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Full Length Research Paper

Modified cumulative sum quality control scheme

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A modified two-sided cumulative sum (MOCUSUM) quality control scheme is proposed that improves CUSUM (Cumulative-Sum) in Average Run Length (ARL) properties. The MOCUSUM scheme is based on single cumulative sum that may be positive or negative and hence can be generalized to multivariate MOCUSUM scheme easily. In this scheme, the 'large' CUSUM values are shrunk towards zero, but the 'small' CUSUM values are updated away from zero. The design and monitoring procedure for MOCUSUM is discussed along with signal resistance and ARL through Markov chain approximation. Algorithms (in R-language) for the MOCUSUM monitoring and computing of ARL are provided.

Key words: ARMA, average run length, CUSUM, inertia, signal resistance, modified CUSUM.

INTRODUCTION

One of the very effective alternatives to the Shewhart control schemes used in detection of small shifts is the Cumulative-Sum (CUSUM) control scheme. CUSUM quality control scheme was first proposed by Page (1961) and have been studied by many authors (Ewan, 1963; Gan, 1991; Hawkins, 1993; Woodall and Adams, 1993). Several improvement and suggestions have been made by researchers from time to time. Yaschin (1987) recommended that the CUSUM should be preferred over the Exponentially Weighted Moving Average (EWMA) chart. He proved that the possibility of an EWMA statistic being in a disadvantageous position is serious than a two-sided CUSUM, which use resets and do not have a significant inertia problem. Crosier (1986) defined a two-sided CUSUM method that requires only one cumulative sum for monitoring lower and upper shifts. The main advantage of Crosier CUSUM is the possible generalization to multivariate CUSUM scheme useful in case of multi variable processes.

In this paper, a modified two-sided cumulative sum quality control scheme is proposed by modifying the Crosier CUSUM statistic. MOCUSUM chart is defined in section of the CUSUM Control Charts for Process Mean. The use of MOCUSUM chart is illustrated in section of illustrations. The properties of MOCUSUM scheme are discussed in section of Average Run Length Properties of MOCUSUM scheme and comparisons with CUSUM schemes.

A comparison of run length distributions and ARL values for MOCUSUM scheme with standard CUSUM and Crosier CUSUM is given. The comparison indicates that MOCUSUM has better ARL properties. Further, in this article we bring to the notice of users that MOCUSUM (also CUSUM) can be used for normal ARMA(1,1) process when the AR(1) and MA(1) parameters have the same sign and differ by not more than 0.3. In this case, also, MOCUSUM is more efficient than CUSUM in detecting small shifts.

The ARL properties of MOCUSUM scheme are derived using Markov chain approach and is described in Appendix A. For the benefit of users and readers, a computational algorithm (in R-language) for monitoring a process by MOCUSUM is given in Appendix B. Algorithm for ARL computation is also listed.

THE CUSUM CONTROL CHARTS FOR PROCESS MEAN

The variable Y may be a single instrument reading or the average of several readings resulting in $Y_1, Y_2, \ldots, Y_n$ independent normally distributed observations from a production process having μ_0 and s as the target mean and standard deviation. The standard two-sided CUSUM scheme detects change in the mean (that is, a shift to the high side/ low side) of a variable Y based on two kinds of cumulative sums. The high-side S_H and the low-side S_L CUSUM are defined as:

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$S_{H(n)} = \max(0, S_{H(n-1)} + Y_n - \mu_0 - ks)$, $S_{H(0)} \geq 0$, $k > 0$,
$S_{L(n)} = \min(0, S_{L(n-1)} + Y_n - \mu_0 + ks)$, $S_{L(0)} \leq 0$, n = 1, 2, 3, ….(1)

The standard CUSUM scheme indicates that the mean has increased at the first n when $S_{H(n)} > hs$, and that the mean has decreased at the first n when $S_{L(n)} < -hs$, where $h > 0$. The parameter h is called the decision interval of the scheme and the parameter k is called the reference value of the scheme or the allowable slack for the process. Crosier (1986) defined a CUSUM scheme based on C_n, $C_n = | S_{n-1} + Y_n - \mu_0 |$, a single cumulative sum. The variable C_n is the absolute value of the updated, but unshrunk CUSUM. Then the high side and low side Crosier CUSUM are expressed in terms of C_n, and the shrinkage is always toward zero by sk_1:

$S_n = 0$ if $C_n \leq sk_1$

$S_n = (S_{n-1} + Y_n - \mu_0)(1 - sk_1/C_n)$ if $C_n > s$ ……. (2)

where $S_0 = 0$, $k_1 > 0$ is the reference value and h_1 is the decision interval of the scheme. The scheme signals that the mean has increased at the first n such that $S_n > sh_1$ or that the mean has decreased at the first n such that $S_n < -sh_1$. Crosier CUSUM statistic mimics the standard two-sided CUSUM; it sets the CUSUM to zero if the updated CUSUM falls is in the interval $[-sk_1, sk_1]$, thus ignores small fluctuations. However, it fails to mimic in case of worst case scenario; the CUSUM is close to its upper (lower) limit and a lower (upper) side shift occur. We modify CUSUM statistic for improving the ability of shift detection. We incorporate single cumulative sum feature so that multivariate generalization can be discussed subsequently.

Modified CUSUM (MOCUSUM) control chart

The two-sided MOCUSUM scheme is based on a single cumulative sum that may be positive, negative or zero, just as a two-sided CUSUM may be. Let $D_n = |T_{n-1} + Y_n - \mu_0|$, $T_0 = 0$ be the updated cumulative sum. It is shrunk towards zero only if updated CUSUM is not smaller than a specified reference amount, otherwise it is raised away from zero. The two-sided MOCUSUM statistic at n^{th} ($n \geq 1$) observation is defined as follows:

$T_n = (T_{n-1} + Y_n - \mu_0)(1 - sk_2/D_n)$ if $D_n \geq sk_2$

$T_n = (T_{n-1} + Y_n - \mu_0)(1 + sk_2/D_n)$ if $D_n < sk_2$ ……….(3)

where, μ_0 is target mean, s is standard deviation, k_2 is reference value, sk_2 the specified reference amount and h_2 specifies decision interval. An MOCUSUM scheme signals that the mean has increased at the first n such that $T_n > sh_2$ or that the mean has decreased at the first n such that $T_n < -sh_2$.

The MOCUSUM statistic $T_n = D_n - sk_2$ or $-D_n + sk_2$; if $D_n \geq sk_2$ is just like a standard CUSUM. But unlike standard CUSUM, MOCUSUM statistic $T_n = D_n + sk_2$ or $-D_n - sk_2$; if $0 < D_n < sk_2$ and $T_n = 0$; when $D_n = 0$. That is, CUSUM is shrink towards zero by sk_2 if CUSUM magnitude is larger than sk_2, and CUSUM is raised by sk_2 and if it falls in the set $(-sk_2, 0)\cup(0, sk_2)$. These directional updates of 'small' CUSUM get cancelled out in the subsequent CUSUM value if the next fluctuation is in the same direction and hence the cumulative sum remains updated by each small fluctuation. On the other hand, if the next fluctuation is in opposite direction then CUSUM gets shrink towards zero by $2sk_2$. In other words, every same sign process fluctuations following a 'small' CUSUM gets cumulated exactly without shrinkage towards zero and every opposite sign process fluctuations next to 'small' CUSUM gets cumulated truncated; shrunk by twice amount. Therefore, the shrinkage is done reasonably and in reasonable numbers of runs, the total shrinkage towards zero remains the same as in case of a standard CUSUM. As a result, MOCUSUM attains the state of process earlier than standard CUSUM, as well as Crosier CUSUM and takes lesser time for shift detection. Our modification improves the accuracy of CUSUM and hence its run length distribution. It also improves the SR property by the shrinkage amount, which is usually the reference amount. The improvement effect magnifies in the multivariate version of MOCUSUM (will be presented in subsequent article).

MOCUSUM for special normal ARMA (1,1) process

The performance of control charts is greatly determined by the corresponding underlying assumptions. Yashchin (1993) cautioned that the ARL property of CUSUM control schemes is considerably affected when it is applied to serially correlated observations. It may considerably reduce the degree of protection against false alarms. However, CUSUM schemes may be applied when the autoregressive parameter and the moving average parameter are equal in sign and close in magnitude (AR (1) = MA (1)). Since serially correlated contain several occurrences of 'small' fluctuations, MOCUSUM scheme is more efficient than the CUSUM schemes in such ARMA processes.

Monitoring by MOCUSUM can easily be carried out using computer algorithm given in Appendix B. The monitoring performance of MOCUSUM is illustrated for detection of small shift (Example 1), detection of abrupt large shifts (Example 2) in normal iid process and for small shifts in special normal ARMA(1,1) process (Example 3) along with performance of standard CUSUM and Crosier CUSUM.

ILLUSTRATIONS

Example 1

Table 1 shows a set of 19 normal iid observations (Lucas and Saccucci, 1990) having zero target mean and unit standard deviation. Lucas (1976) has shown that CUSUM schemes with reference value as k = 1/2 (employed often in practice) outperform Shewhart chart to detect shift of one standard deviation in the process mean.

This example illustrates the performance of the CUSUM charts and MOCUSUM chart for detection of small shift and performance in worst case situations (runs 6-11). Woodall and Mahmoud (2005) have defined a simple measure of inertia, SR (Signal to Resistance); the largest standardized shift from target not resulting in an immediate out of control signal. The worst case SR for EWMA scheme is almost twice that of a two-sided CUSUM. Two-sided CUSUM experience inertia problem particularly when two opposite direction abrupt shifts occur consecutively. MOCUSUM scheme detects shift one run ahead of both the CUSUM schemes. During runs 6-11 the process mean fluctuated negative-positive in

Table 1. Monitoring performance of Crosier's CUSUM, MOCUSUM, and standard CUSUM for the sample data.

Observation		Crosier CUSUM (h_1 = 3.73)		MOCUSUM (h_2 = 3.705)		Standard CUSUM (h = 4)	
		(k = 1/2, μ_0 = 0, s = 1)					
n	Y_n	C_n	S_n	D_n	T_n	$S_{H(n)}$	$S_{L(n)}$
			0		0	0	0
1	1	1	0.5	1	0.5	0.5	0
2	-0.5	0	0	0	0	0	0
3	0	0	0	0	0	0	0
4	-0.8	0.8	-0.3	0.8	-0.3	0	-0.3
5	-0.8	1.1	-0.6	1.1	-0.6	0	-0.6
6	-1.2	1.8	-1.3	1.8	-1.3	0	-1.3
7	1.5	0.2	0	0.2	0.7	1	0
8	-0.6	0.6	-0.1	0.1	0.6	0	-0.1
9	1	0.9	0.4	1.6	1.1	0.5	0
10	-0.9	0.5	0	0.2	0.7	0	-0.4
11	1.2	1.2	0.7	1.9	1.4	0.7	0
12	0.5	1.2	0.7	1.9	1.4	0.7	0
13	2.6	3.3	2.8	4	3.5	2.8	0
14	0.7	3.5	3	4.2	3.7	3	0
15	1.1	4.1	3.6	4.8	4.3*	3.6	0
16	2	5.6	5.1*	6.3	5.8*	5.1*	0
17	1.4	6.5	6*	7.2	6.7*	6*	0
18	1.9	7.9	7.4*	8.6	8.1*	7.4*	0
19	0.8	8.2	7.7*	8.9	8.4*	7.7*	0

*Shift.

succession. Standard CUSUM updated all greater-than half fluctuations shrunk by half in respective CUSUMs.

Crosier CUSUM also updated all greater–than-half fluctuations shrunk by half, while failing to update 7th run fluctuation (inertia problem). MOCUSUM updated all the process fluctuations adding 8th run shrunk towards zero by 1 (twice reference amount), while adding 10th and 11th run exactly. It was observe that Crosier CUSUM got reset to zero ignoring small fluctuations in 7th and 10th runs, while MOCUSUM was updated regularly.

Example 2

The data set is adapted from Hawkins and Maboudou-Tchao (2008). The wearer's heart rate was measured and recorded every 15 min for 6 years. Monitoring of up and down fluctuating sample data levels for target mean μ_0 = 80.95, having standard deviation s =1 is carried by CUSUM type schemes, the standard CUSUM, the Crosier CUSUM and MOCUSUM. This exemplifies the performance in case of large fluctuating process and worst case inertia problems. Standard CUSUM detected shift at observations 4th to 19th and 21st to 24th. Crosier CUSUM method detected shift at 4th to 18th and 21st to 24th observations, while MOCUSUM method detected shift at 4th to 18th and 20th to 24th observations. The process data in Table 2 shows that two abrupt changes in opposite directions, lower side change followed by high side occurred at 19 to 20 observations. All the schemes faced inertia problem, standard CUSUM and MOCUSUM faced it in either case, but Crosier CUSUM faced that in both instances.

We notice that none of the three CUSUM charts could detect both the changes. Standard CUSUM detected lower side change at the 19th observation, but failed in detecting immediate higher change due to inertia problem, MOCUSUM suffered inertia problem for the lower side shift, but could detect the following higher shift because of not ignoring 19th run small fluctuation, while Crosier CUSUM failed in the both cases.

Since MOCUSUM is based on single cumulative sum, its worst case SR (WSR) almost double the WSR of standard CUSUM, but it is smaller than WSR of Crosier CUSUM. Therefore, choice of MOCUSUM over CUSUM is desirable when worst case situations are unlikely or not risky for a process.

Example 3

Several papers have been written to explore the performance of CUSUM chart for serially correlated data. Yashchin (1993) suggested applying CUSUM on i.i.d.

Table 2. Monitoring performance of Crosier's CUSUM, MOCUSUM and standard CUSUM for the mean of heart rate data.

Observation (Mean of Heart Rate – HR)		Crosier Two-Sided CUSUM (h_1 = 3.73)		MOCUSUM (h_2 = 3.705)		Standard CUSUM (h = 4)	
		(k = 0.5, μ_0 = 80.95, s = 1)					
n	Y_n	C_n	S_n	D_n	T_n	$S_{H(n)}$	$S_{L(n)}$
			0		0	0	0
1	79.020	1.93	-1.43	1.93	-1.43	0.0	-1.43
2	81.730	0.65	-0.15	0.65	-0.15	0.28	-0.15
3	81.746	0.65	0.15	0.65	0.15	0.576	0.0
4	87.121	6.32	5.82*	6.32	5.82*	6.247*	0.0
5	83.401	8.27	7.77*	8.27	7.77*	8.198*	0.0
6	80.547	7.36	6.86*	7.36	6.86*	7.295*	0.0
7	81.975	7.89	7.39*	7.89	7.39*	7.82*	0.0
8	81.642	8.08	7.58*	8.08	7.58*	8.012*	0.0
9	82.293	8.92	8.42*	8.92	8.42*	8.855*	0.0
10	80.900	8.37	7.87*	8.37	7.87*	8.305*	0.0
11	81.876	8.80	8.30*	8.80	8.30*	8.731*	0.0
12	83.393	10.74	10.24*	10.74	10.24*	10.674*	0.0
13	80.747	10.04	9.54*	10.04	9.54*	9.971*	0.0
14	82.212	10.80	10.30*	10.80	10.30*	10.733*	0.0
15	80.523	9.88	9.38*	9.88	9.38*	9.806*	0.0
16	79.443	7.87	7.37*	7.87	7.37*	7.799*	-1.007
17	81.222	7.64	7.14*	7.64	7.14*	7.571*	-0.235
18	79.061	5.25	4.75*	5.25	4.75*	5.182*	-1.624
19	76.604	0.41	0.0	0.41	0.91	0.336	-5.470*
20	84.957	4.01	3.51	4.91	4.41*	3.842	-0.963
21	83.823	6.38	5.88*	7.29	6.69*	6.216*	0.0
22	82.672	7.60	7.10*	8.51	8.01*	7.438*	0.0
23	82.948	9.10	8.60*	10.01	9.51*	8.936*	0.0
24	78.917	6.57	6.07*	7.47	6.97*	6.403*	-1.533

*Shift.

transformation of serially correlated data. But Harris and Ross (1991), Longnecker and Ryan (1992) and Zhang (1997) have demonstrated that the residual chart may have poor capability to detect the process mean shift. Table 3 displays the part of measurements on temperature column taken every minute from a chemical process that is working in control and out of control situations (small shifts occur). These observations are ARMA(1,1) with AR (1) = 0.9998 and MA(1) = 0.8632 and normal residuals. The target mean is 95.56°C, and the process standard deviation (is dependent AR(1), MA(1) and the standard deviation of the error terms) is 1.03°C.

Lucas and Crosier (1982) discussed the fast initial response feature for CUSUM scheme that permits a more rapid response to an initial out-of-control situation than standard CUSUM quality control scheme does. MOCUSUM chart detected small shifts ahead by three runs (832nd run) because it considered the small deviation that occurred in the first run. It also detected shifts at 1324th and 1325th observations because of improved decision limit. The Crosier CUSUM chart detected shifts at observations 834th to 1323rd and the standard CUSUM detected shifts at 835th to 1323rd observations.

Average run length properties of MOCUSUM scheme and comparisons with CUSUM schemes

All the ARL computations were carried out using Markov-chain approach described in Appendix A, using a square matrix resulting from the number of state, t = 15.

ARL values for the Crosier CUSUM and the standard CUSUM are given for the sake of comparisons in Table 4.

The ARL values of MOCUSUM schemes have been computed for different choices of decision limit and in control ARL with reference value 1/2, for detection of

Table 3. Monitoring performance of Crosier's CUSUM, MOCUSUM, and standard CUSUM for the mean of temperature data in a chemical process.

Observation (Temperature)		Crosier Two-Sided CUSUM (h_1 = 4.713)		MOCUSUM (h_2 = 4.696)		Standard CUSUM (h = 5)	
		(k = 0.5, μ_0 = 95.56, s = 1)					
n	Y_n	C_n	S_n	D_n	T_n	$S_{H(n)}$	$S_{L(n)}$
0			0		0	0	0
1	95.164	0.396	0	0.396	-0.896	0	0
…	…	…	…	…	…	…	…
831	96.319	4.447	3.947	4.993	4.493	3.947	0
832	96.328	4.715	4.215	5.261	4.761*	4.215	0
833	96.336	4.991	4.491	5.537	5.037*	4.491	0
834	96.345	5.276	4.776*	5.882	5.322*	4.776	0
835	96.354	5.57	5.07*	6.116	5.616*	5.07*	0
…	…	…	…	…	…	…	…
1323	95.604	5.622	5.122*	6.168	5.668*	5.122*	0
1324	95.602	5.164	4.664	5.71	5.21*	4.664	0
1325	95.599	4.703	4.203	5.249	4.749*	4.203	0
1326	95.597	4.24	3.74	4.786	4.286	3.74	0
…	…	…	…	…	…	…	…
1439	95.291	0.269	0	1.003	-0.503	0	0

*Shift.

Table 4. ARL values for the Crosier CUSUM and standard CUSUM.

Crosier CUSUM, k_1 = 0.5, (Conditional steady-state ARL, Crosier; 1986 (Table 4))											
δ	0.00	0.25	0.50	0.75	1.0	1.50	2.0	2.50	3.0	4.0	5.0
h_1 = 3.73	164.0	69.0	24.3	12.1	7.69	4.39	3.12	2.46	2.07	1.60	1.29
h_1 = 4	219.0	82.7	27.1	13.1	8.21	4.66	3.30	2.60	2.18	1.69	1.36
h_1 = 4.713	460.0	130.0	35.1	15.8	9.62	5.36	3.77	2.95	2.45	1.91	1.57
h_1 = 5	618.0	155.0	38.6	16.9	10.2	5.65	3.96	3.09	2.57	1.99	1.66
Standard CUSUM scheme, k = 0.5 (Steady-state ARL, Crosier; 1986 (Table 5, p-190)											
h = 4	163.0	71.6	25.2	12.3	7.68	4.31	3.03	2.38	2.0	1.55	1.22
h = 5	459.0	136.0	36.4	16.0	9.62	5.28	3.68	2.86	2.38	1.86	1.53

Table 5. The ARL values of MOCUSUM schemes computed for different choices of decision limit and in control ARL with reference value ½.

ARL of MOCUSUM, k_2 = 0.50, (Conditional steady-state ARL)											
δ	0.00	0.25	0.50	0.75	1.0	1.50	2.0	2.50	3.0	4.0	5.0
h_2 = 3.705	164.66	60.16	22.17	11.15	6.78	3.42	2.15	1.61	1.32	1.25	1.22
h_2 = 3.916	217.17	72.05	25.68	11.82	7.22	3.64	2.28	1.65	1.30	1.26	1.25
h_2 = 4	260.65	75.41	26.21	12.25	7.42	3.72	2.33	1.68	1.31	1.25	1.25
h_2 = 4.696	460.82	115.70	32.79	14.57	8.67	4.42	2.79	1.96	1.45	1.22	1.27
h_2 = 5	692.98	156.48	33.52	15.74	9.14	4.69	2.99	2.11	1.57	1.22	1.22

small and moderate shifts. These are given in Table 5.

The comparison of ARL values in Tables 4 and 5 indicate that MOCUSUM schemes has improved in control ARL, as well as out of control ARL. The in control ARL for the values of MOCUSUM schemes are greater, while all the out of control ARL values are smaller than the corresponding ARLs of equivalent Crosier CUSUM schemes. For example, for decision limit value 4 (h_1 = 4 =h_2), the in control ARL of MOCUSUM is 260, while Crosier CUSUM in control ARL is 219; an out of control

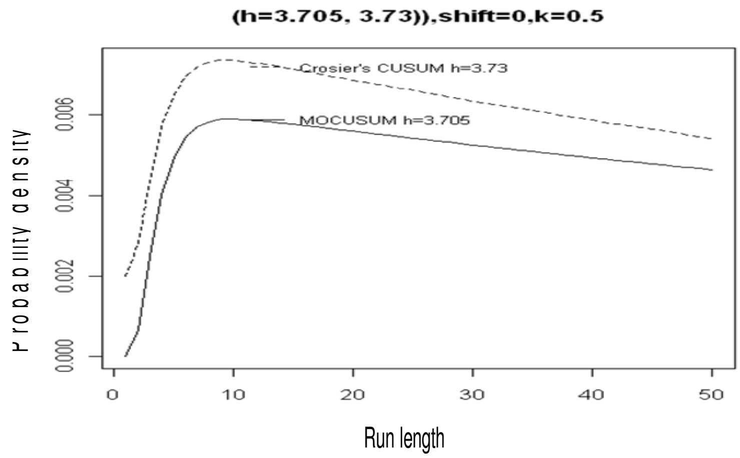

Figure 1. Comparison for run length distribution for shift = 0.

ARL of MOCUSUM is 75, while Crosier CUSUM out of control ARL is 83. So, for this decision limit, MOCUSUM scheme will on average detect small shift earlier than equivalent Crosier scheme by eight runs with lower false alarm rates. The Crosier CUSUM scheme has slightly better average run length (ARL) properties than the standard (Page's) CUSUM scheme, but has worsen inertia problem and there is scope of some improvement.

The run length distributions of MOCUSUM and Crosier CUSUM are plotted in Figure 1 for no shift case.

The run length distributions of MOCUSUM and Crosier CUSUM are plotted in Figure 2 for one standard deviation shift case. This confirms that an MOCUSUM chart will perform better than Crosier CUSUM scheme and hence, also the standard CUSUM scheme, in terms of ARL.

The Inertial properties of CUSUM type control charts

Yashchin (1987, 1993) have suggested that the inertial properties of control charts be considered along with ARL properties in control chart selection. Woodall and Mahmoud (2005) have proposed the measure of inertia, the signal resistance (SR).

In the case of monitoring the mean in the univariate case, they referred to the largest standardized deviation of the sample mean from the target value not leading an immediate out of control signal as the signal resistance of a chart. This measure is most relevant when there is interest in detecting assignable causes that affect the distribution of only one sample mean.

Run length performance is not relevant when an assignable cause affects only a single sample. Further, that determining the value of the signal resistance does not require any distributional assumptions, although if one makes distributional assumptions, then it would be straight forward to calculate the probability of an immediate signal for a particular value of the control chart statistic and an assumed process mean shift. They have shown that the amount of inertia depends on the value of the chart statistic. The measure of inertia could be low if the chart statistic is close to the appropriate boundary when a shift occurs, so, there is no frequent inertia problem.

Standard CUSUM chart

The SR for the standard CUSUM chart is $h + k - w$, where w is the upper CUSUM statistic value. Thus, for the two-sided standard CUSUM chart, the maximum SR is $(h + k)$ standard errors.

Crosier CUSUM chart

The SR for the Crosier CUSUM chart is $h_1 + k_1 - w$, where w is the upper CUSUM statistic value, similar to a standard CUSUM. But the worst case SR ($w = -h_1$) for the Crosier CUSUM chart is $(2h_1 + k_1)$ standard errors.

MOCUSUM chart

The SR for the MOCUSUM chart is $(h_2 + k_2 - w)$, if $w \geq k_2$ and is $(h_2 - w)$ if $w < k_2$, where w is the upper MOCUSUM statistic value. The worst case SR ($w = -h_2$) is $(2h_2 + k_2)$ standard errors.

Crosier CUSUM method shrink all the CUSUM values towards zero, MOCUSUM shrink only 'large' CUSUM values towards zero and updates the 'small' CUSUM values away from zero. The 'small' CUSUM is the zero case of standard CUSUM that may be negative or positive. By this move, an MOCUSUM cumulates all the fluctuations happening in a process and thus possess

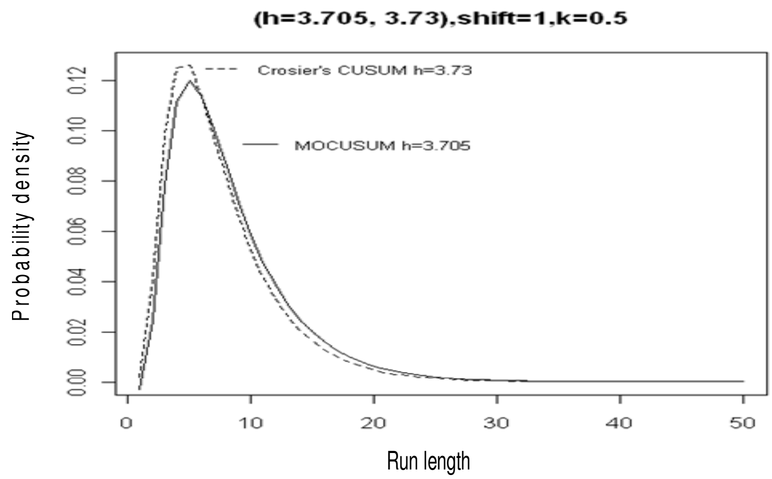

Figure 2. Comparison for run length distribution for shift = 1.

Table 6. Comparisons of SR for standard CUSUM, Crosier CUSUM and MOCUSUM.

Schemes	h	k	Statistic w	SR
Standard CUSUM	4	0.5	0	4.5
	5	0.5	0	5.5
Crosier CUSUM	3.73	0.5	0	4.23
	4.713	0.5	0	5.213
MOCUSUM	3.705	0.5	0	4.205
	4.698	0.5	0	5.198
	3.705	0.5	(0, 0.5)	3.705
	4.698	0.5	(0, 0.5)	4.698

improved run length and slightly improved SR properties. Charts based on single CUSUM statistic will be in disadvantageous position in case when it is positive or zero and a negative large shift occur. However, Page's standard CUSUM will be able to detect it through negative CUSUM. Crosier CUSUM will be in more disadvantageous position than MOCUSUM when nonzero CUSUM statistic is smaller than the reference amount. Example 2 shows this clearly. Table 6 shows the SR values for aforementioned cumulative sum schemes, corresponding to w=0 and 0< w <0.5.

Conclusion

We suggest that the study's modification of CUSUM statistic is helpful in improving detection of small and moderate shift.

MOCUSUM chart may be used for iid normal process, as well as a special normal ARMA (1,1) process. MOCUSUM chart will also minimize the rate of false alarms. There is a noteworthy difference in the number of runs needed to detect small shifts via a CUSUM scheme and a MOCUSUM scheme.

As a single cumulative sum statistic chart, MOCUSUM can not have good inertial properties, but the suggested modification can be extended to improve multivariate CUSUM (MCUSUM).

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APPENDICES

Appendix A

Average run lengths of modified CUSUM (MOCUSUM) scheme using Markov-Chain approach

The use of Markov chains to approximate for one-sided CUSUM schemes was discussed by Brook and Evans (1972). Crosier (1986) extended this methodology for a two-sided CUSUM scheme. The procedure consists of replacing the continuous CUSUM scheme by a discrete CUSUM scheme with transition probability matrix P = [P_{ij}], which gives the probability of moving from state i to state j. The CUSUM is discretized into $2t + 1$ states, t is a suitable positive integer. Two of these are absorbing states representing CUSUM < $-sh_2$ and CUSUM > sh_2. The remaining 2t – 1 states are numbered 1 – t, 2 – t, …, -1, 0, 1, …, t – 1, and they represent CUSUM values between – sh_2 and sh_2. The matrix R is obtained from transition probability matrix P by deleting the rows and columns corresponding to absorbing states. The R matrix is therefore (2t – 1) x (2t -1), and the width of the grouping interval is w = [$2h_2$ / (2t-1)] standard deviations. The probability of moving from state i to state j, 1 – t ≤ i, j ≤ t – 1, under normal distribution, is given as:

Case 1: For j < 0

$P_{ij} = \Pr\{-k_2 + (j - i - ½)\, w < Z \le -k_2 + (j - i + ½)\, w\} = \Phi[-k_2+(j-i+1/2)w)-\delta] - \Phi[-k_2+(j-i-1/2)w)-\delta]$, where δ is shift

Case 2: For j = 0

$P_{i0} = \Pr\{-k_2 + (j - i - ½)w < Z \le +k_2 + (j - i + ½)w\} = \Phi[k_2+(j - i +1/2)w)-\delta] - \Phi[-k_2+(j-i-1/2)w)-\delta]$

Case 3: for j > 0

$P_{ij} = \Pr\{+k_2 + (j - i - ½)\, w < Z \le +k_2 + (j - i + ½)\, w\} = \Phi[k_2+ (j-i+1/2)\, w)-\delta] - \Phi[k_2+ (j-i-1/2)\, w)-\delta]$

where Z has a normal distribution with variance 1 and mean δ.

Then vector u is calculated as:

$$u = [I - R]^{-1}\, I \quad \text{(A.1)}$$

A steady state ARL is found as:

$$ARL = q'\, u, \quad \text{(A.2)}$$

where q is the cyclical (or conditional) steady state distribution of CUSUM values.

The cyclical steady state distribution is obtained from a modified R matrix; the rows of R are made to sum to 1 by adding an adjustment to the column representing initial state of the scheme. The initial q consists of zeros except for a single 1 corresponding to the initial state of the scheme. q is found by M iterations of q = R'q.

The differences of cumulative sum in Crosier CUSUM and MOCUSUM are due to near zero states. The transition probability matrix for MOCUSUM scheme is obtained by modifying the transition probability of near zero state (valued ±0w ±w or ±2w); that is, around center of the range {1-t, t-1}. The near zero states are updated by ±sk_2 amount. So that, the probability of CUSUM being in state ±1, ±2 has increased and that of being in zero state has decreased or is ruled out. Accordingly, we modify the transition probabilities in (i, j = 0, ±1, ±2) cases, preserving the stochastic nature of matrix.

Case 1.1: In particular, for j = -1,-2, i = 0, -1, -2 probability $P_{i\,j}$ are:

$P_{(-2,-2)} = P_{(-2,-2)} + (P_{(-2,0)} / 2)$, $P_{(-2,-1)} = P_{(-2,-1)} + (P_{(-2,0)} / 2)$;
$P_{(-1,-1)} = P_{(-1,-1)} + (P_{(-1,0)} / 2)$, $P_{(-1,-2)} = P_{(-1,-2)} + (P_{(-1,0)} / 2)$;
$P_{(0,-2)} = P_{(0,-2)} + P_{(0,-1)}$, $P_{(0,-1)} = 0$.

Case 2.1: In particular, for j = 0, and i = -2, -1, 1, 2 transition probabilities are:

$P_{(-2,0)} = 0$, $P_{(-1,0)} = 0$, $P_{(1,0)} = 0$, $P_{(2,0)} = 0$.

Case 3.1: In particular, for j = 1, 2 and i = 0, 1, 2 transition probabilities are:

$P_{(2,2)} = P_{(2,2)} + (P_{(2,0)} / 2)$, $P_{(2,1)} = P_{(2,1)} + (P_{(2,0)} / 2)$;
$P_{(1,1)} = P_{(1,1)} + (P_{(1,0)} / 2)$, $P_{(1,2)} = P_{(1,2)} + (P_{(1,0)} / 2)$;
$P_{(0,2)} = P_{(0,2)} + P_{(0,1)}$, $P_{(0,1)} = 0$.

Appendix B: Computational algorithms

MOCUSUM control chart monitoring procedure (With R-language terminology)

Step 1: Let n = number of observations

Step 2: Let Y_n = n process observations
In R-language we write

```
Y<-read.table("Data.txt", header=TRUE) Y<-as.matrix (Y[1:n], ])
OR
Y<-matrix(c(1,2,…,n), nrow = n, ncol = 1)
```

Step 3: Let $D_n = | T_{n-1} + Y_n - \mu_0 |$,the variable D_n is now the absolute value of the MOCUSUM.

Step 4: Let statistic

$T_n = (T_{n-1} + Y_n - \mu_0)(1 + s*k_2/D_n)$ if $D_n < s*k_2$

$T_n = (T_{n-1} + Y_n - \mu_0)(1 - s*k_2/D_n)$ if $D_n \geq s*k_2$

$T_n = 0$ if $D_n = 0$

where $T_0 = 0$ and k > 0.

In R-language we write

```
Dn<-matrix(0,n,1)
Tn<-matrix(0,n,1)
for(i in 1:n)
{  Dn[i]<-Mod(Y[i]- μ0)
 if(Dn[i]> s*k2)    Tn[i]<-(Y[i]- μ0)%*%(1-( s*k2/Dn[i]))
 if(Dn[i]< s*k2)    Tn[i]<-(Y[i]- μ0)%*%(1+( s*k2/Dn[i]))
if((Dn[i]== s*k2)||(Dn[i]==0)) Tn[i]<-0
  if(i>1)
 {  Dn[i]<-Mod(Tn[i-1]+Y[i]- μ0)
  if(Dn[i]>   s*k2)          Tn[i]<-(Tn[i-1]+Y[i]-   μ0)%*%(1-( s*k2/Dn[i]))
  if(Dn[i]<   s*k2)          Tn[i]<-(Tn[i-1]+Y[i]-   μ0)%*%(1+( s*k2/Dn[i]))
  if((Dn[i]== s*k2)||(Dn[i]==0)) Tn[i]<-0 }}
  round(Dn,2)
  round(Tn,2)
```

Step 5: For detection of shift, define column recording values of shift

1: mean has shifted

0: mean has not shifted.

In R-language we write

```
shift<-matrix(0,n,1)
hp<-matrix(sh2,n,1) ## positive h
hn<-(-hp) ## negative h
for(i in 1:n)
{  if(Tn[i]>=0)
{  if(Tn[i]>hp[i])
{  shift[i]<-1   } }
if(Tn[i]<0)
{ if(Tn[i]<hn[i])
          {  shift[i]<-1   }}}
Shift
```

Step 6. End of steps.

ARL values of MOCUSUM control scheme

Step 1: Define values for number of state t, reference value k_2, standard deviation s, decision interval h_2.

Step 2: Compute weight $w = \frac{2h_2}{2t-1}$.

Step 3: Let shift δ, for transition matrix let number of row m = 2t-1 and number of column n = 2t-1.

Step 4: Let transition matrix P_{ij} or r, and adjusted (steady state) matrix r_1.

In R-language we write

```
Pij <- matrix(0,m,n)  or r<- matrix(0,m,n)
 r1<- matrix(0,m,n)
```

Step 5: Let transition matrix P_{ij}. The transition probabilities for the Markov chain are then as follows, 2t – 1 states are numbered 1 – t, 2 – t, …, -1, 0, 1, …, t – 1, and they represent CUSUM values between –hs and hs. The probability of moving from state i to state j, $1 - t \leq i, j \leq t - 1$, is

if j <0

$P_{ij} = \Phi[-k_2+(j-i+1/2)w)-\delta]- \Phi[-k_2+(j-i-1/2)w)-\delta]$

if j=0

$P_{i0} = \Phi[k_2+(j-i+1/2)w)-\delta]- \Phi[-k_2+(j-i-1/2)w)-\delta]$, where δ is shift if j>0

$P_{ij} = \Phi [k_2+ (j-i+1/2) w)-\delta] - \Phi [k_2+ (j-i-1/2) w)-\delta]$ and

```
p[-2,-2]=p[-2,-2]+(p[-2,0]/2)
p[-2,-1]=p[-2,-1]+(p[-2,0]/2)
p[-2,0]=0
p[-1,-1]=p[-1,-1]+(p[-1,0]/2)
p[-1,-2]=p[-1,-2]+(p[-1,0]/2)
p[-1,0]=0
p[2,2]=p[2,2]+(p[2,0]/2)
p[2,1]=p[2,1]+(p[2,0]/2)
p[2,0]=0
p[1,1]=p[1,1]+(p[1,0]/2)
p[1,2]=p[1,2]+(p[1,0]/2)
p[1,0]=0
p[0,2]=p[0,2]+p[0,1]
p[0,1]=0
p[0,-2]=p[0,-2]+p[0,-1]
p[0,-1]=0
```

```
## Transition Matrix
  m<-2*t-1
  n<-2*t-1
  d1<-rep(0,2*t-1)
  for(i in 1:m)
  { d1[i]<-(-t+i)
  for(j in 1:n)
  {d1[j]<-(-t+j)
  if(d1[j]<0)
  p[i,j]<-round(pnorm(-k2+(d1[j]-d1[i]+(1/2))%*%w-d)-
            pnorm(-k2+(d1[j]-d1[i]-1/2)%*%w-d),3)
  else if(d1[j]==0)
p[i,j]<-round(pnorm(k2+(d1[j]-d1[i]+(1/2))%*%w-d)-
pnorm(-k2+(d1[j]-d1[i]-1/2)%*%w-d),3)
  else if(d1[j]>0)
   p[i,j]<-round(pnorm(k2+(d1[j]-d1[i]+(1/2))%*%w-d)-
 pnorm(k2+(d1[j]-d1[i]-1/2)%*%w-d),3)}}
## Changes, z1= zero, middle term
z1<-((2*t-1)+1)/2
##P-2,-2
p[z1-2,z1-2]<-p[z1-2,z1-2]+(p[z1-2,z1]/2)
```

```
##P-2,-1
p[z1-2,z1-1]<-p[z1-2,z1-1]+(p[z1-2,z1]/2)
##P-2,0
p[z1-2,z1]<-0
##P02, p01,p0-2,p0-1
p[z1,z1+2]<-p[z1,z1+2]+p[z1,z1+1]
p[z1,z1+1]<-0
p[z1,z1-2]<-p[z1,z1-2]+p[z1,z1-1]
p[z1,z1-1]<-0
r<-p
```

Step 9: Adjust the transition probability matrix r (say r_1), such that row sums are unity.

$$r_1 = \left(\frac{1-RowSum}{t}\right) + r$$

In R-language we write

```
k1<-rowSums(p)
## Adjusted r1 matrix
r1<-round(((1-k1)/t)+p,3)
k2<-rowSums(r1)
```

Step 10: Let Identity matrix I

In R-language we write

```
I<-matrix(0,m,n)
diag(I)<-1
```

Step 11: Let the vector u gives the ARL of the discrete CUSUM scheme $u = [I - R]^{-1} I$.

In R-language we write

```
u2<-matrix(0,m,n)
u1<-I1-r
u2<-solve(u1)
one<-matrix(1,m,1)
u<-u2%*%one
```

Step 12: Compute $q = r_1'{}^{*}I$, where I is the identity matrix, ARL = q' u, where q is the steady state distribution of CUSUM values.

In R-language we write

```
q<-matrix(0,m,n)
## Arl of steady state
q<-t(r1)%*%I
arls<-t(q)%*%u
arls
```

Step13: End of steps.