 1998). Polygon graphs are constructed by connecting odd number of copies of

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polygons of same size in a delicate manner. These graphs are defined only for polygons with an even number of sides and denoted by $G_P(m,p)$, where *2m* is the number of sides and *2p +1* is the number of copies of polygon P. These graphs are bicubic with some interesting properties. Principle of mathematical induction is used to prove the existence of Hamiltonian cycles in these graphs. It is shown that, these graphs have girth 6. It is a simple fact that cubic Hamiltonian graphs have at least two Hamiltonian cycles. Finding such a cycle is NP-hard in general, and no polynomial time algorithm is known for the problem of finding a second Hamiltonian cycle, when one such cycle is given as part of the input (Cristina, 1999). However, in the case of polygon graph, it is easy to find Hamiltonian cycles.

METHODS OF CONSTRUCTING POLYGON GRAPHS

Let P be a polygon with $2m\ (m \geq 2)$ sides. Place *2p+1* copies of *P* denoted by $P, P_1', P_2', \cdots P_p', P_1, P_2, \cdots P_p$ in parallel, such that *P* is in the middle, *p* copies on the right side of *P* say $P_1, P_2, \cdots P_p$ and *p* copies on the left side of *P* say $P_1', P_2', \cdots P_p'$ as follows. $P_1', P_2', \cdots P_p', P, P_1, P_2, \cdots P_p$. Vertices of *P*, P_i' and P_i are denoted by $v_1^{(0)}, v_2^{(0)}, \cdots, v_{2m}^{(0)}$, $u_1^{(i)}, u_2^{(i)}, \cdots, u_{2m}^{(i)}$, $w_1^{(i)}, w_2^{(i)}, \cdots, w_{2m}^{(i)}$, $i = 1, \cdots, p$ respectively. Simple connected graph G_P is constructed by drawing edges such that, an even vertex is connected with odd vertex and an odd vertex with an even one in the following manner.

Algorithm

Edges between vertices of different polygons

a) For $i < p$ connect *m* vertices of P_i with *m* vertices of P_{i-1} and *m* vertices of P_i with *m* vertices of P_{i+1}. Similarly, connect *m* vertices of P_i' with *m* vertices of P_{i-1}' and *m* vertices of P_i' with *m* vertices of P_{i+1}.
b) For $i = p$, *m* vertices of P_i are already joined with *m* vertices of P_{i-1}, where the remaining *m* vertices of P_i are joined with the remaining *m* vertices of P_i'.

Edges between vertices within a polygon

Only sides of polygon represent edges in G_P. This simple graph is denoted by $G_P(m, p)$ and is named as a polygon graph.

Example 1: If *m* = 2, *p* = 2, G_P *(2 , 2)* (Figure 1)

$G_P(3,1)$ is isomorphic to a famous bicubic symmetric distance-regular Pappus graph with 18 vertices. It has representations as presented in Figure 2.

RESULTS

Theorem 1

Let $G_P(m, p)$ be a polygon graph then:

i) $|V_{G_P}|$ = *2m* (*2p+1*), $|E_{G_P}|$ = *3m* (*2p+1*);
ii) $G_P(m, p)$ is a cubic graph;
iii) $G_P(m, p)$ is a bipartite graph.

Proof

i) Since each polygon has *2m* number of vertices and there are (*2p+1*) copies of polygons in $G_P(m, p)$. Hence, total number of vertices equals:

$|V_{G_P}|$ = *2m* (*2p+1*)

$$|E_{G_P}| = \frac{1}{2}\sum_{k=1}^{|V_{G_P}|} d(v_k)$$

Where $d(v_k)$ is the degree of kth vertex.

$$= \frac{1}{2}\sum_{i=1}^{2m(2p+1)} d(v_i) = \frac{1}{2}3\{2m(2p+1)\} = 3m(2p+1)$$

ii) By definition of $G_P(m, p)$, it is clear that each vertex of a polygon is connected with two vertices of the same polygon and with one vertex of a polygon, either on its right or on its left. Hence, degree of each vertex becomes three. So, $G_P(m, p)$ is a regular simple graph of degree three, that is $G_P(m, p)$ is a cubic graph.
iii) There exists a vertex labeling such that, each even vertex is connected with three odd vertices and each odd vertex is connected with three even vertices, therefore vertices of G_P can be colored using only two colors. As every two colorable graph is bipartite. Hence, G_P is a bipartite graph with equal number of vertices in each part.

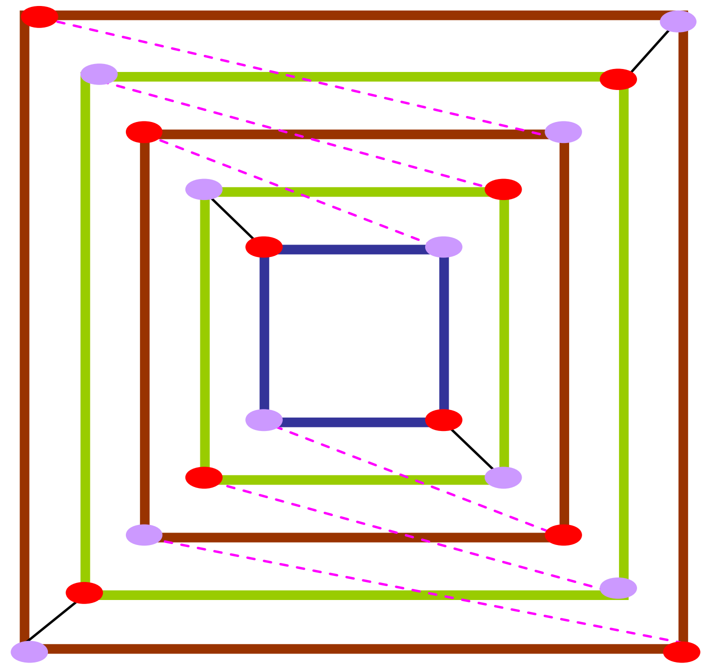

Figure 1. Polygon graph G_p (2 , 2).

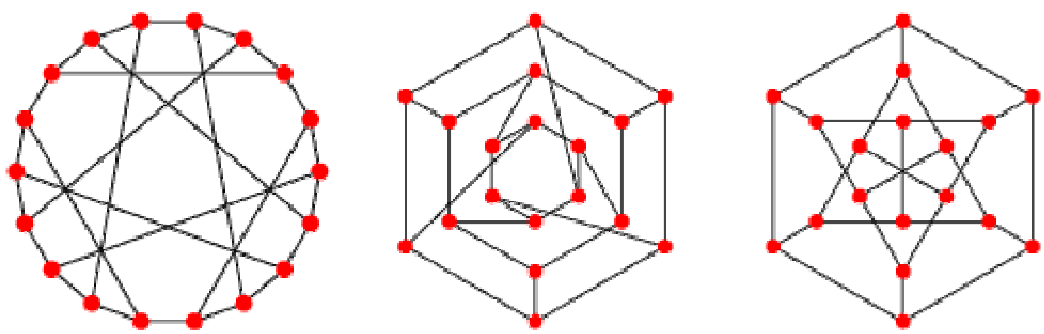

Figure 2. Polygon graph G_p (3 , 1).

Theorem 2

Let $G_P(m,p)$ be a polygon graph, then $\chi\left(G_P(m,p)\right) \leq \Delta\left(G_P(m,p)\right)$

Proof

Since $G_P(m,p)$ is a bipartite graph, therefore there is no odd cycle in it. Also, since by definition $G_P(m,p)$ is not complete, so according to Brooks theorem (1914).

$$\chi\left(G_P(m,p)\right) \leq \Delta\left(G_P(m,p)\right)$$

$\chi\left(G_P(m,t)\right) = 2$, ($G_P(m,p)$ is bipartite) $\Delta\left(G_P(m,t)\right) = 3$ ($G_P(m,p)$ is cubic).

Theorem 3

$G_P(m,p)$ *is a* Hamiltonian graph that is, it contains a Hamiltonian cycle!.

Proof

Let m = 2, we use mathematical induction on p, to prove the result. For p= 1, G_P *(2,1)* contains the Hamiltonian

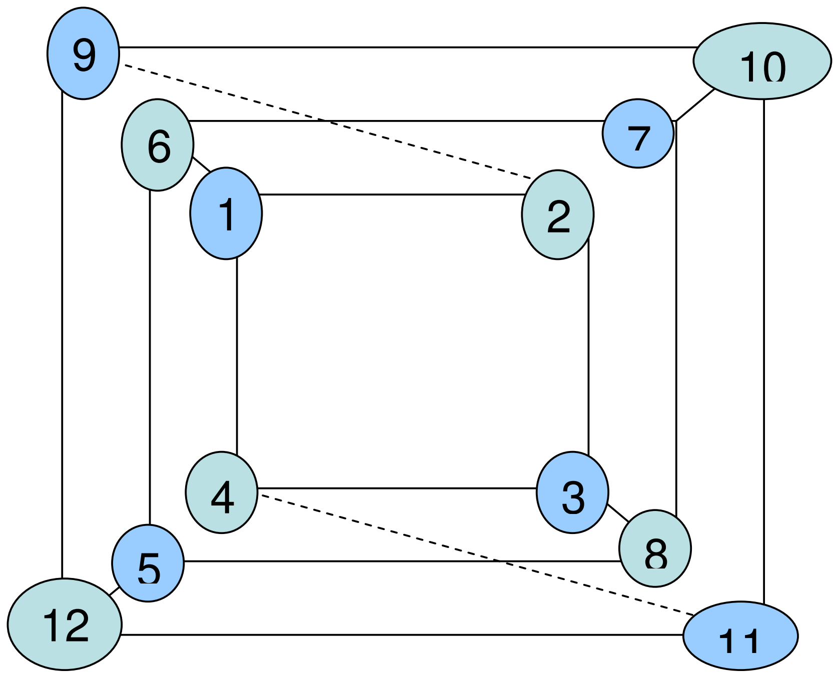

Figure 3. Polygon graph $G_p(2,1)$.

path. 1 – 2 – 3 – 4 – 11 – 12 – 9 – 10 – 7 – 8 – 5 – 6 as shown in Figure 3.

Let $G_P(2,p)$ contain Hamiltonian cycle for $p \le k$ that is;

$$v_1^{(0)}, v_2^{(0)}, v_3^{(0)}, v_4^{(0)}, u_3^{(1)}, u_2^{(1)}, u_1^{(1)}, u_4^{(1)} \ldots\ldots\ldots$$
$$u_3^{(k)}, u_2^{(k)}, u_1^{(k)}, u_4^{(k)}, w_1^{(k)}, w_2^{(k)}, w_3^{(k)}, w_4^{(k)}, \ldots\ldots\ldots\ldots$$
$$w_1^{(k)}, w_2^{(k)}, w_3^{(k)}, w_4^{(k)}\ v_1^{(0)}$$

Now we prove that $G_P(2,p)$ contains Hamiltonian cycle for $p = k+1$. Replace edges between P_k and P_k' by drawing edges between P_k , P_{k+1} and P_k', P_{k+1}' . Now we have two vertices of P_{k+1} and two vertices of P_{k+1}' each of degree two; now connect these vertices to make the following path.

$$v_1^{(0)}, v_2^{(0)}, v_3^{(0)}, v_4^{(0)}, u_3^{(1)}, u_2^{(1)}, u_1^{(1)}, u_4^{(1)} \ldots\ldots\ldots$$
$$u_3^{(k)}, u_2^{(k)}, u_1^{(k)}, u_4^{(k)}, u_3^{(k+1)}, u_2^{(k+1)}, u_1^{(k)+1}, u_4^{(k+1)}$$
$$w_1^{(k+1)}, w_2^{(k+1)}, w_3^{(k+1)}, w_4^{(k+1)}\ w_1^{(k)}, w_2^{(k)}, w_3^{(k)}, w_4^{(k)}, \ldots\ldots\ldots\ldots$$
$$w_1^{(k)}, w_2^{(k)}, w_3^{(k)}, w_4^{(k)}\ v_1^{(0)}$$

is a Hamiltonian path. Similarly, it could be proved that for arbitrary m , $G_P(m,p)$ has Hamiltonian cycle.

Theorem 4

Let $G_P(m,p)$ be a simple graph, for $m \ge 3$, girth of $G_P(m,p)$ is 6 that is, $g(G_P(m,p)) = 6.$

Proof

Since $G_P(m,p)$ is a bipartite graph. The girth must be an even number. There are two types of cycles in $G_P(m,p)$.

Cycles containing vertices of one polygon

Since there are *2m* (with $m \ge 3$) vertices in each polygon, the length of the cycle must be greater than or equal to 6.

$g(G_P(m,p)) = 6$

Cycles involving the vertices of more than one polygon

In this case, the shortest cycle must involve at least two vertices from two adjacent polygons that is, two edges from each polygon.

Moreover, one edge is required to switch from one polygon to the other and one edge to come back making a total of 6.

$$g\left(G_P\left(m,p\right)\right)=6$$

Theorem 5

Let $G_P\left(m,p\right)$ be a polygon graph. Then there are at least $\left(2p+1\right)$ number of cycles of length $2m$.

Proof

All cycles containing the vertices of a polygon are of length $2m$. The cycles containing vertices of more than one polygon of length $2m$ may also exist.

However the number of cycles which involve vertices of only one polygon equals $\left(2p+1\right)$,(since $G_P\left(m,p\right)$, contains $\left(2p+1\right)$ polygons). Hence, there will be at least $2p+1$ cycles of length $2m$.

Theorem 6

Let $G_P\left(m,p\right)$ be a polygon simple graph. Then, there is at least one cycle of length

$2m\left(2p+1\right)$.

Proof

From theorems 5 and 3, $G_P\left(m,p\right)$ is a Hamiltonian graph for all possible values of m and p, so there is a Hamiltonian cycle that is a closed path contain each vertex (except first) exactly once. Since there are $2m\left(2p+1\right)$ vertices in $G_P\left(m,p\right)$. So, there must be at least one cycle of length $2m\left(2p+1\right)$.

Corollary 1

Let $G_P\left(m,p\right)$ be a simple graph. The circumference of $G_P\left(m,p\right)$ is $2m\left(2p+1\right)$.

Theorem 7

Let $G_P\left(m,p\right)$ be a simple graph. Then, there are at most $m\left(2p+1\right)\left[2m\left(2p+1\right)-1\right]$ cycles in G_P.

Proof

Since $G_P\left(m,p\right)$ is a cubic graph. So when a path enters a vertex v_i there are two options for the path to proceed to the next vertex, since the total number of vertices is $2m\left(2p+1\right)$. Therefore, total number of different paths from v_i to v_i equals

$$\binom{2m(2p+1)}{2}=m\ (2p+1)\ [2m(2p+1)\text{-}1]$$

Since at initial vertex that is v_i, there is one more option, at the last there is one less option. Hence, total number of cycles is $m\ (2p+1)\ [2m(2p+1)\text{-}1]$.

Theorem 8

Every edge of $G_P\left(m,p\right)$ is contained in an even number of Hamiltonian cycles.

Proof

Since $G_P\left(m,p\right)$ is a cubic graph that is, degree of each vertex is three let $v_iv_j\in E(G_P\left(\left(m,p\right)\right)$

Let $G_P^1\left(m,p\right)=G_P-v_iv_j$. Now in G_P^1 except v_i and v_j all vertices are of degree three, whereas v_i and v_j are the only vertices of degree two. G_P^1 Contains two longest paths from v_i and v_j, hence $G_P\left(m,p\right)$ posses two Hamiltonian cycles containing v_iv_j □

Conclusions

Polygon graph is a new discrete structure. We worked out basic properties. Polygon graphs are bipartite, therefore could be used as Tanner graphs, to generate low density parity check codes. Moreover, polygon graphs would be attractive due to their simple construction algorithm with a wide range of a choice of parameters. In this paper, we presented polygon graphs of girth 6. It could be extended to the graphs of girth more than 6.

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